Parameter Estimation with Multiple Sources and Levels of Uncertainties

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***Abstract*—Least-squares designs are sensitive to errors in the data, which can be due to several factors including the approximation of complex models by simpler ones, the presence of unavoidable experimental errors when collecting data, or even due to unknown or unmodeled effects. In this paper, we formulate a new design criterion that treats multiple sources of uncertainties in the data with possibly varied degrees of intensity. We show that the solution has a regularized form, with one regularization parameter for each source of uncertainty. The parameters turn out to be model dependent and can be determined optimally as the nonnegative roots of certain coupled equations. Applications in array signal processing and image processing are considered.**

***Index Terms*—Cross validation, modeling errors, parameter estimation, regularization, robust estimation, total least squares.**

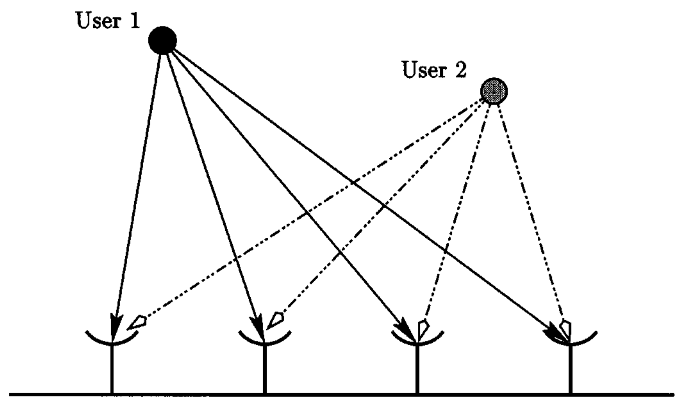

Fig. 1. Spatial processing with multiple users and antennas.

I. Introduction

IN MANY applications in signal processing and communications, data are collected from several sources and are subject to different levels of noise, distortion, or interference. A typical example arises in the context of co-channel interference cancelation in array signal processing, as depicted in simplified form in Fig. 1 for the case of two sources (or users) and four antenna elements.

The figure shows two emitters transmitting signals from different directions to an antenna array. The signals interfere with each other at the antenna array, and it is desired to recover them by annihilating co-channel interference (CCI) and by suppressing noise. A difficulty that is encountered by most techniques that are used for this purpose is that they require good initial estimates of the channel gains (i.e., of the gains from each source to the various antennas). These gains are often estimated before signal recovery by a variety of methods (see, e.g., [1] and [2]), and they are, therefore, subject to errors. Such errors, or uncertainties, in the gain estimates can ultimately influence the accuracy of signal recovery (see the discussion in Section III).

Several robust design techniques have been developed in the estimation and control literature in order to minimize the effect of uncertainties in the data or model on the performance of the overall design. Among the most widely used techniques are those based on the total-least-squares method (TLS) [3], the generalized cross-validation method (GCV) [4], [5], and the $\mathcal{H}_\infty$ method (see, e.g., [6] and the many references therein). These methods perform deregularization in general (e.g., TLS and $\mathcal{H}_\infty$), and their formulations do not usually incorporate different sources of uncertainties. For example, in the aforementioned co-channel interference application, it might be the case that the gains recovered for User 1 are less certain than the gains recovered for User 2. It is therefore useful to study a design procedure that allows the user to incorporate into its formulation information about the number of sources of uncertainties in a model and about how large (or how damaging) each uncertainty can be. Such issues, of course, arise in many other settings and applications other than the CCI cancelation example of Fig. 1.

Given the above, the purpose of this paper is to study perturbed models that involve *multiple* sources of uncertainties in the data with possibly *different* levels of intensity (or interference). More specifically, we develop an estimation technique for models with multiple sources and levels of bounded data uncertainties. For reasons of brevity, we will refer to the resulting method as a BDU estimation method.

The new estimation technique is derived by formulating and solving a constrained game-type problem with multiple opponents of different strengths. Although the solution can be obtained algebraically, we resort instead to geometric arguments (such as orthogonality conditions and projections). These arguments provide powerful insights into the nature of the solution (as explained in [7] for the single source case), and they also establish important connections with classical least-squares theory, where such geometric insights are prevalent.

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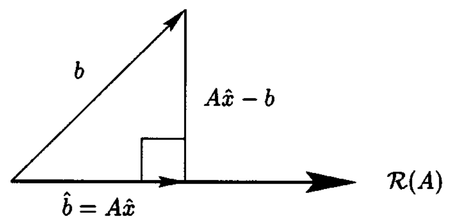

Fig. 2. Orthogonality principle: $A\hat{x} - b \perp \mathcal{R}(A)$.

II. Least-Squares Method and the BDU Formulation

In this section, we review the least-squares criterion and formulate the BDU problem of this paper.

A. The Least-Squares Criterion

Consider an $N \times n$ matrix A with $N \geq n$.[1] Let b denote a measurement vector that is related to an unknown n-dimensional vector x via the linear model $b = Ax + v$, for some additive noise component v. The least-squares criterion estimates x by solving

$$\min_x \|Ax - b\| \tag{1}$$

where the notation $\|\cdot\|$ stands for the Euclidean norm of its vector argument (it will also be used to denote the maximum singular value of a matrix argument).

Due to v, the vector b does not, in general, lie in the column span of A, which is denoted by $\mathcal{R}(A)$. The least-squares problem then seeks the vector $\hat{b} = A\hat{x}$ in $\mathcal{R}(A)$ that is closest to b in the Euclidean norm sense. The solution of (1) is obtained by requiring the residual vector $(A\hat{x} - b)$ to be orthogonal to the data matrix A (see Fig. 2), i.e.,

$$A^T(A\hat{x} - b) = 0 \tag{2}$$

or, equivalently, by solving the normal equations

$$(A^TA)\hat{x} = A^Tb. \tag{3}$$

We will see later in Section V that this useful geometric interpretation of the least-squares solution extends to the BDU formulation of this paper.

Now, least-squares methods are well known to be sensitive to errors in the data. More specifically, a least-squares design that is based on A can perform poorly if the vector b has been actually generated by a perturbed version of A, say $b = (A + \delta A)x + v$ for some unknown δA. In this case, if we persist in using the solution $\hat{x}$ from (3), which is based solely on the nominal data matrix A, then the actual residual norm will be $\|(A + \delta A)\hat{x} - b\|$. This norm satisfies, in view of the triangle inequality of norms

$$\|(A + \delta A)\hat{x} - b\| \leq \underbrace{\|A\hat{x} - b\|}_{\text{LS residual}} + \underbrace{\|\delta A\hat{x}\|}_{\text{additional term.}} \tag{4}$$

The first term on the right-hand side is equal to the least-squares residual norm that is associated with $(A, b, \hat{x})$. The second term is due to the perturbation in the data. Such perturbation errors in the data are very common in practice, and they can be due to several factors including the approximation of complex models by simpler ones, the presence of unavoidable experimental errors when collecting data, or even due to unknown or unmodeled effects. Regardless of their source, they can degrade the performance of least-squares designs. Several examples to this effect, and comparisons with alternative robust design methods, are provided in [7] as well as later in this paper (see Section III).

B. The BDU Formulation

Motivated by the above discussion, we formulate below a new optimization problem. Thus, let A be a given $N \times n$ matrix, which we shall refer to as the *nominal matrix* or the nominal data. Assume further that A is partitioned column-wise into several submatrices $\{A_j\}$, say K of them

$$A = [\, A_1 \quad A_2 \quad \cdots \quad A_K \,].$$

With each A_j, we associate an unknown perturbation matrix of the same dimension δA_j, and denote the overall perturbation matrix by

$$\delta A = [\, \delta A_1 \quad \delta A_2 \quad \cdots \quad \delta A_K \,].$$

We assume that a bound η_j is available on the size of each perturbation δA_j, viz., $\|\delta A_j\| \leq \eta_j$.

