RLS-Laguerre Lattice Adaptive Filtering: Error-Feedback, Normalized, and Array-Based Algorithms

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***Abstract*—This paper develops several lattice structures for RLS Laguerre adaptive filtering including *a posteriori* and *a priori* based lattice filters with error-feedback, array-based lattice filters, and normalized lattice filters. All structures are efficient in that their computational cost is proportional to the number of taps, albeit some structures require more multiplications or divisions than others. The performance of all filters, however, can differ under practical considerations, such as finite-precision effects and regularization. Simulations are included to illustrate these facts.**

***Index Terms*—Array algorithm, error feedback, Laguerre network, lattice filter, normalized lattice, order-recursive filter, regularized least-squares, RLS algorithm.**

I. Introduction

In recent work [1], the authors have addressed the problem of developing fast lattice (i.e., order-recursive) RLS filters for Laguerre structures. The regression vectors that arise in such filters do not exhibit the standard shift structure that is characteristic of tapped-delay-line implementations. In other words, successive regression vectors are not shifted versions of each other. Still, the authors showed in [1] that a more general form of data structure exists and that it can be exploited to derive a fast order-recursive filter. In related works [2], [3], the authors have further shown that fixed-order (as opposed to order-recursive) fast RLS Laguerre filters can also be derived by relying on certain structural data constraints established in [4] and [5].

The usefulness of these fast RLS Laguerre filters stems from the fact that it has been realized for some time that Laguerre networks offer superior modeling capabilities than FIR networks, at a reduced number of tap coefficients and with a guaranteed stable performance (see, e.g., [6]–[9] and the references therein). However, the computational cost of existing RLS Laguerre filters has been of the order of M^2 operations per iteration, where M is the order of the filter (number of taps). The fast versions reduce this complexity to M operations per iteration.

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Now, the RLS lattice algorithm that was derived in [1] appears in a form that is based on *a posteriori* errors, and it does not involve an error feedback mechanism. Although this is a common lattice form (see, e.g., [10]–[12]), several other equivalent lattice forms can be derived such as error-feedback forms, array-based forms, and normalized forms. All these algorithms are, of course, theoretically equivalent. However, they tend to differ in performance under different operating conditions that arise, for example, in finite-precision implementations or as a result of noise and regularization. These facts are well understood for RLS lattice filters that result from tapped-delay implementations (see, e.g., [13]–[18]).

The purpose of this paper is to develop similar lattice variants for Laguerre structures. Due to the special form of the regression vectors in the Laguerre case, and due to the lack of shift-structure in the regression vectors, the derivation of these alternative lattice filters is not a direct extension of what has been done before for tapped-delay lines. Some care is needed in deriving the new forms from the *a posteriori* recursions of [1]. In particular, it will be useful to first introduce an extended RLS algorithm; it is a simple yet very convenient extension of the classical RLS algorithm. Once this is done, we will then derive, in sequence, an *a priori*-based lattice filter, error-feedback-based lattice filters, and a normalized lattice filter. In a later section, we will compare the performance of these different forms using fixed-point implementations. Several simulation results are included.

II. Modified RLS Algorithm

Consider a column vector $y_N \in \mathbb{C}^{N+1}$ and a data matrix $H_N \in \mathbb{C}^{(N+1)\times M}$. The exponentially weighted least squares problem seeks the column vector $w \in \mathbb{C}^M$ that solves

$$\min_w [\mu\lambda^{N+1}\|w\|^2 + (y_N - H_N w)^* W_N (y_N - H_N w)] \quad (1)$$

where μ is a scalar positive regularization parameter (usually small), and

$$W_N = \text{diag}\{\lambda^N, \lambda^{N-1}, \ldots, \lambda, 1\}.$$

The so-called forgetting factor λ satisfies $0 \ll \lambda < 1$.

The vector y_N is a growing length vector whose entries are assumed to change according to the following rule:

$$y_N = \begin{bmatrix} a y_{N-1} \\ d(N) \end{bmatrix} \quad (2)$$

for some scalar a.[1] The individual rows of H_N will be denoted by $\{u_i\}$

$$H_N = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_N \end{bmatrix}.$$

Let w_N denote the optimal solution of (1). It is given by

$$w_N \triangleq P_N H_N^* W_N y_N \quad (3)$$

where we introduced the coefficient matrix

$$P_N = (\mu\lambda^{N+1} I + H_N^* W_N H_N)^{-1}. \quad (4)$$

We will further denote the estimate of y_N by $\hat{y}_N = H_N w_N$. We will refer to it as the (regularized) projection of y_N onto H_N.

Now, let w_{N-1} be the solution to a similar problem with the variables $\{y_N, H_N, W_N, \lambda^{N+1}\}$ in (1) replaced by $\{y_{N-1}, H_{N-1}, W_{N-1}, \lambda^N\}$. That is

$$w_{N-1} = (\mu\lambda^N I + H_{N-1}^* W_{N-1} H_{N-1})^{-1} H_{N-1}^* W_{N-1} y_{N-1}.$$

Using (2) and the fact that

$$H_N = \begin{bmatrix} H_{N-1} \\ u_N \end{bmatrix}$$

in addition to the matrix inversion formula, it is straightforward to verify that the following recursions hold:

$$\gamma^{-1}(N) = 1 + \lambda^{-1} u_N P_{N-1} u_N^* \quad (5)$$
$$g_N = \lambda^{-1} P_{N-1} u_N^* \gamma(N) \quad (6)$$
$$w_N = a w_{N-1} + g_N \epsilon(N) \quad (7)$$
$$\epsilon(N) = d(N) - a u_N w_{N-1} \quad (8)$$
$$P_N = \lambda^{-1} P_{N-1} - g_N \gamma^{-1}(N) g_N^* \quad (9)$$

with $w_{-1} = 0$ and $P_{-1} = \mu^{-1} I$. It also holds that $g_N = P_N u_N^*$. These recursions tell us how to update the weight estimates $\{w_N\}$ in time. The well-known exponentially weighted RLS algorithm corresponds to the special choice $a = 1$. The introduction of a nonunity scalar a, however, allows for a level of generality that is sufficient for our purposes in the coming sections.

Notation: Since, in this paper, we deal primarily with order-recursive least-squares problems, it becomes important to explicitly indicate the size of all quantities involved (in addition to a time index). For example, we will write $w_{M,N}$ from now on instead of w_N in order to indicate that it is a vector of order M that is computed by using data up to time N. We will also write $H_{M,N}$ instead of H_N to indicate that it is a matrix with row vectors of size M and with data up to time N.

III. *A Posteriori* RLS-Laguerre Lattice Filter

Before deriving the new lattice variants, we briefly review and summarize the *a posteriori*-based lattice algorithm of [1]. This algorithm was motivated by a (complex) Laguerre structure of the form shown in Fig. 1, where

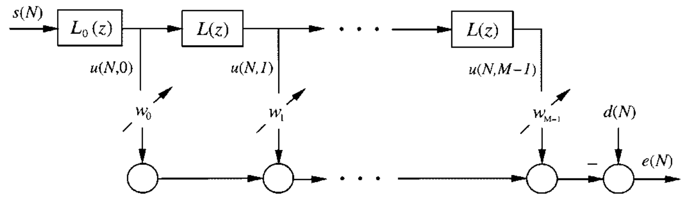

Fig. 1. Transversal Laguerre structure for adaptive filtering.

$$L_0(z) = \frac{\sqrt{1-|a|^2}}{1-az^{-1}} \quad \text{and}$$
$$L(z) = \frac{z^{-1}-a^*}{1-az^{-1}}, \quad 0 < |a| < 1. \quad (10)$$

In this structure, successive regression vectors $\{u_i\}$ are not shifted versions of each other. However, they still satisfy certain structural properties that can be exploited to derive an efficient RLS lattice filter. The resulting algorithm is listed in Table I , and the resulting lattice structure is shown in Fig. 2.

The recursions of Table I assume $\lambda = 1$, and they are based on the propagation of certain *a posteriori* backward and forward estimation errors, which are denoted by

$$\{f_{M+1}(N), b_{M+1}(N), e_{M+1}(N), \tilde{c}_{M+1}(N)\}.$$

In addition, five reflection coefficients, which are denoted by

$$\left\{\kappa_M(N), \kappa_M^f(N), \kappa_M^b(N), \kappa_M^c(N), \kappa_M^{\bar{b}}(N)\right\}$$

are evaluated as ratios of certain quantities for which recursions are also provided. Comparing Fig. 2 with the conventional lattice structure for shift-structured data [19], we see that the new lattice filter is still fundamentally simple, although it now consists of two lattice structures running in parallel.

Although the definitions of the forward and backward errors are standard in the adaptive filtering literature (e.g., [16], [19]), subtle differences do arise in the Laguerre case in view of the fact that the regression vectors do not possess shift structure. For this reason, and for the sake of reference, we briefly reproduce here the definitions introduced in [1]. Thus, consider the data matrix $H_{M+2,N}$ (whose rows are $(M+2)$-dimensional) and partition it as

$$H_{M+2,N} = [H_{M+1,N} \quad x_{M+1,N}]$$
$$= [x_{0,N} \quad \bar{H}_{M,N} \quad x_{M+1,N}].$$

Note that we are denoting the last column of $H_{M+2,N}$ by $x_{M+1,N}$ and its first column by $x_{0,N}$. Note also that while in the case of regressors with shift structure, there exists a simple relation between $\bar{H}_{M,N}$ and $H_{M,N}$, this relation is less obvious in the Laguerre case.