Let b be a measurement vector that is generated by the model $b = (A + \delta A)x + v$. That is, b is produced by a perturbed version of the nominal matrix A. The exact value of the perturbation δA is not known. What is known are bounds on the how large the individual submatrices of δA can be. We now pose the problem of determining x optimally by solving

$$\min_x \max_{\left(\substack{\|\delta A_i\| \leq \eta_i \\ 1 \leq i \leq K}\right)} \|[\, A_1 + \delta A_1 \quad \cdots \quad A_K + \delta A_K \,]x - b\|. \tag{5}$$

This is a constrained game-type problem where the uncertainties $\{\delta A_j\}$ are treated as opponents with varied strengths (or sizes); the designer tries to minimize the cost through the selection of x, whereas the opponents try to maximize the cost. In this way, the solution $\hat{x}$ will be such that it performs best in the worst-possible scenario. The game problem is constrained since it imposes a limit on how large (or how damaging) each opponent can be. This further limits how "bad" the worst-possible scenario can be and, in this way, overly conservative designs are avoided.

In applications, the submatrices $\{A_j\}$ can refer to different components in a model or a system. For example, returning to the co-channel interference application of Fig. 1, we shall see later in Section III that the matrix A will consist of two columns: One column contains the gains for User 1, whereas the second column contains the gains for User 2. Since the paths from the two users will, in general, have different levels of uncertainties, we will thus be reduced to a BDU formulation as in (5) with two submatrices $\{A_1, A_2\}$ and two uncertainty levels $\{\eta_1, \eta_2\}$.

The special case $K = 1$, i.e., the case of a single source of uncertainty in the data

$$\min_x \max_{\|\delta A_1\| \leq \eta_1} \|(A_1 + \delta A_1)x - b\| \tag{6}$$

[1] We use the capital letter N to denote the larger dimension of A and the letter n to denote the smaller dimension of A.

was studied in detail in [7]–[10]. This case was independently formulated and solved in [9] and [10] by using very different (algebraic) solution techniques; one was based on LMI techniques [9], whereas the other was based on SVD techniques [10]. It turns out that for problem (6), the LMI technique is more costly. In [7], the same problem was solved from a purely geometric point of view by extending to the BDU context several of the projection arguments that are widely used for least-squares problems. We will adopt this geometric approach in this paper in order to solve the more general problem (5).

For notational convenience in the remainder of the paper, we will partition the vector x accordingly with A and write

$$x = \text{col}\{x_1, x_2, \ldots, x_K\}. \tag{7}$$

We will also assume that

$$\text{rank}(A) = n, \quad b \neq 0, \quad \text{and} \quad b \notin \mathcal{R}(A). \tag{8}$$

That is, we will assume that A is a full-rank matrix and that b is a nonzero vector that does not belong to its column span (this requires $N > n$). The analysis can be extended to cases where assumptions (8) are violated but, for simplicity, we will focus in this paper on (8) in order to highlight the main ideas.

C. Form and Properties of the Solution

Before discussing the solution of problem (5) and in order to not to overburden the reader with the derivations, we choose to summarize here the main conclusions of the coming sections for ease of reference.

Thus, let $\hat{x}$ denote a solution of (5). We will show that $\hat{x}$ has the following properties.

1) The solution $\hat{x}$ exists and is unique.
2) The solution $\hat{x}$ is zero only if the uncertainties $\{\eta_i\}$ are large enough. Otherwise, it is nonzero. This is established in Lemma 1.
3) When the solution $\hat{x}$ is nonzero, it has an interesting regularized form. For example, in a so-called regular case to be studied later, the expression for $\hat{x}$ will be given by (see Section V-A)

$$\hat{x} = (A^T A + D_\alpha)^{-1} A^T b$$

where D_α is a diagonal matrix of the form

$$D_\alpha = \text{diag}\{\hat{\alpha}_1 I, \hat{\alpha}_2 I, \ldots, \hat{\alpha}_K I\}$$

and the $\{\hat{\alpha}_i\}$ are certain nonnegative scalar regularization parameters. Such regularized solutions have been used extensively in signal and image processing applications (see, e.g., [11]–[14]) and in adaptive filtering (see, e.g., [15] and [16]).
4) A major issue in applications (see, e.g., the titles of [12]–[14]) is always how to select the regularization parameters. It turns out that the BDU solution leads to an *automatic* selection of the parameters $\{\hat{\alpha}_i\}$. More explicitly, these parameters will be shown to be the unique non-negative roots of certain coupled equations that are fully determined from the given data $\{A, b, \eta_i\}$ (see Section VI).

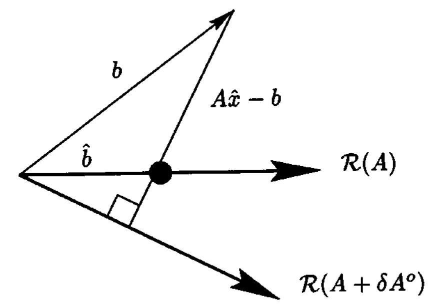

Fig. 3. Orthogonality condition for BDU estimation.

5) Recall from Fig. 2 that the least-squares solution satisfies the orthogonality condition (2), which states that the residual vector has to be orthogonal to the given data. Interestingly enough, the BDU solution $\hat{x}$ has a similar interpretation. More specifically, we shall show in Section V that the residual vector $(A\hat{x} - b)$ has to be orthogonal not to A but to a *rank-one modification of* A. That is

$$[A + \delta A^o]^T (A\hat{x} - b) = 0$$

for some matrix δA^o that is rank one (and dependent on $\hat{x}$). This is depicted in Fig. 3. Equivalently, the BDU solution can be regarded as performing an oblique projection onto A rather than an orthogonal projection.

Some of the above properties may not be straightforward to establish as the reader will be able to verify from some of the arguments in the appendices. Nevertheless, when all is said and done, it is interesting to note that the final solution of the BDU problem (5) turns out to share some desirable properties with classical least-squares designs (such as uniqueness of solution, regularization, orthogonality properties) in addition to new distinguishing features (such as robustness to errors, automatic regularization, oblique projection, and multiple levels of regularization).

Before establishing the above properties, we will demonstrate the application of the BDU solution to two problems in array signal processing and image separation.

III. Applications in Array Processing and Image Separation

Our first example is the co-channel interference cancelation problem of Fig. 1. The figure shows two emitters transmitting, at time i, the signals $\{x_{1,i}, x_{2,i}\}$ from different directions to an antenna array; the signal transmitted by User 1 is denoted by $x_{1,i}$, and the signal transmitted by User 2 is denoted by $x_{2,i}$. The antenna array has four elements that are equally spaced.[2] The signal received by the elements of the antenna array can be expressed in vector form as

$$b_i = A_1 x_{1,i} + A_2 x_{2,i} + v_i \tag{9}$$

where v_i denotes a 4 × 1 measurement noise vector, and where A_1 and A_2 are 4 × 1 column vectors. The jth entry of A_1 is

[2]Although our discussion is general and applies to a higher number of sources and antenna elements, we limit ourselves to an example with two sources and four antenna elements in order to convey the main ideas without an overburden of notation.

the gain from User 1 to the jth antenna. Likewise, the jth entry in A_2 is the gain from User 2 to the jth antenna. As mentioned before, these gains can be estimated by a variety of methods (see [1] and [2]).

Once the $\{A_1, A_2\}$ are known (or estimated), the common techniques in the literature proceed to recover the transmitted signals $\{x_{1,i}, x_{2,i}\}$ by solving a least-squares problem of the form

$$\min_{x_{1,i},x_{2,i}} \left\| \begin{bmatrix} A_1 & A_2 \end{bmatrix} \begin{bmatrix} x_{1,i} \\ x_{2,i} \end{bmatrix} - b_i \right\|.$$

Assuming a full-rank data matrix $A = [A_1 \ \ A_2]$, the unique solution is given by

$$\begin{bmatrix} \hat{x}_{1,i} \\ \hat{x}_{2,i} \end{bmatrix} = \left(\begin{bmatrix} A_1^T \\ A_2^T \end{bmatrix} \begin{bmatrix} A_1 & A_2 \end{bmatrix} \right)^{-1} \begin{bmatrix} A_1^T \\ A_2^T \end{bmatrix} b_i.$$

In situations where the data matrix may not be well conditioned, a regularized least-squares problem can be solved, say

$$\min_{x_{1,i},x_{2,i}} \left[\gamma(|x_{1,i}|^2 + |x_{2,i}|^2) + \left\| A \begin{bmatrix} x_{1,i} \\ x_{2,i} \end{bmatrix} - b_i \right\|^2 \right] \tag{10}$$

for some regularization parameter $\gamma > 0$ that is chosen by the designer. The solution in this case is given by

$$\begin{bmatrix} \hat{x}_{1,i} \\ \hat{x}_{2,i} \end{bmatrix} = \left(\begin{bmatrix} A_1^T \\ A_2^T \end{bmatrix} \begin{bmatrix} A_1 & A_2 \end{bmatrix} + \gamma I_2 \right)^{-1} \begin{bmatrix} A_1^T \\ A_2^T \end{bmatrix} b_i.$$

Choosing γ is usually not a trivial task. One method that has been proposed for the choice of γ is the so-called generalized cross-validation (GCV) method [4], [5]. It picks γ by minimizing the following cost function:

$$\min_{\gamma} \frac{\|(I - A(A^TA + \gamma I_2)^{-1}A^T)b_i\|^2}{\text{Trace}[I_4 - A(A^TA + \gamma I_2)^{-1}A^T]^2}.$$

[Note that the term in the numerator is the norm of the residual vector $\tilde{b}_i = b_i - \hat{b}_i$ with $\hat{b}_i = A(A^TA + \gamma I_2)^{-1}A^T b_i$.]

Another popular technique that has been used in the literature in order to address uncertainties in the data matrix A is the total-least-squares (TLS) method [3]. This method first replaces the given A by another estimate $\bar{A}$ and the given vector b_i by another vector $\bar{b}_i$ that lies in the range space of $\bar{A}$. It then solves the consistent linear system of equations $\bar{A}\hat{x}_i = \bar{b}_i$ in order to determine the estimates $\{\hat{x}_{1,i}, \hat{x}_{2,i}\}$. In the so-called nondegenerate case, this construction amounts to the following. We determine the smallest singular value of the extended matrix $[A \ \ b_i]$, say $\sigma_{\min}$, and then use it as a "deregularization" parameter to find $\{\hat{x}_{1,i}, \hat{x}_{2,i}\}$

$$\begin{bmatrix} \hat{x}_{1,i} \\ \hat{x}_{2,i} \end{bmatrix} = \left(\begin{bmatrix} A_1^T \\ A_2^T \end{bmatrix} \begin{bmatrix} A_1 & A_2 \end{bmatrix} - \sigma_{\min} I_2 \right)^{-1} \begin{bmatrix} A_1^T \\ A_2^T \end{bmatrix} b_i. \tag{11}$$

In this method of solution, there is no *a priori* bound on how much correction can be made to the matrix A in order to obtain the $\bar{A}$. For this reason, the solution can at times be overconservative.

To apply the BDU formulation of this paper, we start with bounds on the sizes of the uncertainties in $\{A_1, A_2\}$ (these bounds can usually be estimated from the identification procedure that led to A_1 and A_2), say $\|\delta A_1\| \le \eta_1$ and $\|\delta A_2\| \le \eta_2$. We then recover the $\{x_{1,i}, x_{2,i}\}$ by solving

$$\min_{x_{1,i},x_{2,i}} \max_{\substack{\|\delta A_1\| \le \eta_1 \\ \|\delta A_2\| \le \eta_2}} \left\| \begin{bmatrix} A_1 + \delta A_1 & A_2 + \delta A_2 \end{bmatrix} \begin{bmatrix} x_{1,i} \\ x_{2,i} \end{bmatrix} - b_i \right\|$$

which is a special case of (5). The solution, as mentioned in the previous section, will be of the form

$$\begin{bmatrix} \hat{x}_{1,i} \\ \hat{x}_{2,i} \end{bmatrix} = \left(\begin{bmatrix} A_1^T \\ A_2^T \end{bmatrix} \begin{bmatrix} A_1 & A_2 \end{bmatrix} + \begin{bmatrix} \hat{\alpha}_1 & \\ & \hat{\alpha}_2 \end{bmatrix} \right)^{-1} \begin{bmatrix} A_1^T \\ A_2^T \end{bmatrix} b_i$$

with two regularization parameters $\{\hat{\alpha}_1, \hat{\alpha}_2\}$ that are determined by solving two coupled equations of the form (see Section VI)

$$\alpha_1 = f_1(\alpha_1, \alpha_2), \quad \alpha_2 = f_2(\alpha_1, \alpha_2)$$

where the functions $\{f_1, f_2\}$ are determined by the data $\{A_1, A_2, b_i, \eta_1, \eta_2\}$. Such equations can be solved by any appropriate zero-finding technique (e.g., the command fsolve of Matlab[3] was used to generate our simulation results).

In our simulations, the transmitted signals $\{x_{1,i}, x_{2,i}\}$ were chosen uniformly from the 4PAM distribution $\{-3, -1, 1, 3\}$ so that the variance of each source was $\sigma_x^2 = 5$. The variance of the noise vector v_i was taken as $\sigma_v^2 I_4$, and the signal-to-noise ratio of each source was defined as

$$\text{SNR} = 10 \log\left(\frac{\sigma_x^2}{\sigma_v^2}\right).$$

The simulations were carried out for different values of the SNR. At each SNR, 300 data points $\{x_{1,i}\}$ and 300 data points $\{x_{2,i}\}$ were generated and transmitted through nominal channels with path gains

$$A_1 = \text{col}\{1, -0.5, 0.2, 0.1\}$$
$$A_2 = \text{col}\{-0.32, 0.76, 0.08, 0.024\}$$

and with relative uncertainties 15% and 25%, respectively. That is, $\eta_1 \approx 0.1710$ and $\eta_2 \approx 0.2072$ (by relative uncertainties, we mean $\eta_1/\|A_1\|$ and $\eta_2/\|A_2\|$). We simulated two scenarios. First, for each pair of transmitted symbols $\{x_{1,i}, x_{2,i}\}$, the perturbations $\{\delta A_1, \delta A_2\}$ were *randomly* generated within the permissible bounds. Second, the nominal paths $\{A_1, A_2\}$ were perturbed maximally by 15% and 25%, respectively, for each SNR (and then fixed at these values during the transmission of the 300 symbol pairs $\{x_{1,i}, x_{2,i}\}$). The first scenario allows us to observe the behavior of the different estimators on average, whereas the second one allows us to observe their behavior under a worst-case condition.

Estimates $\{\hat{x}_{1,i}, \hat{x}_{2,i}\}$ were determined by using the least-squares method, regularized least-squares (with $\gamma = 0.02$), generalized cross-validation, and BDU.[4] For each method, and at

[3]Matlab© is a trademark of the MathWorks, Inc. Matlab codes for the technique developed in this paper can be obtained from sayed@ee.ucla.edu.

[4]The TLS method did not perform well in this case, and its MSE curves are not shown in Fig. 4.