Further, let c_N denote the column vector

$$c_N^* \triangleq a^* \sqrt{1-|a|^2}[a^{N-1} \quad a^{N-2} \quad \cdots \quad 1 \quad a^{-1}]$$

[1]Why we introduce the scalar a will be understood very soon. The classical recursive least-squares (RLS) problem corresponds to the special choice $a = 1$.

TABLE I
A POSTERIORI-BASED RLS-LAGUERRE LATTICE FILTER

Initialization

For $m = 0$ to $M-1$ set:

μ is a small positive number.

$$\delta_m(-1) = \rho_m(-1) = \tau_m(-1) = 0$$
$$\zeta_m^f(-1) = \zeta_m^b(-1) = \zeta_m^{\bar{b}}(-1) = \mu$$
$$\zeta_m^c(-1) = 1$$

For $N \geq 0$, repeat:

$$u(N) = au(N-1) + \sqrt{1-|a|^2}s(N)$$

$$\gamma_0(N) = 1 \qquad e_0(N) = d(N)$$
$$\bar{\gamma}_0(N) = 1 \qquad f_0(N) = u(N)$$
$$\tilde{c}_0(N) = \frac{a}{a^*}\sqrt{1-|a|^2} \qquad b_0(N) = u(N)$$

For $m = 0$ to $M-1$, repeat:

$$\tau_m(N) = a^*\tau_m(N-1) + \frac{\tilde{c}_m^*(N)b_m(N)}{\gamma_m(N)}$$
$$\zeta_m^c(N) = |a|^2\zeta_m^c(N-1) + \frac{|\tilde{c}_m(N)|^2}{\gamma_m(N)}$$
$$\kappa_m^{\bar{b}}(N) = \frac{\tau_m(N)}{\zeta_m^c(N)}$$
$$\bar{b}_m(N) = -\frac{1}{a}\left(b_m(N) - \kappa_m^{\bar{b}}(N)\tilde{c}_m(N)\right)$$

$$\zeta_m^f(N) = \zeta_m^f(N-1) + \frac{|f_m(N)|^2}{\bar{\gamma}_m(N)}$$
$$\zeta_m^b(N) = \zeta_m^b(N-1) + \frac{|b_m(N)|^2}{\gamma_m(N)}$$
$$\zeta_m^{\bar{b}}(N) = \zeta_m^{\bar{b}}(N-1) + \frac{|\bar{b}_m(N)|^2}{\bar{\gamma}_m(N)}$$

$$\delta_m(N) = \delta_m(N-1) + \frac{f_m^*(N)\bar{b}_m(N)}{\bar{\gamma}_m(N)}$$
$$\rho_m(N) = \rho_m(N-1) + \frac{e_m^*(N)b_m(N)}{\gamma_m(N)}$$

$$\gamma_{m+1}(N) = \gamma_m(N) - \frac{|b_m(N)|^2}{\zeta_m^b(N)}$$
$$\bar{\gamma}_{m+1}(N) = \bar{\gamma}_m(N) - \frac{|\bar{b}_m(N)|^2}{\zeta_m^{\bar{b}}(N)}$$

$$\kappa_m^c(N) = \frac{\tau_m^*(N)}{\zeta_m^b(N)} \qquad \kappa_m(N) = \frac{\rho_m^*(N)}{\zeta_m^b(N)}$$
$$\kappa_m^b(N) = \frac{\delta_m(N)}{\zeta_m^f(N)} \qquad \kappa_m^f(N) = \frac{\delta_m^*(N)}{\zeta_m^{\bar{b}}(N)}$$

$$\tilde{c}_{m+1}(N) = \tilde{c}_m(N) - \kappa_m^c(N)b_m(N)$$
$$e_{m+1}(N) = e_m(N) - \kappa_m(N)b_m(N)$$
$$b_{m+1}(N) = \bar{b}_m(N) - \kappa_m^b(N)f_m(N)$$
$$f_{m+1}(N) = f_m(N) - \kappa_m^f(N)\bar{b}_m(N)$$

Alternative order-updates:

$$\zeta_{m+1}^f(N) = \zeta_m^f(N) - \frac{|\delta_m(N)|^2}{\zeta_m^{\bar{b}}(N)}$$
$$\zeta_{m+1}^b(N) = \zeta_m^{\bar{b}}(N) - \frac{|\delta_m(N)|^2}{\zeta_m^f(N)}$$
$$\zeta_{m+1}^c(N) = \zeta_M^c(N) \, \mathfrak{g} \, \frac{|\tau_m(N)|^2}{\zeta_m^b(N)}$$
$$\zeta_m^{\bar{b}}(N) = \zeta_m^b(N) - \frac{|\tau_m(N)|^2}{\zeta_m^c(N)}$$
$$\zeta_{m+1}(N) = \zeta_m(N) - \frac{|\rho_m(N)|^2}{\zeta_m^b(N)}$$
$$\gamma_{m+1}(N) = \bar{\gamma}_m(N) - \frac{|f_m(N)|^2}{\zeta_m^f(N)}$$

which is defined in terms of the pole a of each section $L(z)$. Now, the filter listed in Table I provides order-recursive updates for *a posteriori* errors that arise from the problems of projecting (in a regularized manner) the vectors $\{x_{0,N}, x_{M+1,N}\}$ onto $\bar{H}_{M,N}$ and $H_{M+1,N}$ and the vectors $\{y_N, c_{M,N}\}$ onto $H_{M,N}$. More specifically, consider the error vectors

$$\begin{aligned} b_{M+1,N} &= x_{M+1,N} - H_{M+1,N}w_{M+1,N}^b \\ \bar{b}_{M,N} &= x_{M+1,N} - \bar{H}_{M,N}w_{M,N}^{\bar{b}} \\ f_{M+1,N} &= x_{0,N} - \bar{H}_{M,N}w_{M,N}^f \\ e_{M,N} &= y_N - H_{M,N}w_{M,N} \\ \tilde{c}_{M,N} &= c_N - H_{M,N}w_{M,N}^c \end{aligned}$$

where, for example, $w_{M,N}^f$ is the solution to

$$\min_{w_M}[\mu\lambda^{N+1}\|w_M\|^2 + (x_{0,N} - \bar{H}_{M,N}w_M)^* \cdot W_N(x_{0,N} - \bar{H}_{M,N}w_M)]. \tag{11}$$

This problem projects the first column $x_{0,N}$ onto $\bar{H}_{M,N}$. Similarly, we define $\{w_{M,N}^b, w_{M,N}^c, w_{M,N}\}$ (see [1]). The last entries of the above error vectors are denoted by

$$\{b_{M+1}(N), \bar{b}_M(N), f_{M+1}(N), e_M(N), \tilde{c}_M(N)\}.$$

These are the quantities that appear in the recursions of Table I.

IV. Error-Feedback Lattice Filters

As mentioned above, the order updates for the *a posteriori* estimation errors in Table I are described in terms of certain reflection coefficients, which are in turn computed as ratios of certain quantities. For example, the reflection coefficient $\kappa_M^f(N)$ is computed as the ratio $\delta_M^*(N)/\zeta_M^b(N)$, with separate recursions available for both the numerator and the denominator.

An error-feedback form of the algorithm can be obtained by deriving explicit recursions for the reflection coefficients themselves. We shall arrive at this form in three steps. First, we define certain *a priori* errors; then, we derive order-update relations for them, and finally, we derive time-update relations for the reflection coefficients.

A. *A Priori* Estimation Errors

We first introduce *a priori*, as opposed to *a posteriori*, estimation errors. Thus, define the *a priori* error vectors

$$\begin{aligned} \beta_{M+1,N} &= x_{M+1,N} - H_{M+1,N}w_{M+1,N-1}^b \\ \bar{\beta}_{M,N} &= x_{M+1,N} - \bar{H}_{M,N}w_{M,N-1}^b \\ \alpha_{M+1,N} &= x_{0,N} - \bar{H}_{M,N}w_{M,N-1}^f \\ \epsilon_{M,N} &= y_N - H_{M,N}w_{M,N-1} \end{aligned}$$

where now, $w_{M,N-1}^f$, for example, is the solution to a problem similar to (11) with all Ns by $N-1$. Comparing these expressions for the *a priori* error vectors with the expressions above for the *a posteriori* error vectors $\{b_{M+1,N}, \bar{b}_{M,N}, f_{M+1,N}, e_{M,N}, \tilde{c}_{M,N}\}$, the only differences lie in the use of the prior weight vector estimates.

The last entries of these *a priori* error vectors are denoted by

$$\{\beta_{M+1}(N), \bar{\beta}_M(N), \alpha_{M+1}(N), \epsilon_M(N)\}$$

and they play a fundamental role in all future developments. In particular, by following the same argument as in [1, Sec.

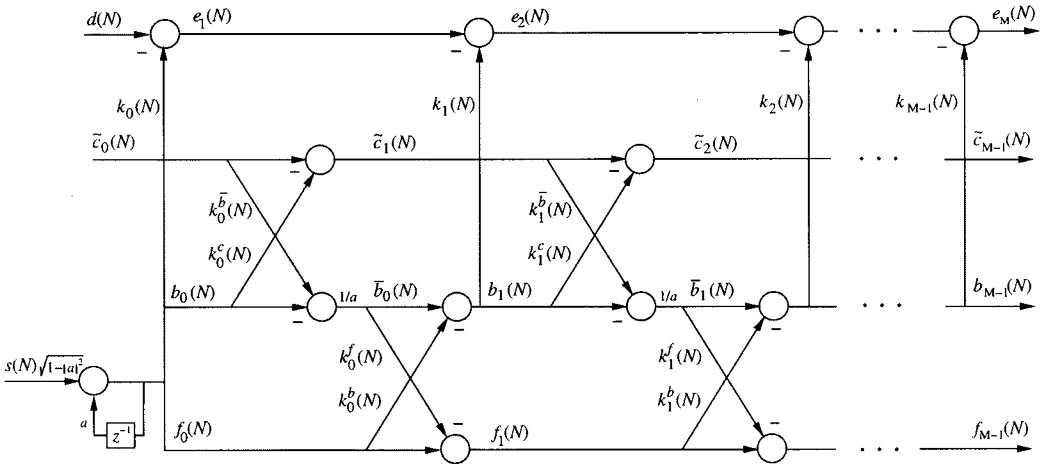

Fig. 2. *A posteriori*-based RLS-Laguerre lattice filter, where $\tilde{c}_0(N) = (a/a^*)\sqrt{1-|a|^2}$.

III], it can be verified that these errors satisfy the following order-update relations in terms of the same reflection coefficients $\{\kappa_M(N), \kappa_M^f(N), \kappa_M^b(N)\}$ that are defined in Table I:

$$\epsilon_{M+1}(N) = \epsilon_M(N) - \kappa_M(N-1)\beta_M(N) \tag{12}$$

$$\beta_{M+1}(N) = \bar{\beta}_M(N) - \kappa_M^b(N-1)\alpha_M(N) \tag{13}$$

$$\alpha_{M+1}(N) = \alpha_M(N) - \kappa_M^f(N-1)\bar{\beta}_M(N). \tag{14}$$

A recursion for the *a priori* error that corresponds to $\tilde{c}_M(N)$ requires a little more effort. For this purpose, we first recall from [1] that $\tilde{c}_M(N)$ satisfies the order-update relation

$$\tilde{c}_{M+1}(N) = \tilde{c}_M(N) - \kappa_M^c(N) b_M(N).$$

However, since the vector c_N evolves in time according to the relation

$$c_N = \begin{bmatrix} ac_{N-1} \\ c_M(N) \end{bmatrix}$$

then, by resorting to the modified form of the RLS algorithm of Section II, we readily conclude that the *a priori* error that is associated with $\tilde{c}_M(N)$ should have the form

$$\begin{aligned} \nu_M(N) &\triangleq c_M(N) - au_{M,N}w_{M,N-1}^c \\ &= \frac{a}{a^*}\sqrt{1-|a|^2} - au_{M,N}w_{M,N-1}^c. \end{aligned} \tag{15}$$

Using this definition, the following order update for $\nu_M(N)$ can be established by again relying on the arguments of [1]:

$$\nu_{M+1}(N) = \nu_M(N) - a\kappa_M^c(N-1)\beta_M(N). \tag{16}$$

B. Relating $\beta_M(N)$ and $\bar{\beta}_M(N)$

In order to complete the recursions (12)–(14) for the *a priori* errors, we still need to know how to update $\bar{\beta}_M(N)$. This can be done as follows. We first recall from [1] that the optimal weight vectors $w_{M,N-1}^{\bar{b}}$ and $w_{M,N-1}^b$ are related by

$$\begin{aligned} w_{M,N-1}^{\bar{b}} &= w_{M,N-1}^b \\ &\quad - \frac{P_{M,N-1}H_{M,N-1}^*c_{N-1}c_{N-1}^*}{1 - c_{N-1}^*H_{M,N-1}P_{M,N-1}H_{M,N-1}^*c_{N-1}} b_{M,N-1} \end{aligned} \tag{17}$$

where

$$c_{N-1} \triangleq a^*\sqrt{1-|a|^2}[a^{N-2} \quad a^{N-3} \quad \cdots \quad a \quad 1]^T.$$

It was further shown in [1] that the data matrices $\bar{H}_{M,N}$ and $H_{M,N}$ are related via $\bar{H}_{M,N} = \Phi_N H_{M,N}$, where Φ_N is the $(N+1)\times(N+1)$ lower triangular Toeplitz matrix (here for the complex case)

$$\Phi_N = \begin{bmatrix} -a^* & & & & \\ 1-|a|^2 & -a^* & & & \\ a(1-|a|^2) & 1-|a|^2 & -a^* & & \\ \vdots & \vdots & \ddots & \ddots & \\ a^{(N-1)}(1-|a|^2) & \cdots & \cdots & 1-|a|^2 & -a^* \end{bmatrix} \tag{18}$$

Thus, multiplying (17) from the left by $\bar{H}_{M,N} = \Phi H_{M,N}$ and subtracting $x_{M+1,N}$ from both sides, we obtain

$$\bar{\beta}_M(N) = \phi_N\beta_{M,N} + \frac{c_{N-1}^*b_{M,N-1}}{1 - c_{N-1}^*\hat{c}_{M,N-1}}\phi_N H_{M,N}w_{M,N-1}^c \tag{19}$$

where ϕ_N denotes the last row of Φ_N and is given by

$$\begin{aligned} \phi_N &\triangleq [a^{N-1}(1-|a|^2) \quad \cdots \quad (1-|a|^2) \quad -a^*] \\ &= \left[\frac{a}{a^*}\sqrt{1-|a|^2}c_{N-1}^* \quad -a\right] \end{aligned}$$

and $\hat{c}_{M,N-1}$ denotes the leading N entries of $H_{M,N}w_{M,N-1}^c$, which we partition as follows:

$$H_{M,N}w_{M,N-1}^c = \begin{bmatrix} \hat{c}_{M,N-1} \\ u_{M,N}w_{M,N-1}^c \end{bmatrix}$$

$$\beta_{M,N} = \begin{bmatrix} b_{M,N-1} \\ \beta_M(N) \end{bmatrix}.$$

If we substitute these partitionings into (19) and expand, we obtain, after some manipulations, the relation

$$\bar{\beta}_M(N) = -a\beta_M(N) + \kappa_M^{\bar{b}}(N-1)\nu_M(N) \quad (20)$$

where $\nu_M(N)$ is the *a priori* error defined in (15). Having derived the order-update relations for the various *a priori* estimation errors, it only remains to derive time-update relations for the reflection coefficients

$$\left\{\kappa_M(N), \kappa_M^f(N), \kappa_M^b(N), \kappa_M^c(N), \kappa_M^{\bar{b}}(N)\right\}.$$

C. Time Updates for the Reflection Coefficients

Unlike conventional derivations of error-feedback lattice algorithms for tapped-delay line structures, we will obtain time-update relations for the reflection coefficients in a more direct way by exploiting the fact that these coefficients can be regarded as solutions to least-squares problem of first order [5].

We start with the reflection coefficient

$$\kappa_M(N) = \frac{\rho_M^*(N)}{\zeta_M^b(N)} \quad (21)$$

where, from Table I, the numerator and denominator quantities satisfy the time updates

$$\rho_M(N) = \rho_M(N-1) + \frac{e_M^*(N)b_M(N)}{\gamma_M(N)}$$
$$\zeta_M^b(N) = \zeta_M^b(N-1) + \frac{|b_M(N)|^2}{\gamma_M(N)}.$$

Now, define the *angle normalized* errors

$$b'_M(N) \triangleq \frac{b_M(N)}{\gamma_M^{1/2}(N)} = \beta_M(N)\gamma_M^{1/2}(N)$$
$$e'_M(N) \triangleq \frac{e_M(N)}{\gamma_M^{1/2}(N)} = \epsilon_M(N)\gamma_M^{1/2}(N)$$

in terms of the square-root of the conversion factor $\gamma_M(N)$. It then follows from the above time updates for $\rho_M(N)$ and $\zeta_M^b(N)$ that $\{\rho_M(N), \zeta_M^b(N)\}$ can be recognized as the inner products

$$\rho_M^*(N) = b'^*_{M,N}e'_{M,N}$$
$$\zeta_M^b(N) = \mu + b'^*_{M,N}b'_{M,N}$$

which are written in terms of the following vectors of angle normalized prediction errors:[2]

$$b'_{M,N} \triangleq \begin{bmatrix} b'_M(0) \\ b'_M(1) \\ \vdots \\ b'_M(N) \end{bmatrix}, \quad e'_{M,N} \triangleq \begin{bmatrix} e'_M(0) \\ e'_M(1) \\ \vdots \\ e'_M(N) \end{bmatrix}.$$