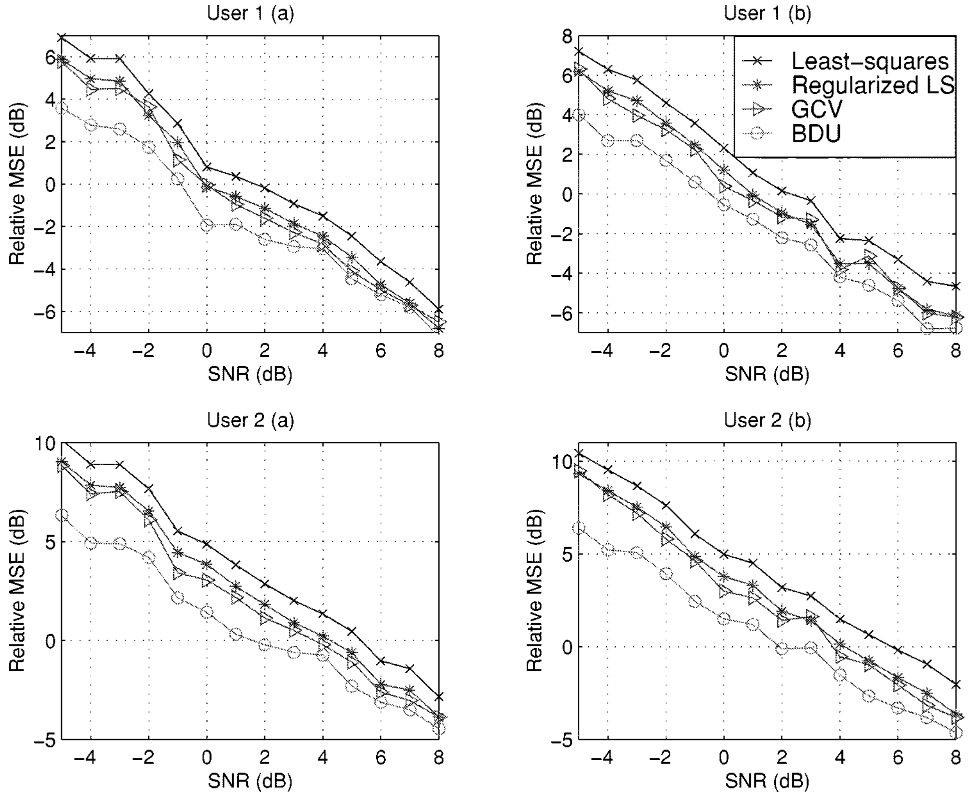

Fig. 4. MSE curves for both users 1 and 2 in a 4PAM modulation scenario with perturbation bounds $\eta_1 \approx 15\%$, $\eta_2 \approx 25\%$. The plots (a) correspond to random perturbations within the permissible bounds. The plots (b) correspond to maximal perturbations (on the boundary). Each point in the plots was obtained by averaging over 300 transmissions.

each SNR, a relative mean-square error measure (in dB) was computed as

$$\text{Relative MSE} = 10 \log \left(\frac{\frac{1}{300} \sum_{i=1}^{300} |x_{1,i} - \hat{x}_{1,i}|^2}{\sigma_x^2} \right).$$

Fig. 4 compares the resulting MSE curves for both Users 1 and 2 as a function of the SNR, and for both cases of random perturbations $\{\delta A_1, \delta A_2\}$ [plots (a)] and maximal perturbations that lie on the boundary [plots (b)]. Recall that the uncertainty in the paths for User 2 is significantly more than that in the paths for User 1; the MSE curves for User 2 show how the BDU solution is more robust in this case.

Figs. 5–7 show the results of a similar experiment in a different context. Now, the transmitted signals $\{x_{1,i}, x_{2,i}\}$ represent the pixels of two 256 $\times$ 256 images that are being transmitted over the different paths. Here, the purpose is to identify and separate the superimposed images. In this example, the nominal paths were chosen as

$$A_1 = \text{col}\{1, 0.5, 0.1\}$$
$$A_2 = \text{col}\{0.4, 0.9, 0.2\}$$

and the relative uncertainties were $\eta_1 = 15\%$ and $\eta_2 = 25\%$. Fig. 5 shows the original images (a clock and a chart), and Figs. 6 and 7 show the received image at the left-most antenna in addition to the recovered images by means of five different methods (followed by 3 $\times$ 3 median filtering):

1) least-squares;
2) regularized least-squares (with $\gamma = 0.02$);
3) TLS;
4) GCV;
5) BDU.

In this example, the chart image was transmitted through the more erroneous channel.

Fig. 7 shows that the BDU solution recovers more fine details of this image than the other methods. A special feature of the chart image is that it exhibits many edges, and its pixels are generally at extreme values (black and white). The purpose of this image processing example is, of course, not to show that the BDU method is always superior to other methods since the other methods will perform reasonably well in many situations. The purpose of the example is to show the superior robustness of the BDU solution in situations that involve perturbed data.

IV. Equivalent Estimation Problem

In the remaining sections of this paper, we return to problem (5) and provide the full details for its solution. That is, we establish analytically all the properties that we mentioned before (such as uniqueness of solution, orthogonality, and automatic regularization). The arguments in these sections are mostly geometric and rely on several useful and accessible concepts from linear algebra and matrix theory such as projections, nullspaces, and range spaces. Those interested in the final statement of the solution can move directly to Theorem 2.

We start with the following result, which gives the exact condition under which the solution of the BDU problem is the zero

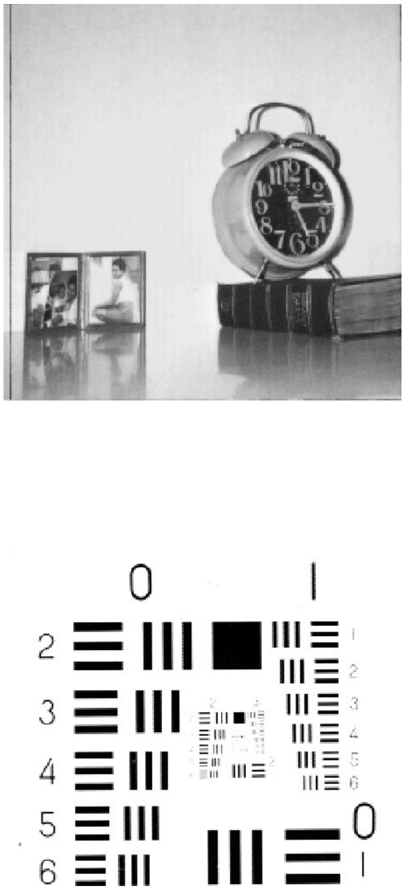

Fig. 5. Original 256 × 256 clock and chart images.

vector. Loosely speaking, the result states that the solution is zero if the uncertainties in the data are large enough. This makes sense from a practical point of view; if the data is sufficiently corrupted, then the safest, or most robust, estimate should be the zero estimate. Observe further that the condition in the lemma is both necessary and sufficient.

Lemma 1 (Zero Solution Vector): The BDU estimation problem (5) has a unique solution at $\hat{x} = 0$ if and only if

$$\eta_i \geq \frac{\left\|A_i^T b\right\|}{\|b\|} \quad \text{for all } 1 \leq i \leq K. \tag{12}$$

Proof: One direction of the lemma is straightforward to establish, whereas the proof of the other direction is more complex. We provide the details in Appendix A. □

It then follows that a nonzero solution $\hat{x}$ will exist if and only if (12) is violated at least for some i, say

$$\eta_{i_o} < \frac{\left\|A_{i_o}^T b\right\|}{\|b\|} \quad \text{for some } i_o. \tag{13}$$

We will in fact show that the nonzero solution is also unique in this case. In order to arrive at this uniqueness result, we first need to solve the maximization step in (5). Thus, let $J(x)$ denote the solution of the maximum in (5), i.e.,

$$J(x) \triangleq \max_{\{\|\delta A_i\| \leq \eta_i\}} \|(A + \delta A)x - b\|.$$

Then, the following result can be established (in terms of the partitioning in (7)).

Lemma 2 (Maximum Residual): It holds that

$$J(x) = \|Ax - b\| + \eta_1\|x_1\| + \cdots + \eta_K\|x_K\|. \tag{14}$$

Moreover, one choice for the worst-case perturbations that achieve $J(x)$ is (these perturbations are functions of x)[5]

$$\delta A_i^o(x) \triangleq \frac{\eta_i\|x_i\|^\dagger}{\|Ax - b\|}\,(Ax - b)\,x_i^T. \tag{15}$$

In addition, the following facts hold.

1) The resulting perturbed matrix

$$A + \delta A^o(x) \triangleq A + \frac{(Ax - b)\,q^T(x)}{\|Ax - b\|} \tag{16}$$

has full column rank for all x, where $q(x)$ denotes the column vector

$$q(x) \triangleq \text{col}\{\eta_1\|x_1\|^\dagger x_1, \eta_2\|x_2\|^\dagger x_2, \ldots, \eta_K\|x_K\|^\dagger x_K\}. \tag{17}$$

2) For any perturbation matrix δA that achieves the maximum residual norm $J(x)$, the residual vectors $(A + \delta A)x - b$ and $Ax - b$ are collinear. They also point in the same direction (i.e., one is a positive multiple of the other).

Proof:: The argument requires that we first identify the perturbations that achieve the worst-case residual. The details are given in Appendix B. □

The important fact to note is that we are now reduced to studying the equivalent problem

$$\min_x \left(\|Ax - b\| + \eta_1\|x_1\| + \cdots + \eta_K\|x_K\|\right) \tag{18}$$

where the $\{x_i\}$ are defined as in (7). Observe that (18) is a *distance* problem, involving sums of distances rather than sums of squared distances (as is prevalent in least-squares designs; see, e.g., (10)). We can now establish the uniqueness of the nonzero solution.

Lemma 3 (Uniqueness of Nonzero Solution): A unique nonzero solution $\hat{x}$ of (5) (or, equivalently, (18)) exists if, and only if (13) holds.

Proof:: The equivalence between problems (5) and (18) holds for all x. Now, we verify that the cost function $J(x)$ in (18) is strictly convex in x in view of the condition $b \notin \mathcal{R}(A)$ [recall (8)]. Indeed, for any two distinct vectors x and z, and for any real number $0 < \gamma < 1$

$$\begin{aligned} J[\gamma x + (1-\gamma)z] &= \|A[\gamma x + (1-\gamma)z] - b\| \\ &\quad + \sum_{i=1}^K \eta_i\|\gamma x_i + (1-\gamma)z_i\| \\ &\leq \gamma\left(\|Ax - b\| + \sum_{i=1}^K \eta_i\|x_i\|\right) \\ &\quad + (1-\gamma)\left(\|Az - b\| + \sum_{i=1}^K \eta_i\|z_i\|\right) \end{aligned}$$

[5]Here, $a^\dagger$ denotes the pseudo-inverse of the scalar a. It is equal to a^{-1} if a is nonzero; otherwise, it is zero.

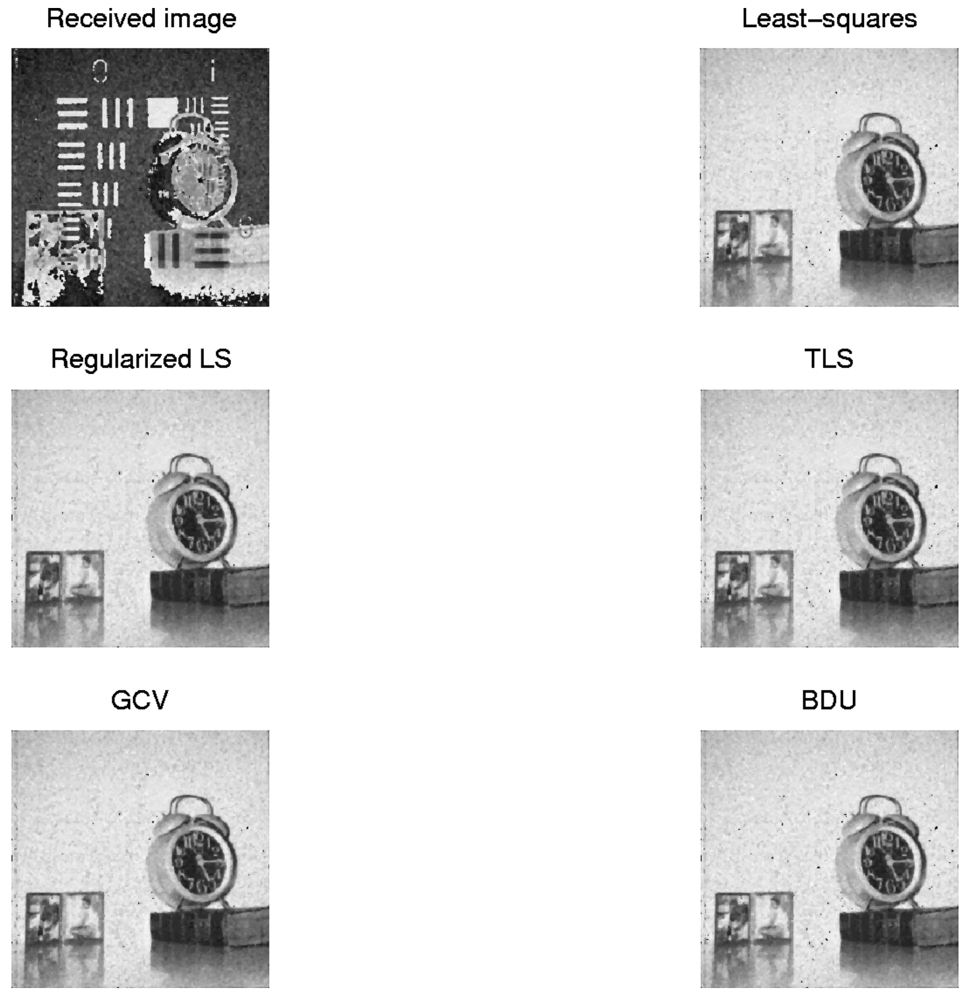

Fig. 6. Received image at the left-most antenna in addition to the clock images that are recovered by the LS, regularized LS, TLS, GCV, and BDU methods followed by 3×3 median filtering. The clock image is the one that is transmitted over the nominal channel A_1 with relative uncertainty bound 15%. In this case, the performances of the methods are comparable.

with equality only if x_i and z_i are parallel, as well as $(Ax - b)$ and $(Az - b)$. This last condition violates the assumption of a full-rank matrix A with a nonzero b that satisfies $b \notin \mathcal{R}(A)$. Therefore

$$J[\gamma x + (1-\gamma)z] < \gamma J(x) + (1-\gamma)J(z)$$

which shows that $J(x)$ is strictly convex so that it must have a single global minimum.

When (13) holds, we already know that $\hat{x} = 0$ cannot be the global minimum. Therefore, the unique global minimum is necessarily nonzero. Conversely, when the unique solution is nonzero, then in view of Lemma 1, (13) must hold. □

V. Geometry of the BDU Problem

We therefore know that problem (5) always has a solution and that this solution is unique. Let us now verify that the nonzero BDU solution admits an interpretation in terms of an orthogonality condition in much the same way as classical least-squares solutions do.

To see this, we first note that we can rewrite the BDU estimation problem (18) in the equivalent form

$$\min_x \|[A + \delta A^o(x)]x - b\| \tag{19}$$

where, as we already know from (16), the perturbed matrix $(A + \delta A^o)$ attains the maximum residual norm. For compactness of notation, we will further denote the worst-case perturbed matrix used in (19) by $A(x)$

$$A(x) \triangleq A + \delta A^o(x) = A + \frac{(Ax - b)q^T(x)}{\|Ax - b\|}$$

so that (19) becomes

$$\min_x \|A(x)x - b\|. \tag{20}$$

This statement looks similar to a least-squares problem with two important distinctions. First, the coefficient matrix A of (1) is replaced by a perturbed version of it, i.e., $A(x) = A + \delta A^o(x)$, and second, the new coefficient matrix $A(x)$ is dependent on the unknown x as well. Hence, what we have is a nonlinear least-squares problem with a special form for the coefficient matrix $A(x)$. If $A(x)$ were a constant matrix and therefore not dependent on x, then we know from the geometry of least-squares estimation that the residual vector must be orthogonal to A [cf. (2)]. In the BDU case (20), however, the coefficient matrix $A(x)$ is a nonlinear function of x. Interestingly enough, it turns out