In this way, the defining relation (21) for $\kappa_M(N)$ becomes

$$\kappa_M(N) = (\mu + b'^*_{M,N}b'_{M,N})^{-1}b'^*_{M,N}e'_{M,N}$$

which shows that $\kappa_M(N)$ can be interpreted as the regularized least-squares solution of a first-order least-squares problem, namely, that of projecting (in a regularized manner) the vector $e'_{M,N}$ onto the vector $b'_{M,N}$. This simple observation shows that $\kappa_M(N)$ can be readily time updated via a standard RLS recursion of the form[3]

$$\begin{aligned}\kappa_M(N) &= \kappa_M(N-1) + \frac{b'^*_M(N)}{\zeta_M^b(N)}[e'_M(N) \\ &\quad - b'_M(N)\kappa_M(N-1)] \\ &= \kappa_M(N-1) + \frac{b^*_M(N)}{\zeta_M^b(N)}[\epsilon_M(N) \\ &\quad - \beta_M(N)\kappa_M(N-1)] \\ &= \kappa_M(N-1) + \frac{\beta^*_M(N)\gamma_M(N)}{\zeta_M^b(N)}\,\epsilon_{M+1}(N).\end{aligned}$$

The last equation is obtained from the order update for $\epsilon_M(N)$ in (12). Similarly, using (13) and (14), we can justify the following updates for $\kappa_M^f(N)$ and $\kappa_M^b(N)$:

$$\kappa_M^f(N) = \kappa_M^f(N-1) + \frac{\bar{\beta}^*_M(N)\bar{\gamma}_M(N)}{\zeta_M^{\bar{b}}(N)}\,\alpha_{M+1}(N) \quad (22)$$
$$\kappa_M^b(N) = \kappa_M^b(N-1) + \frac{\beta^*_M(N)\gamma_M(N)}{\zeta_M^f(N)}\,\beta_{M+1}(N). \quad (23)$$

Now, let us examine the reflection coefficient $\kappa_M^c(N)$. Defining the angle-normalized quantity $\tilde{c}'_M(N) = \tilde{c}_M(N)/\gamma_M^{1/2}(N)$, we can express $\kappa_M^c(N)$ as a least-squares solution of the form

$$\kappa_M^c(N) = (\mu + b'^*_{M,N}b'_{M,N})^{-1}b'^*_{M,N}A_N\tilde{c}'_{M,N}$$

where $A_N = \text{diag}\{a^N, a^{N-1}, \ldots, a, 1\}$. This means that we can again time update $\kappa_M^c(N)$ via an RLS recursion of the general form (7), i.e.,

$$\kappa_M^c(N) = a\kappa_M^c(N-1) + \frac{\beta^*_M(N)\gamma_M(N)}{\zeta_M^b(N)}\,\nu_{M+1}(N).$$

Finally, we need to update the reflection coefficient $\kappa_M^{\bar{b}}(N)$. Again, using the time updates for $\zeta_M^c(N)$ and $\tau_M(N)$, it can be expressed as a least-squares solution of the form

$$\kappa_M^{\bar{b}}(N) = \left(1 + \tilde{c}'^*_{M,N}A_N^2\tilde{c}'_{M,N}\right)^{-1}\tilde{c}'^*_{M,N}A_Nb'_{M,N}$$

or, equivalently

$$\kappa_M^{\bar{b}}(N) = \left(1 + \tilde{c}'^*_{M,N}A_N^2\tilde{c}'_{M,N}\right)^{-1}\tilde{c}'^*_{M,N}A_N^2\left(A_N^{-1}b'_{M,N}\right).$$

Writing the RLS recursion for this variable, we obtain

$$\begin{aligned}\kappa_M^{\bar{b}}(N) &= \frac{1}{a}\,\kappa_M^{\bar{b}}(N-1) + \frac{\tilde{c}'^*_M(N)}{\zeta_M^c(N)} \\ &\quad \times \left[b'_M(N) - \frac{1}{a}\tilde{c}'_M(N)\kappa_M^{\bar{b}}(N-1)\right] \\ &= \frac{1}{a}\left(\kappa_M^{\bar{b}}(N-1) - \frac{\nu^*_M(N)\gamma_M(N)}{\zeta_M^c(N)}\,\bar{\beta}_M(N)\right)\end{aligned}$$

where the last equation is obtained from (20). Fig. 3 illustrates the resulting RLS-Laguerre lattice structure that is based on *a priori* errors. The corresponding recursions are listed in Table II.

[2] Observe that the vectors $b_{M,N}$ and $b'_{M,N}$ differ in a fundamental way. The entries of $b_{M,N}$ cannot be interpreted as $\{b_M(0), b_M(1), \ldots, b_M(N)\}$, that is, all the elements of $b_{M,N}$ change with a change in N and likewise for $e_{M,N}$ and $e'_{M,N}$.

[3] This geometric argument avoids some typical algebraic derivations.

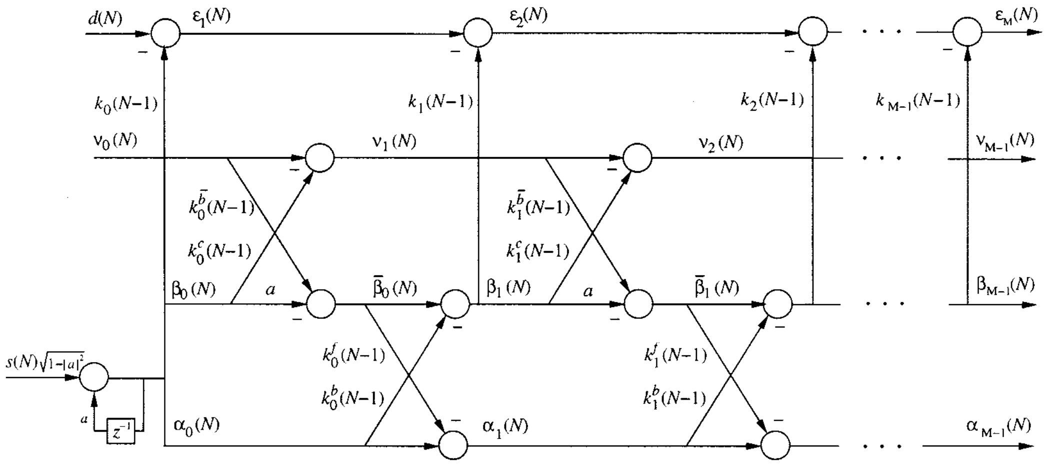

Fig. 3. RLS-Laguerre lattice network based on *a priori* errors with error feedback.

TABLE II
ERROR-FEEDBACK RLS-LAGUERRE LATTICE FILTER BASED ON *A PRIORI* ERRORS

Initialization

For $m = 0$ to $M-1$ set:

μ is a small positive number.

$$\kappa_m^b(-1) = \kappa_m^{\bar{b}}(-1) = \kappa_m^f(-1) = \kappa_m^c(-1) = \kappa_m(-1) = 0$$
$$\zeta_m^f(-1) = \zeta_m^b(-1) = \zeta_m^{\bar{b}}(-1) = \mu$$
$$\zeta_m^c(-1) = 1$$

For $N \geq 0$, repeat:

$$u(N) = au(N-1) + \sqrt{1-|a|^2}s(N)$$

$$\gamma_0(N) = 1 \qquad \epsilon_0(N) = d(N)$$
$$\bar{\gamma}_0(N) = 1 \qquad \alpha_0(N) = u(N)$$
$$\nu_0(N) = \tfrac{a}{a^*}\sqrt{1-|a|^2} \qquad \beta_0(N) = u(N)$$

For $m = 0$ to $M-1$, repeat:

$$\zeta_m^c(N) = |a|^2\zeta_m^c(N-1) + |\nu_m(N)|^2\gamma_m(N)$$
$$\bar{\beta}_m(N) = -a^*\beta_m(N) + \kappa_m^{\bar{b}}(N-1)\nu_m(N)$$
$$\kappa_m^{\bar{b}}(N) = \tfrac{1}{a}\left(\kappa_m^{\bar{b}}(N-1) - \tfrac{\nu_m(N)\gamma_m(N)}{\zeta_m^c(N)}\bar{\beta}_m(N)\right)$$

$$\zeta_m^{\bar{b}}(N) = \zeta_m^{\bar{b}}(N-1) + |\bar{\beta}_m(N)|^2\bar{\gamma}_m(N)$$
$$\zeta_m^f(N) = \zeta_m^f(N-1) + |\alpha_m(N)|^2\bar{\gamma}_m(N)$$
$$\zeta_m^b(N) = \zeta_m^b(N-1) + |\beta_m(N)|^2\gamma_m(N)$$

$$\nu_{m+1}(N) = \nu_m(N) - a\kappa_m^c(N-1)\beta_m(N)$$
$$\epsilon_{m+1}(N) = \epsilon_m(N) - \kappa_m(N-1)\beta_m(N)$$
$$\beta_{m+1}(N) = \bar{\beta}_m(N) - \kappa_m^b(N-1)\alpha_m(N)$$
$$\alpha_{m+1}(N) = \alpha_m(N) - \kappa_m^f(N-1)\bar{\beta}_m(N)$$

$$\kappa_m^c(N) = a\kappa_m^c(N-1) + \tfrac{\beta_m^*(N)\gamma_m(N)}{\zeta_m^b(N)}\nu_{m+1}(N)$$
$$\kappa_m^f(N) = \kappa_m^f(N-1) + \tfrac{\bar{\beta}_m^*(N)\bar{\gamma}_m(N)}{\zeta_m^{\bar{b}}(N)}\alpha_{m+1}(N)$$
$$\kappa_m^b(N) = \kappa_m^b(N-1) + \tfrac{\beta_m^*(N)\gamma_m(N)}{\zeta_m^f(N)}\beta_{m+1}(N)$$
$$\kappa_m(N) = \kappa_m(N-1) + \tfrac{\beta_m^*(N)\gamma_m(N)}{\zeta_m^b(N)}\epsilon_{m+1}(N)$$

$$\gamma_{m+1}(N) = \gamma_m(N) - \tfrac{|b_m(N)|^2}{\zeta_m^b(N)}$$
$$\bar{\gamma}_{m+1}(N) = \bar{\gamma}_m(N) - \tfrac{|\bar{b}_m(N)|^2}{\zeta_m^b(N)}$$