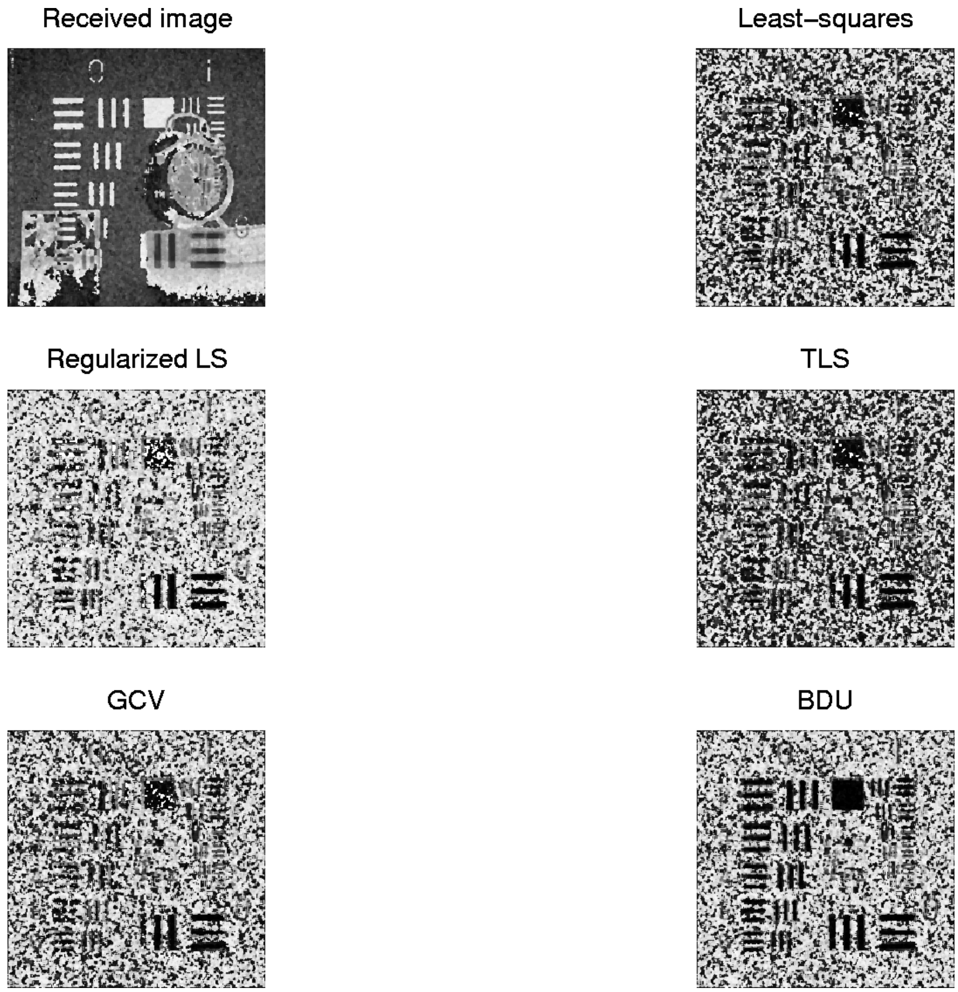

Fig. 7. Received image at the left-most antenna in addition to the chart images that are recovered by the LS, regularized LS, TLS, GCV, and BDU methods followed by 3 × 3 median filtering. The chart image is the one that is transmitted over the nominal channel A_2 with relative uncertainty bound 25%. In this case, the BDU solution recovers more details.

that the solution $\hat{x}$ can still be characterized by a similar orthogonality condition.

We establish this fact by distinguishing between two classes of vectors x.[6]. Recall that we are partitioning every x into individual components $\{x_i, i = 1, \ldots, K\}$ in accordance with the partitioning of the coefficient matrix A itself. A nonzero vector x, however, can still have one or more zero components x_i. We thus let $\mathcal{X}$ denote the set of all vectors x with nonzero components $\{x_i\}$ (we will refer to these vectors are regular vectors)

$$\mathcal{X} = \left\{ x \in \mathcal{R}^n \text{ with all components } \begin{pmatrix} x_i \neq 0, \\ 1 \leq i \leq K \end{pmatrix} \right\}.$$

The unique nonzero solution $\hat{x}$ of (18) can either be in $\mathcal{X}$ (i.e., has all its components $\hat{x}_i$ nonzero) or in $\mathcal{R}^n - \mathcal{X} - \{0\}$ (i.e., has some zero components $\hat{x}_i$). We refer to the first case as a regular solution and to the second case as a boundary solution. We study the regular case first.

[6]This distinction is not necessary in the single source case $(K = 1)$, as explained in [7] and [10]

A. A Unique Nonzero Regular Solution

The next result establishes that the unique solution $\hat{x}$ of (18) belongs to $\mathcal{X}$ if and only if an element $\hat{x}$ of $\mathcal{X}$ can be found that satisfies the following orthogonality condition (see Fig. 8):

$$A^T(\hat{x})[A(\hat{x})\hat{x} - b] = 0. \tag{21}$$

Since, from fact 2 in Lemma 2, the residual vector $A(\hat{x})\hat{x} - b$ is collinear with $A\hat{x} - b$, we obtain the equivalent orthogonality condition

$$A^T(\hat{x})[A\hat{x} - b] = 0$$

or, equivalently

$$\left[A + \frac{(A\hat{x} - b)q^T(\hat{x})}{\|A\hat{x} - b\|} \right]^T [A\hat{x} - b] = 0 \tag{22}$$

where $q(\hat{x})$ is as defined in (17). That is, the residual vector $A\hat{x} - b$ has to be orthogonal not to A but to a rank-one perturbation of A that is equal to $A + \delta A^o(\hat{x})$.

Compared with least-squares theory, we can interpret the result (22) as an oblique projection onto A rather than an or-

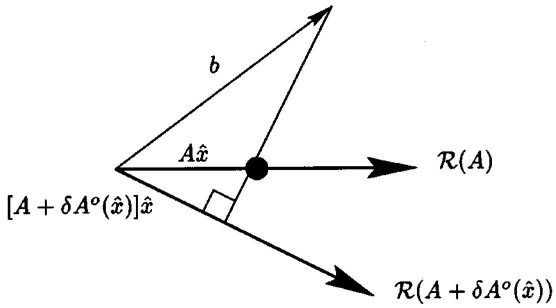

Fig. 8. Orthogonality condition for BDU estimation.

thogonal projection. We now establish the validity of the above claims.

Theorem 1 (Orthogonality Condition): Assume (13) holds. Then, an $\hat{x} \in \mathcal{X}$ is the unique solution of (18) or, equivalently, (20) for all x if and only if the residual vector $A\hat{x} - b$ is orthogonal to the following rank-one modification of the data matrix A:

$$A(\hat{x}) = A + \frac{(A\hat{x} - b)q^T(\hat{x})}{\|A\hat{x} - b\|} \tag{23}$$

that is, if and only if either (21) or (22) hold.

Proof: See Appendix C. □

It further follows in this case that the solution of the BDU problem can be expressed in a regularized form. Thus, introduce the auxiliary non-negative numbers

$$\hat{\alpha}_i \triangleq \eta_i \|A\hat{x} - b\| \|\hat{x}_i\|^\dagger \tag{24}$$

and define the diagonal matrix $D_\alpha \triangleq \text{diag}\{\hat{\alpha}_1 I, \ldots, \hat{\alpha}_K I\}$. Then, we can rewrite (22) in the form

$$(A^T A + D_\alpha)\hat{x} = A^T b. \tag{25}$$

Expressions (24) and (25) define a system of equations in the unknowns $\{\hat{x}, \hat{\alpha}_i\}$. The mapping between the variables $\hat{x}$ and $\hat{\alpha}_i$ is bijective. Given $\hat{x}$, we can evaluate the $\{\hat{\alpha}_i\}$ uniquely via (24), and given the $\{\hat{\alpha}_i\}$, we can evaluate $\hat{x}$ uniquely via (25). Hence, since the regular solution $\hat{x}$ is nonzero and unique, when it exists, we conclude that the above-coupled nonlinear equations in the $\{\hat{\alpha}_i\}$ have a unique non-negative solution $\{\hat{\alpha}_i\}$.

The regularization parameters $\{\hat{\alpha}_i\}$ are determined by the BDU solution rather than specified by the designer. In this sense, we can say that the BDU problem (18) performs automatic regularization.

B. A Unique Nonzero Boundary Solution

If a vector $\hat{x} \in \mathcal{X}$ satisfying the orthogonality condition (23) does not exist, i.e., one with all its entries $\{\hat{x}_i\}$ nonzero, then the unique minimizer belongs to the set $\mathcal{R}^n - \mathcal{X} - \{0\}$. That is, we need to examine the possibility of a solution $\hat{x}$ with one or more zero entries $\{\hat{x}_i\}$. In this case, the search for the solution can be obtained by considering smaller order problems.

We illustrate this point by considering the simple case of $K = 2$, in which case, the cost $J(x)$ that we wish to minimize in (18) becomes

$$J(x) = \left\| \begin{bmatrix} A_1 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\| + \eta_1 \|x_1\| + \eta_2 \|x_2\|. \tag{26}$$

Recall that we are assuming that (13) holds. This means that either $\eta_1 < \|A_1^T b\|/\|b\|$, or $\eta_2 < \|A_2^T b\|/\|b\|$, or both.