D. *A Posteriori*-Based Error Feedback Filter

A similar error feedback filter can be derived by relying on *a posteriori* errors rather than *a priori* errors. This can be achieved simply by expressing the updates for the reflection coefficients in terms of *a posteriori* quantities and rearranging terms. For example, consider the first recursion shown previously for $\kappa_M(N)$, which can be written as

$$\kappa_M(N) = \left(1 - \frac{|\tilde{b}'_M(N)|^2}{\zeta_M^b(N)}\right)\kappa_M(N-1) + \frac{\tilde{b}'^*_M(N)e'_M(N)}{\gamma_M(N)\zeta_M^b(N)}$$
$$= \frac{\gamma_{M+1}(N)}{\gamma_M(N)}\left[\kappa_M(N-1) + \frac{\tilde{b}^*_M(N)e_M(N)}{\gamma_M(N)\zeta_M^b(N-1)}\right].$$

Here we used the following relation from Table I:

$$\frac{\gamma_{M+1}(N)}{\gamma_M(N)} = \frac{\zeta_M^b(N-1)}{\zeta_M^b(N)}.$$

In a similar manner, we can derive time updates for the other reflection coefficients. The resulting equations are summarized in Table III.

V. ARRAY-BASED LATTICE ALGORITHM

We now derive another equivalent lattice form, albeit one that is described in terms of compact arrays. This form involves only orthogonal rotations and tends to exhibit good numerical properties.

To arrive at the array form, we first define the following quantities:

$$q_M^b(N) \triangleq \frac{\delta_M(N)}{\zeta_M^{\bar{b}/2}(N)}, \quad q_M^f(N) \triangleq \frac{\delta_M^*(N)}{\zeta_M^{f/2}(N)}$$
$$q_M^{\bar{b}}(N) \triangleq \frac{\tau_M(N)}{\zeta_M^{c/2}(N)}, \quad q_M^c(N) \triangleq \frac{\tau_M^*(N)}{\zeta_M^{\bar{b}/2}(N)}.$$

The second step is to rewrite all the recursions in Table II in terms of these quantities and in terms of the angle normalized prediction errors $\{b'_M(N), e'_M(N), \tilde{c}'_M(N), \bar{b}'_M(N)\}$ defined before, e.g.,

$$\tau_M(N) = a\tau_M(N-1) + \tilde{c}'^*_M(N)b'_M(N) \tag{24}$$
$$\zeta_M^c(N) = |a|^2\zeta_M^c(N-1) + |\tilde{c}'_M(N)|^2 \tag{25}$$
$$\zeta_M^b(N) = \zeta_M^b(N-1) + |b'_M(N)|^2 \tag{26}$$
$$\zeta_M^{\bar{b}}(N) = \zeta_M^b(N) - \left|q_M^{\bar{b}}(N)\right|^2. \tag{27}$$

TABLE III
ERROR FEEDBACK RLS-LAGUERRE LATTICE FILTER BASED ON *A POSTERIORI* ERRORS

Initialization

For $m = 0$ to $M-1$ set:

μ is a small positive number.
$\delta_m(-1) = \rho_m(-1) = \tau_m(-1) = 0$
$\zeta_m^f(-1) = \zeta_m^b(-1) = \zeta_m^{\bar{b}}(-1) = \mu$
$\zeta_m^c(-1) = 1$

For $N \geq 0$, repeat:

$u(N) = au(N-1) + \sqrt{1-|a|^2}s(N)$

$\gamma_0(N) = 1 \quad e_0(N) = d(N)$
$\bar{\gamma}_0(N) = 1 \quad f_0(N) = u(N)$
$\tilde{c}_0(N) = \frac{a}{a^*}\sqrt{1-|a|^2} \quad b_0(N) = u(N)$

For $m = 0$ to $M-1$, repeat:

$\zeta_m^c(N) = |a|^2\zeta_m^c(N-1) + \frac{|\tilde{c}_m(N)|^2}{\gamma_m(N)}$
$\kappa_m^{\bar{b}}(N) = \frac{1}{\zeta_m^c(N)}\left[a\zeta_m^c(N-1)\kappa_m^{\bar{b}}(N-1) + \frac{\tilde{c}_m^*(N)b_m(N)}{\gamma_m(N)}\right]$
$\bar{b}_m(N) = -\frac{1}{a}\left(b_m(N) - \kappa_m^{\bar{b}}(N)\tilde{c}_m(N)\right)$

$\zeta_m^f(N) = \zeta_m^f(N-1) + \frac{|f_m(N)|^2}{\bar{\gamma}_m(N)}$
$\zeta_m^b(N) = \zeta_m^b(N-1) + \frac{|b_m(N)|^2}{\gamma_m(N)}$
$\zeta_m^{\bar{b}}(N) = \zeta_m^{\bar{b}}(N-1) + \frac{|\bar{b}_m(N)|^2}{\bar{\gamma}_m(N)}$

$\gamma_{m+1}(N) = \gamma_m(N) - \frac{|b_m(N)|^2}{\zeta_m^b(N)}$
$\bar{\gamma}_{m+1}(N) = \bar{\gamma}_m(N) - \frac{|\bar{b}_m(N)|^2}{\zeta_m^{\bar{b}}(N)}$

$\kappa_m^c(N) = \frac{\gamma_{m+1}(N)}{\gamma_m(N)}\left[a\kappa_m^c(N-1) + \frac{b_m^*(N)\tilde{c}_m(N)}{\gamma_m(N)\zeta_m^b(N-1)}\right]$

$\kappa_m^b(N) = \frac{\gamma_{m+1}(N)}{\bar{\gamma}_m(N)}\left[\kappa_m(N-1) + \frac{f_m^*(N)\bar{b}_m(N)}{\bar{\gamma}_m(N)\zeta_m^f(N-1)}\right]$

$\kappa_m^f(N) = \frac{\bar{\gamma}_{m+1}(N)}{\bar{\gamma}_m(N)}\left[\kappa_m^f(N-1) + \frac{\bar{b}_m^*(N)f_m(N)}{\bar{\gamma}_m(N)\zeta_m^{\bar{b}}(N-1)}\right]$

$\kappa_m(N) = \frac{\gamma_{m+1}(N)}{\gamma_m(N)}\left[\kappa_m(N-1) + \frac{b_m^*(N)e_m(N)}{\gamma_m(N)\zeta_m^b(N-1)}\right]$

$\tilde{c}_{m+1}(N) = \tilde{c}_m(N) - \kappa_m^c(N)b_m(N)$
$e_{m+1}(N) = e_m(N) - \kappa_m(N)b_m(N)$
$b_{m+1}(N) = \bar{b}_m(N) - \kappa_m^b(N)f_m(N)$
$f_{m+1}(N) = f_m(N) - \kappa_m^f(N)\bar{b}_m(N)$

The third step is to implement a unitary transformation matrix Θ that lower triangularizes the following prearray of numbers:

$$\begin{bmatrix} a\zeta_M^{c/2}(N-1) & \tilde{c}_M^{\prime *}(N) \\ q_M^{\bar{b}*}(N-1) & b_M^{\prime *}(N) \end{bmatrix}\Theta = \begin{bmatrix} m & 0 \\ n & p \end{bmatrix}$$

for some $\{m, n, p\}$. The values of the resulting $\{m, n, p\}$ can be determined from the equality shown at the bottom of the page.

$$\begin{bmatrix} a\zeta_M^{c/2}(N-1) & \tilde{c}_M^{\prime *}(N) \\ q_M^{\bar{b}*}(N-1) & b_M^{\prime *}(N) \end{bmatrix}\underbrace{\Theta\Theta^*}_{I}\begin{bmatrix} a\zeta_M^{c/2}(N-1) & \tilde{c}_M^{\prime *}(N) \\ q_M^{\bar{b}*}(N-1) & b_M^{\prime *}(N) \end{bmatrix}^* = \begin{bmatrix} m & 0 \\ n & p \end{bmatrix}\begin{bmatrix} m & 0 \\ n & p \end{bmatrix}^*$$

Using (25) and (26), we can make the identifications

$$m = \zeta_M^{c/2}(N), \quad n = q_M^{\bar{b}*}(N), \quad p = \bar{b}_M^{\prime *}(N).$$

Proceeding similarly, we can derive three additional array transformations, all of which are listed in Table IV. The resulting algorithm is also represented schematically in Fig. 4. The matrices $\{\Theta_M^{\bar{b}}(N), \Theta_M^c(N), \Theta_M^f(N), \Theta_M^b(N)\}$ are 2×2 unitary (Givens) transformations that introduce the zero entries in the post-arrays at the desired locations.