From Theorem 1, a minimizer $\hat{x}$ with $\hat{x}_1 \neq 0$ and $\hat{x}_2 \neq 0$ exists if and only if it satisfies

$$\left[\begin{bmatrix} A_1 & A_2 \end{bmatrix} + \frac{\left(\begin{bmatrix} A_1 & A_2 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} - b \right) \begin{bmatrix} \frac{\eta_1 \hat{x}_1^T}{\|\hat{x}_1\|} & \frac{\eta_2 \hat{x}_2^T}{\|\hat{x}_2\|} \end{bmatrix}}{\left\| \begin{bmatrix} A_1 & A_2 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} - b \right\|} \right]^T \cdot \left(\begin{bmatrix} A_1 & A_2 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} - b \right) = 0.$$

If such an $\hat{x}$ does not exist, then we need to check for solutions of the form $\{0, \hat{x}_2\}$ or $\{\hat{x}_1, 0\}$. We will refer to these as boundary solutions. In the first case, with $x_1 = 0$, the cost function $J(x)$ collapses to

$$J_1(x) = \|A_2 x_2 - b\| + \eta_2 \|x_2\|. \tag{27}$$

A unique nonzero minimum of this cost exists if and only if $\eta_2 < \|A_2^T b\|/\|b\|$, in which case, it is given by the solution of the orthogonality condition

$$\left[A_2 + \frac{(A_2\hat{x}_2 - b)\frac{\eta_2 \hat{x}_2^T}{\|\hat{x}_2\|}}{\|A_2\hat{x}_2 - b\|} \right]^T (A_2\hat{x}_2 - b) = 0. \tag{28}$$

We should stress that this orthogonality condition is a necessary and sufficient condition for the existence of a nonzero minimizer for the above cost $J_1(x)$. It is, however, *only* a necessary condition for the corresponding $\{0, \hat{x}_2\}$ to be the minimum of the original cost $J(x)$. Indeed, if $\{0, \hat{x}_2\}$ is the minimum of $J(x)$, then by differentiation of $J(x)$ with respect to x_2, we obtain (28). On the other hand, if (28) holds, it would not follow in general that $\{0, \hat{x}_2\}$ is the minimizer of $J(x)$. This is because the other boundary solution $\{\hat{x}_1, 0\}$ can still lead to a smaller cost.

Likewise, in the second case with $x_2 = 0$, the cost function $J(x)$ collapses to

$$J_2(x) = \|A_1 x_1 - b\| + \eta_1 \|x_1\|.$$

A unique nonzero minimum of this cost exists if and only if $\eta_1 < \|A_1^T b\|/\|b\|$, in which case, it is given by the solution of the orthogonality condition

$$\left[A_1 + \frac{(A_1\hat{x}_1 - b)\frac{\eta_1 \hat{x}_1^T}{\|\hat{x}_1\|}}{\|A_1\hat{x}_1 - b\|} \right]^T (A_1\hat{x}_1 - b) = 0.$$

Once the unique minimizers of $J_1(x)$ and $J_2(x)$ have been determined, we pick that solution $\{0, \hat{x}_2\}$ or $\{\hat{x}_1, 0\}$ that has the smallest cost $J(x)$ as the unique minimizer of the original problem (26).

In a similar manner, if K were equal to three, then the determination of the boundary solutions would require that we first seek all minimizers of the form $\{0, \hat{x}_2, \hat{x}_3\}$, $\{\hat{x}_1, 0, \hat{x}_3\}$, $\{\hat{x}_1, \hat{x}_2, 0\}$, $\{0, 0, \hat{x}_3\}$, $\{0, \hat{x}_2, 0\}$, $\{\hat{x}_1, 0, 0\}$ for the collapsed cost function. These are defined by appropriate orthogonality conditions. Then, the desired global minimizer is the boundary solution that has the smallest cost $J(x)$.

C. Statement of the Solution of the BDU Problem

We summarize here the solution of the BDU problem (5) for ease of reference.

Theorem 2 (Solution of BDU Estimation): Consider a full rank matrix $A \in \mathbb{R}^{N\times n}$ with $N > n$ and a nonzero vector b that does not belong to the column span of A. The solution of the BDU estimation problem (5) is always unique. In particular, we have the following.

I) The solution is zero ($\hat{x} = 0$) if and only if each η_i satisfies $\eta_i \geq \|A_i^T b\|/\|b\|$.

II) The solution is nonzero if and only if at least one η_i satisfies $\eta_i < \|A_i^T b\|/\|b\|$. In this case, the solution can be a regular solution or a boundary solution.

II.1) The unique solution $\hat{x}$ is regular (i.e., with all its components $\hat{x}_i \neq 0$) if and only if an $\hat{x} \in \mathcal{X}$ exists (i.e., a regular vector $\hat{x}$ exists) that satisfies (22). Alternatively, this unique $\hat{x}$ can be found by solving the nonlinear system of equations (24) and (25) in $\hat{x}$ and $\hat{\alpha}_i$ (see Section VI).

II.2) If a regular solution $\hat{x}$ does not exist, then the unique minimizer is a boundary solution (i.e., with some $\hat{x}_i$ equal to zero). $\square$

VI. Determining the Regularization Parameters

In this section, we exhibit one method for determining the regularization parameters $\{\hat{\alpha}_i\}$ in the regular case by using the SVD of the nominal data (similar procedures hold in the nonregular case since the orthogonality conditions lead to similar nonlinear equations). To illustrate the main idea, we focus, without loss of generality, on the case $K = 2$

$$\min_x \max_{\left(\substack{\|\delta A_1\| \leq \eta_1 \\ \|\delta A_2\| \leq \eta_2}\right)} \|[\,A_1 + \delta A_1 \quad A_2 + \delta A_2\,]\, x - b\|\,.$$

Assume $\eta_1 < \|A_1^T b\|/\|b\|$ so that a nonzero solution $\hat{x}$ exists. Assume further that the solution is regular: $\hat{x} \in \mathcal{X}$. It is then given by the solution of the coupled equations

$$\left(A^T A + \begin{bmatrix} \hat{\alpha}_1 I & \\ & \hat{\alpha}_2 I \end{bmatrix}\right) \hat{x} = A^T b$$
$$\begin{bmatrix} \hat{\alpha}_1 \\ \hat{\alpha}_2 \end{bmatrix} - \|A\hat{x} - b\| \begin{bmatrix} \eta_1 \|\hat{x}_1\|^\dagger \\ \eta_2 \|\hat{x}_2\|^\dagger \end{bmatrix} = 0.$$

One way to determine the $\{\hat{\alpha}_1, \hat{\alpha}_2\}$ is as follows. Recall that A is $N \times n$ and full rank with $N > n$. Assume A_1 has n_1 columns and A_2 has n_2 columns. We introduce the SVD of the full-rank matrix A

$$A = U \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} V^T$$

with $U(N \times N), V(n \times n), \Sigma(n \times n)$, and partition V, Σ, and the vector $U^T b$ accordingly with A, say

$$V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix}, \quad U^T b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

with $V_1(n_1 \times n), V_2(n_2 \times n), \Sigma_1(n_1 \times n_1), \Sigma_2(n_2 \times n_2), b_1(n \times 1), b_2(N - n \times 1)$. Then, $\hat{x}_1$ is $n_1 \times 1, \hat{x}_2$ is $n_2 \times 1$, and they are given by $\hat{x}_1 = V_1 c$ and $\hat{x}_2 = V_2 c$, where the ($\hat{\alpha}_i$-dependent) vector c is defined by $c = M\Sigma b_1$, with

$$M = \left(\Sigma^2 + V^T \begin{bmatrix} \hat{\alpha}_1 I_{n_1} & \\ & \hat{\alpha}_2 I_{n_2} \end{bmatrix} V\right)^{-1}.$$

It further follows that

$$\|A\hat{x} - b\| = \sqrt{\|b_2\|^2 + \|\Sigma d\|^2}$$

where the ($\hat{\alpha}_i$-dependent) vector d is defined by

$$d = MV^T \begin{bmatrix} \hat{\alpha}_1 I_{n_1} & \\ & \hat{\alpha}_2 I_{n_2} \end{bmatrix} V\Sigma^{-1} b_1.$$

In this way, we obtain the following expressions for $\hat{\alpha}_1$ and $\hat{\alpha}_2$, where the unknowns $\{\hat{x}_1, \hat{x}_2\}$ have been eliminated.

$$\hat{\alpha}_1 = \left(\eta_1 \sqrt{\|b_2\|^2 + \|\Sigma d\|^2}\right) \|V_1 c\|^\dagger$$
$$\hat{\alpha}_2 = \left(\eta_2 \sqrt{\|b_2\|^2 + \|\Sigma d\|^2}\right) \|V_2 c\|^\dagger.$$

These provide two nonlinear coupled equations in the non-negative parameters $\{\hat{\alpha}_1, \hat{\alpha}_2\}$. All other quantities in the equations are known. Such equations can be solved by appropriate zero-finding techniques (e.g., the command fsolve of Matlab).