VI. NORMALIZED RLS-LAGUERRE LATTICE ALGORITHM

The final lattice form that we consider is one that allows us to reduce the number of reflection coefficients. Thus, note that the lattice filters considered this far require the propagation of five reflection coefficients $\{\kappa_M^f(N), \kappa_M^b(N), \kappa_M^c(N), \kappa_M^{\bar{b}}(N), \kappa_M(N)\}$. An equivalent variant can be derived that requires the propagation of only three reflection coefficients. We will denote these new coefficients by $\{\eta_M(N), \omega_M(N), \varphi_M(N)\}$.

A. Recursion for $\eta_M(N)$

We start by defining the coefficient

$$\eta_M(N) \triangleq \frac{\delta_M^*(N)}{\zeta_M^{\bar{b}/2}(N)\zeta_M^{f/2}(N)}$$

along with the normalized prediction errors

$$b_M''(N) \triangleq \frac{b_M(N)}{\gamma_M^{1/2}(N)\zeta_M^{b/2}(N)}$$
$$f_M''(N) \triangleq \frac{f_M(N)}{\bar{\gamma}_M^{1/2}(N)\zeta_M^{f/2}(N)}$$
$$\bar{b}_M''(N) \triangleq \frac{\bar{b}_M(N)}{\bar{\gamma}_M^{1/2}(N)\zeta_M^{\bar{b}/2}(N)}$$
$$\tilde{c}_M''(N) \triangleq \frac{\tilde{c}_M(N)}{\gamma_M^{1/2}(N)\zeta_M^{c/2}(N)}.$$

Now, referring to Table IV, let us substitute the updating equation for $\{\alpha_{M+1}(N)\}$ into the updating equation for $\{\kappa_M^f(N)\}$. This yields

$$\kappa_M^f(N) = \kappa_M^f(N-1)(1-|\bar{b}_M''(N)|^2) + \frac{f_M(N)\bar{b}_M^*(N)}{\zeta_M^{\bar{b}}(N)\bar{\gamma}_M(N)}. \tag{28}$$

Multiplying both sides by the ratio $\zeta_M^{\bar{b}/2}(N)/\zeta_M^{f/2}(N)$, we obtain

$$\eta_M(N) = \frac{\zeta_M^{\bar{b}/2}(N)}{\zeta_M^{f/2}(N)}\kappa_M^f(N-1)(1-|\bar{b}_M''(N)|^2) + f_M''(N)\bar{b}_M^{\prime\prime *}(N). \tag{29}$$

TABLE IV
ARRAY-BASED RLS LAGUERRE LATTICE FILTER

Initialization

For $m = 0$ to $M-1$ set:

μ is a small positive number.
$q_m^f(-1) = q_m^b(-1) = q_m^c(-1) = q_m^d(-1) = \kappa_m^{\bar{b}}(-1) = 0$
$\zeta_m^{f/2}(-1) = \zeta_m^{b/2}(-1) = \zeta_m^{\bar{b}/2}(-1) = \sqrt{\mu}$
$\zeta_m^c(-1) = 1$

For $N \geq 0$, repeat:

$u(N) = au(N-1) + \sqrt{1-|a|^2}s(N)$

$\gamma_0^{1/2}(N) = 1$
$e_0'(N) = d(N)$
$f_0'(N) = b_0'(N) = u(N)$
$\tilde{c}_0'(N) = \frac{a}{a^*}\sqrt{1-|a|^2}$

For $m = 0$ to $M-1$, repeat:

$$\begin{bmatrix} a\zeta_m^{c/2}(N-1) & \tilde{c}'^*_m(N) \\ q_m^{\bar{b}*}(N-1) & b'^*_m(N) \end{bmatrix} \Theta_m^{\bar{b}}(N) = \begin{bmatrix} \zeta_m^{c/2}(N) & 0 \\ q_m^{\bar{b}*}(N) & \bar{b}'^*_m(N) \end{bmatrix}$$

$$\begin{bmatrix} \zeta_m^{b/2}(N-1) & b'^*_m(N) \\ aq_m^{c*}(N-1) & \tilde{c}'^*_m(N) \\ q_m^{d*}(N-1) & e'^*_m(N) \\ 0 & \gamma_m^{1/2}(N) \end{bmatrix} \Theta_m^c(N) = \begin{bmatrix} \zeta_m^{b/2}(N) & 0 \\ q_m^{c*}(N) & \tilde{c}'^*_m(N) \\ q_m^{d*}(N) & e'^*_{m+1}(N) \\ \times & \gamma_{m+1}^{1/2}(N) \end{bmatrix}$$

$$\begin{bmatrix} \zeta_m^{f/2}(N-1) & f'^*_m(N) \\ q_m^{f*}(N-1) & \bar{b}'^*_m(N) \end{bmatrix} \Theta_m^f(N) = \begin{bmatrix} \zeta_m^{f/2}(N) & 0 \\ q_m^{f*}(N) & b'^*_{m+1}(N) \end{bmatrix}$$

$$\begin{bmatrix} \zeta_m^{\bar{b}/2}(N-1) & \bar{b}'^*_m(N) \\ q_m^{b*}(N-1) & f'^*_m(N) \end{bmatrix} \Theta_m^b(N) = \begin{bmatrix} \zeta_m^{\bar{b}/2}(N) & 0 \\ q_m^{b*}(N) & f'^*_{m+1}(N) \end{bmatrix}$$

However, from the time-update recursion for $\zeta_M^{\bar{b}}(N)$ and $\zeta_M^f(N)$, the following relations hold:

$$\zeta_M^{\bar{b}/2}(N) = \frac{\zeta_M^{\bar{b}/2}(N-1)}{\sqrt{1-|\bar{b}_M''(N)|^2}}$$
$$\zeta_M^{f/2}(N) = \frac{\zeta_M^{f/2}(N-1)}{\sqrt{1-|f_M''(N)|^2}}.$$

Substituting these equations into (29), we obtain the desired time-update recursion for the first reflection coefficient:

$$\eta_M(N) = \eta_M(N-1)\sqrt{(1-|\bar{b}_M''(N)|^2)(1-|f_M''(N)|^2)} + f_M''(N)\bar{b}_M''^*(N).$$

This recursion is in terms of the errors $\{b_M''(N), f_M''(N)\}$. We now determine order updates for these errors. Thus, dividing the order update equation for $b_{M+1}(N)$ by $\zeta_{M+1}^{b/2}(N)\gamma_{M+1}^{1/2}(N)$, we obtain

$$b_{M+1}''(N) = \frac{\bar{b}_M(N) - \kappa_M^b(N)f_M(N)}{\zeta_{M+1}^{b/2}(N)\gamma_{M+1}^{1/2}(N)}. \tag{30}$$

Using the order-update relations for $\zeta_M^b(N)$, we also have

$$\zeta_{M+1}^b(N) = \zeta_M^{\bar{b}}(N)(1-|\eta_M(N)|^2). \tag{31}$$

In addition, by following the arguments of [1], $\gamma_M(N)$ can be seen to satisfy the relation

$$\gamma_{M+1}(N) = \bar{\gamma}_M(N) - \frac{|f_M(N)|^2}{\zeta_M^f(N)} \tag{32}$$

which can be written as

$$\gamma_{M+1}(N) = \bar{\gamma}_M(N)(1-|f_M''(N)|^2). \tag{33}$$

Substituting (31) and (33) into (30), we obtain

$$b_{M+1}''(N) = \frac{\bar{b}_M''(N) - \eta_M^*(N)f_M''(N)}{\sqrt{(1-|f_M''(N)|^2)(1-|\eta_M(N)|^2)}}. \tag{34}$$

Similarly, using the order updates for $f_{M+1}(N)$, $\zeta_M^f(N)$, and $\bar{\gamma}_M(N)$, we obtain

$$f_{M+1}''(N) = \frac{f_M''(N) - \eta_M(N)\bar{b}_M''(N)}{\sqrt{(1-|\bar{b}_M''(N)|^2)(1-|\eta_M(N)|^2)}}. \tag{35}$$

B. Recursion for $\omega_M(N)$

In a similar vein, we introduce the normalized error

$$e_M''(N) \triangleq \frac{e_M(N)}{\gamma_M^{1/2}(N)\zeta_M^{1/2}(N)}$$

and the coefficient

$$\omega_M(N) \triangleq \frac{\rho_M^*(N)}{\zeta_M^{b/2}(N)\zeta_M^{1/2}(N)}.$$

Using the order update for $\zeta_M^{1/2}(N)$ and $\gamma_M(N)$, we can establish the following recursion:

$$e_{M+1}''(N) = \frac{e_M''(N) - \omega_M(N)b_M''(N)}{\sqrt{(1-|b_M''(N)|^2)(1-|\omega_M(N)|^2)}}.$$

To obtain a time update for $\omega_M(N)$, we first substitute the recursion for $e_{M+1}(N)$ into the time update for $\kappa_M(N)$. Then, multiplying the resulting equation by the ratio $\zeta_M^{b/2}(N)/\zeta_M^{e/2}(N)$ and using the time updates for $\zeta_M^b(N)$ and $\zeta_M(N)$, we obtain

$$\omega_M(N+1) = \sqrt{(1-|b_M''(N)|^2)(1-|e_M''(N)|^2)}\omega_M(N) + b_M''^*(N)e_M''(N).$$

Note that when $\bar{b}_M''(N) = b_M''(N-1)$, the recursions derived so far collapse to the well-known FIR normalized RLS-lattice algorithm. For Laguerre structures, however, we need to derive a recursion for the normalized variable $\bar{b}_M''(N)$ as well. This can be achieved by normalizing the order-update for $\bar{b}_M(N)$

$$\bar{b}_M''(N) = -\frac{1}{a}\frac{b_M(N) - \kappa_M^{\bar{b}}(N)\tilde{c}_M(N)}{\zeta_M^{\bar{b}/2}(N)\bar{\gamma}_M^{1/2}(N)}. \tag{36}$$

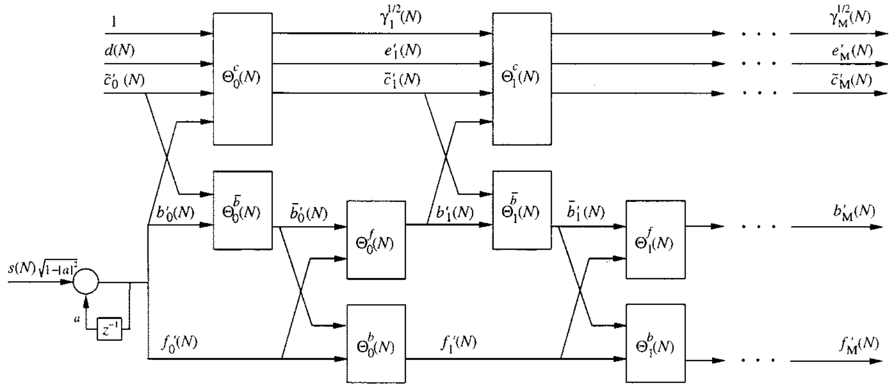

Fig. 4. Schematic representation of the array-based RLS Laguerre lattice filter.

TABLE V
NORMALIZED RLS-LAGUERRE LATTICE FILTER BASED ON A POSTERIORI ERRORS.

Initialization

For $m = 0$ to $M-1$ set:

μ is a small positive number.
$\zeta_m^f(-1) = \zeta_m^b(-1) = \zeta_m(-1) = \mu$

For $N \geq 0$, repeat:

$u(N) = au(N-1) + \sqrt{1-|a|^2}s(N)$
$\zeta_0^b(N) = \zeta_0^b(N-1) + |u(N)|^2$
$\zeta_0(N) = \zeta_0(N-1) + |d(N)|^2$
$b_0''(N) = f_0''(N) = u(N)/\zeta_0^{b/2}(N)$
$e_0''(N) = d(N)/\zeta_0^{1/2}(N)$
$c_0''(N) = \frac{a}{a^*}\sqrt{1-|a|^2}$

For $m = 0$ to $M-1$, repeat:

$r_m^b(N) = \sqrt{1-|b_m''(N)|^2} \quad r_m^f(N) = \sqrt{1-|f_m''(N)|^2}$
$r_m^e(N) = \sqrt{1-|e_m''(N)|^2} \quad r_m^c(N) = \sqrt{1-|c_m''(N)|^2}$

$\varphi_m(N+1) = \frac{a}{|a|} r_m^c(N) r_m^b(N) \varphi_m(N) + b_m^{''*}(N)\tilde{c}_m''(N)$
$r_m^\varphi(N) = \sqrt{1-|\varphi_m(N)|^2}$
$\bar{b}_m''(N) = -\frac{r_m^f(N)}{a r_m^b(N) r_m^\varphi(N)}\left(b_m''(N) - \varphi_m(N)\tilde{c}_m''(N)\right)$
$r_m^{\bar{b}}(N) = \sqrt{1-|\bar{b}_m''(N)|^2}$

$\eta_m(N) = r_m^{\bar{b}}(N) r_m^f(N) \eta_m(N-1) + f_m''(N)\bar{b}_m^{''*}(N)$
$r_m^\eta(N) = \sqrt{1-|\eta_m(N)|^2}$
$\omega_m(N+1) = r_m^b(N) r_m^e(N) \omega_m(N) + b_m^{''*}(N) e_m''(N)$
$r_m^\omega(N) = \sqrt{1-|\omega_m(N)|^2}$

$\tilde{c}_{m+1}''(N) = \frac{1}{r_m^b(N) r_m^\varphi(N)}\left(\tilde{c}_m''(N) - \varphi_m^*(N) b_m''(N)\right)$
$e_{m+1}''(N) = \frac{1}{r_m^b(N) r_m^e(N)}\left(e_m''(N) - \omega_m(N) b_m''(N)\right)$
$b_{m+1}''(N) = \frac{1}{r_m^f(N) r_m^\eta(N)}\left(\bar{b}_m''(N) - \eta_m^*(N) f_m''(N)\right)$
$f_{m+1}''(N) = \frac{1}{r_m^b(N) r_m^\eta(N)}\left(f_m''(N) - \eta_m(N)\bar{b}_m''(N)\right)$

In order to simplify this equation, we need to relate $\zeta_M^{\bar{b}}(N)$ to $\zeta_M^b(N)$ and $\bar{\gamma}_M(N)$ to $\gamma_M(N)$. We have shown in [1] that the following equation holds:

$$\zeta_M^{\bar{b}}(N) = \zeta_M^b(N) - \frac{|\tau_M(N)|^2}{\zeta_M^c(N)} \tag{37}$$

TABLE VI
COMPARISON OF THE COMPUTATIONAL COST OF THE DIFFERENT RLS-LAGUERRE LATTICE ALGORITHMS FOR FILTERS OF ORDER M.

Algorithm	Mult.	Div.	Add.	$\sqrt{\cdot}$
Standard *a posteriori*	$17M+2$	$14M$	$14M+1$	-
A priori error feedback	$24M+2$	$7M$	$16M+1$	-
A posteriori error feedback	$27M+2$	$16M$	$16M+1$	-
Array-based lattice	$48M$	$8M$	$24M$	$4M$
Normalized lattice	$30M+4$	$5M+2$	$16M$	$8M+2$

which implies

$$\zeta_M^{\bar{b}}(N) = \zeta_M^b(N)(1 - |\varphi_M(N)|^2). \tag{38}$$

In order to relate $\{\bar{\gamma}_M(N), \gamma_M(N)\}$, note that $\gamma_{M+1}(N)$ can be written either as in Table I or as in (32). That is

$$\bar{\gamma}_M(N) - \frac{|f_M(N)|^2}{\zeta_M^f(N)} = \gamma_M(N) - \frac{|b_M(N)|^2}{\zeta_M^b(N)}.$$

This leads to the following desired relation:

$$\bar{\gamma}_M(N) = \gamma_M(N)\frac{1-|b_M''(N)|^2}{1-|f_M''(N)|^2}. \tag{39}$$

Substituting (38) and (39) into (36), we obtain

$$\bar{b}_M''(N) = -\frac{\sqrt{1-|f_M''(N)|^2}\left(b_M''(N) - \varphi_M(N)\tilde{c}_M''(N)\right)}{a\sqrt{(1-|b_M''(N)|^2)(1-|\varphi_M(N)|^2)}}.$$

This equation requires an order update for the normalized quantity $\tilde{c}_M''(N)$. From the order update for $\tilde{c}_M(N)$, we can write

$$\tilde{c}_{M+1}''(N) = \frac{\tilde{c}_M(N) - \kappa_M^c(N) b_M(N)}{\zeta_{M+1}^{c/2}(N)\gamma_{M+1}^{1/2}(N)}. \tag{40}$$

From the order update for $\zeta_M^{c/2}(N)$ [1], we have

$$\zeta_{M+1}^{c/2}(N) = \zeta_M^{c/2}(N)\sqrt{(1-|\varphi_M(N)|^2)}.$$

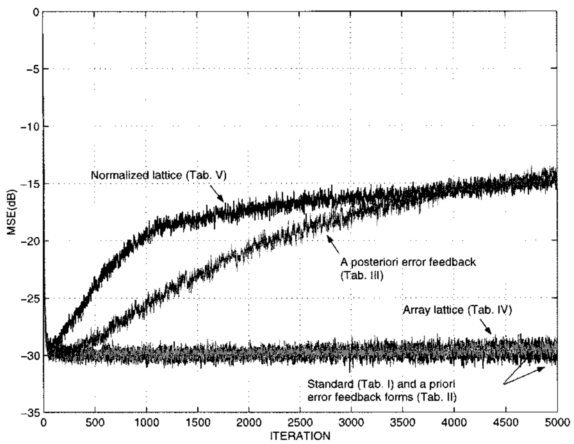

Fig. 5. MSE decay of the various Laguerre lattice forms under finite precision (ten bits).

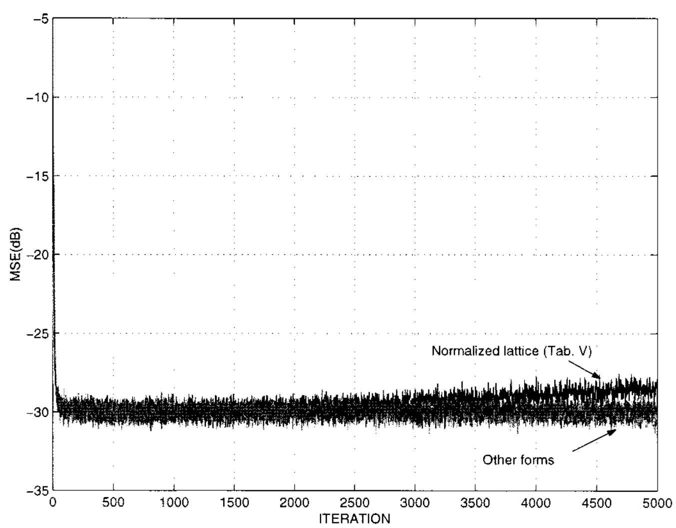

Fig. 6. MSE decay of the various Laguerre lattice forms under finite precision (15 bits).

Substituting this equation, along with the order update for $\gamma_M(N)$ into (40), we get

$$\tilde{c}''_{M+1}(N) = \frac{\tilde{c}''_M(N) - \varphi^*_M(N) b''_M(N)}{\sqrt{(1 - |b''_M(N)|^2)(1 - |\varphi_M(N)|^2)}}.$$

C. Recursion for $\varphi_M(N)$

Finally, we define the coefficient

$$\varphi_M(N) \triangleq \frac{\tau_M(N)}{\zeta_M^{b/2}(N)\zeta_M^{c/2}(N)}.$$

In order to derive an update for it, we proceed similarly to the former recursions. First, we substitute $\nu_{M+1}(N)$ into the recursion for $\kappa^c_M(N)$ from Table II. Then, multiplying the resulting equation by $\zeta_M^{b/2}(N)/\zeta_M^{c/2}(N)$ and using the time updates for $\zeta^b_M(N)$ and $\zeta^c_M(N)$, we obtain

$$\varphi_M(N+1) = \frac{a}{|a|}\sqrt{(1 - |\tilde{c}''_M(N)|^2)(1 - |b''_M(N)|^2)}\varphi_M(N) + b''^*_M(N)\tilde{c}''_M(N).$$

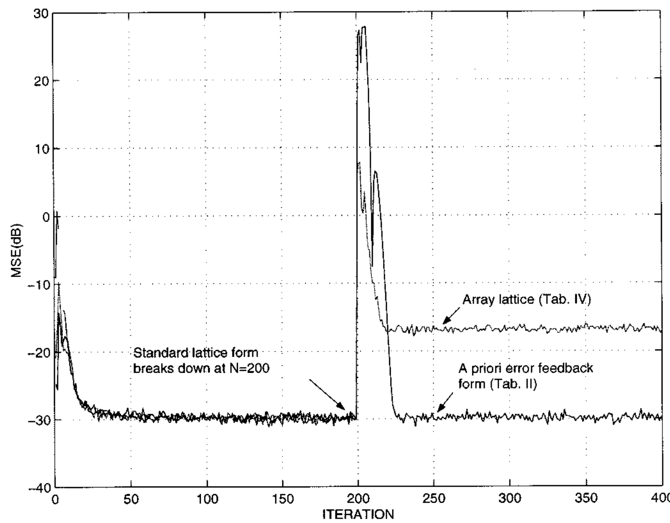

Fig. 7. MSE decay of the various Laguerre lattice forms under finite precision (10 bits) with an impulsive disturbance at $N = 200$.

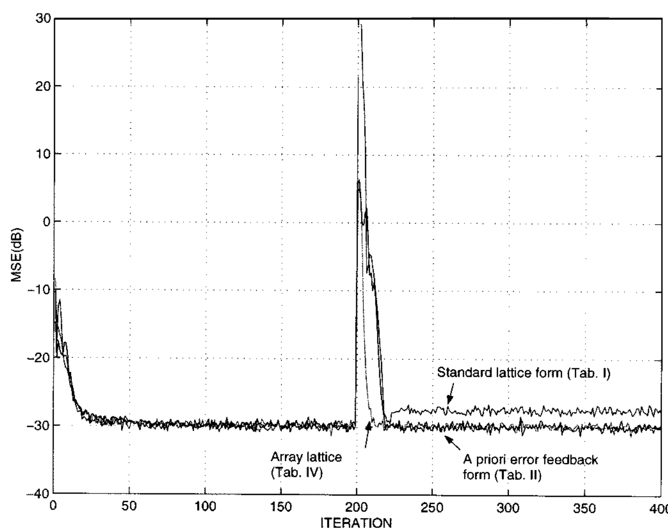

Fig. 8. MSE decay of the various Laguerre lattice forms under finite precision (15 bits) with an impulsive disturbance at $N = 200$.

Table V summarizes the resulting normalized RLS-Laguerre lattice algorithm. Observe that, both for compactness of notation and in order to save in computations, we introduced the variables

$$\left\{ r^b_M(N), r^f_M(N), r^e_M(N), r^c_M(N), r^\varphi_M(N) \right. \\ \left. r^{\bar{b}}_M(N), r^\eta_M(N), r^\omega_M(N) \right\}.$$

Note that the normalized algorithm returns the normalized least-squares residual $e''_{M+1}(N)$. The original error $e_{M+1}(N)$ can be easily recovered since the normalization factor can be computed recursively by

$$\zeta^{1/2}_{M+1}(N)\gamma^{1/2}_{M+1}(N) = r^b_M(N) r^\omega_M(N) \zeta^{1/2}_M(N)\gamma^{1/2}_M(N).$$

VII. SIMULATIONS

Although all the RLS Laguerre lattice variants studied here are theoretically equivalent, they differ in computational cost and perhaps more importantly in robustness to finite-precision effects. Table VI summarizes the computational cost of these algorithms for a least-squares problem of order M. We see that some forms are more costly in terms of multiplications while other are more costly in terms of divisions.

In addition, the algorithms exhibit different behavior under different operating conditions, such as finite-precision implementations and regularization. For example, for small regularization factor μ, the array lattice algorithm exhibits the best performance among all the lattice variants. The other algorithms can break down due to divisions by small numbers (especially for longer filters). We observed this behavior in simulations. The breakdown is a consequence of the fact that in the absence of regularization, the initial least-squares problems become rank-deficient. Moreover, for small regularization, some quantities can become zero if quantization is performed with short wordlength. In this case, divisions by zeros may occur. In the array lattice form, however, no regularization is needed. The behavior is also typical of lattice filters for shift-structured data (see [18]).

Figs. 5 and 6 compare the performance of the lattice filters in finite-precision (fixed-point) implementations with a varying number of bits (and using rounding). A simple fifth-order Laguerre filter was used in a system identification scenario. The regularization factor for the algorithms was set to $\mu = 0.01$. In both simulations, it is seen that the performance of the normalized lattice and the *a posteriori* error feedback versions are the worst. We should also mention that for 10 bits, we noticed in our simulations that the mean square error (MSE) curve starts growing slowly for the array and *a priori* error feedback forms after 5000 iterations, whereas for the standard lattice form, it remains in steady state.

In Figs. 7 and 8, we compare the performance of the different lattice (excluding the normalized and *a posteriori* error feedback) forms for the same filter order when an impulsive disturbance is introduced at $N = 200$. A similar simulation scenario was performed in [18] for tapped-delay lattice filters in order to illustrate the recovery of the MSE convergence following a sudden nonstationarity. We used zero regularization for the array form $(\mu = 0)$ and $\mu = 0.1$ for the other versions. For a ten-bit wordlength, the standard lattice form breaks down after the impulsive disturbance. The error feedback form can still recover its final MSE, whereas the array form achieves a higher MSE value. For a 15–bit wordlength, the array form returns to its final MSE faster than the *a priori* error feedback form, whereas the standard lattice form achieves a higher final MSE.

We have also noticed that for high-order Laguerre lattice filters, the regularization factor has to be high in order to avoid breakdown of the algorithm (in a 15–bit quantization). The array form has shown the best performance in this case since its recursions are valid even for zero regularization.

VIII. Conclusion

In this work, we developed several lattice forms for RLS Laguerre adaptive filtering. One form is based on *a priori* errors with feedback, and it involves propagating the reflection coefficients in time. A second form is based on unitary rotations, and a third form is based on propagating a fewer number of normalized reflection coefficients. The algorithms are all theoretically equivalent but differ in computational cost and in robustness to finite-precision effects and to regularization.

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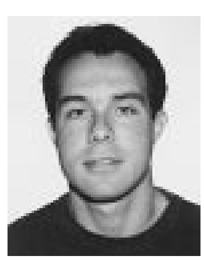

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