VII. Concluding Remarks

This paper developed a geometric framework for BDU estimation in the presence of multiple sources of uncertainties with possibly different levels of intensity. In particular, it was shown that the solution requires a number of regularization parameters that is equal to the number of error sources. It was also shown that these parameters are determined automatically as the non-negative roots of certain coupled nonlinear equations. Two applications were considered in the context of image separation and array signal processing. The results show that there is merit to the new method, but there are also issues and extensions that remain to be addressed. In particular, it would be useful to study the statistical properties of the BDU estimator in terms of bias and consistency. (The curves in Fig. 4, which show that the MSE curve of BDU is on average lower than the other curves, suggest that the BDU solution can have good statistical properties.) It would also be useful to study more general formulations that allow for weightings in the data as well as exploit structure. Some extensions in this direction appear in [17].

Appendix A
Proof of Lemma 1

In order to prove Lemma 1, we first establish a preliminary result. The result states that when (12) holds, i.e., when the uncertainty set $\{\|\delta A_i\| \leq \eta_i\}$ is large enough, then there exists a perturbed matrix $(A + \delta A)$ that is orthogonal to b.

Lemma 4: The uncertainty set $\{\|\delta A_i\| \leq \eta_i\}$ contains a perturbation δA such that $(A+\delta A)$ is orthogonal to b, $(A+\delta A)^T b = 0$, if and only if (12) holds.

Proof: Assume there exists a valid perturbation δA, say $\overline{\delta A}$, with $\{\|\overline{\delta A}_i\| \leq \eta_i\}$ and such that $(A+\overline{\delta A})^T b = 0$. Then, for each i, we also have $(A_i + \overline{\delta A}_i)^T b = 0$ and, consequently, $(\overline{\delta A}_i)^T b = -A_i^T b$. This further means that

$$\|A_i^T b\| = \left\|\overline{\delta A}_i^T b\right\| \leq \left\|\overline{\delta A}_i^T\right\| \cdot \|b\| = \|\overline{\delta A}_i\| \cdot \|b\|$$

which implies that $\|\overline{\delta A}_i\| \geq \|A_i^T b\|/\|b\|$ and, hence, (12) must hold for each i. Conversely, assume (12) holds, and choose

$$\overline{\delta A}_i = -\frac{1}{\|b\|^2} bb^T A_i. \tag{29}$$

Then

$$\|\overline{\delta A}_i\| \leq \frac{1}{\|b\|^2}\|b\|\|b^T A_i\| = \frac{\|A_i^T b\|}{\|b\|} \leq \eta_i.$$

This shows that $\overline{\delta A}_i$ is a valid perturbation for each i. Now note that

$$A_i + \overline{\delta A}_i = A_i - \frac{1}{\|b\|^2} bb^T A_i = \left[I - \frac{bb^T}{\|b\|^2}\right] A_i \tag{30}$$

so that

$$A + \overline{\delta A} = \left[I - \frac{bb^T}{\|b\|^2}\right] A \tag{31}$$

where the matrix $(I - bb^T/\|b\|^2)$ is the projector onto the orthogonal complement space of b. This implies that $(A+\overline{\delta A})^T b = 0$, as desired. □

In fact, we can further verify that the matrix $(A+\overline{\delta A})$ in (31) has full column rank. Indeed, assume otherwise. Then, there should exist a nonzero vector p such that

$$\left[I - \frac{bb^T}{\|b\|^2}\right] Ap = 0.$$

If we denote Ap by w (w is also nonzero since A is full column rank), this means that we must have

$$\left[I - \frac{bb^T}{\|b\|^2}\right] w = 0$$

which is only possible if w is parallel to the vector b, say $w = \alpha b$ for some $\alpha \neq 0$, since the matrix $(I - bb^T/\|b\|^2)$ is the projector onto the orthogonal complement space of b. Hence, we must have $Ap = \alpha b$. This contradicts our assumption that b does not lie in the column span of A. Therefore, the matrix $(A+\overline{\delta A})$ in (31) has full column rank.

We also conclude from the proof of the above lemma that, for some i, whenever $\eta_i \geq \|A_i^T b\|/\|b\|$, then we can find a perturbation $\overline{\delta A}_i$, as in (29), such that

$$(A_i + \overline{\delta A}_i)^T b = 0.$$

It further holds that the resulting $(A_i + \overline{\delta A}_i)$ in (30) has full column rank since otherwise, we would be able to conclude, as above, that b lies in the column span of A_i, which in turn lies in the column span of A, thus leading again to a contradiction.

Proof of Lemma 1: Assume first that (12) holds, and let us show that $\hat{x} = 0$ is the unique solution. Choose

$$\overline{\delta A}_i = -\frac{1}{\|b\|^2} bb^T A_i.$$

We already know from the proof of Lemma 4 that $\overline{\delta A}_i$ is a valid perturbation since $\|\overline{\delta A}_i\| \leq \eta_i$ and that $(A+\overline{\delta A})^T b = 0$. We also know that $(A+\overline{\delta A})$ has full column rank. Now, since b is orthogonal to $(A+\overline{\delta A})$, it follows that

$$\|(A+\overline{\delta A})x - b\| > \|b\|$$

for any nonzero vector x. Therefore

$$\max_{\{\|\delta A_i\| \leq \eta_i\}} \|(A+\delta A)x - b\| \geq \|(A+\overline{\delta A})x - b\| > \|b\|.$$

Now, note that if we set x equal to zero in the BDU cost function (5), we obtain that the cost is equal to $\|b\|$, regardless of δA. Therefore, $\hat{x} = 0$ has to be the unique solution of (5).

The proof of the converse statement is more involved. Thus, assume $\hat{x} = 0$ is the unique solution of (5), and let us establish that (12) must hold. If $\hat{x} = 0$ is the unique solution, then for every x

$$\max_{\{\|\delta A_i\| \leq \eta_i\}} \|(A+\delta A)x - b\|^2 \geq \|b\|^2.$$

That is, for every x

$$\begin{aligned}\max_{\{\|\delta A_i\| \leq \eta_i\}} [&x^T(A+\delta A)^T(A+\delta A)x \\ &- 2b^T(A+\delta A)x] \geq 0.\end{aligned} \tag{32}$$

Choose x as a scaled multiple of $\left[\begin{smallmatrix} A_1^T \\ 0 \end{smallmatrix}\right] b$, say $x = \gamma \left[\begin{smallmatrix} A_1^T \\ 0 \end{smallmatrix}\right] b$, for some positive scalar γ. Then, the above inequality implies that for any such γ

$$\max_{\|\delta A_1\| \leq \eta_1} M_1 \geq 0 \tag{33}$$

where

$$\begin{aligned} M_1 = \gamma^2 b^T A_1 (A_1 + \delta A_1)^T (A_1 + \delta A_1) A_1^T b \\ - 2\gamma b^T (A_1 + \delta A_1) A_1^T b. \end{aligned}$$

We now claim that for the above inequality to hold, it must be true that

$$\max_{\|\delta A_1\| \leq \eta_1} \left[-2b^T(A_1 + \delta A_1)A_1^T b\right] \geq 0. \tag{34}$$

Indeed, assume to the contrary that (34) does not hold, say

$$\max_{\|\delta A_1\| \leq \eta_1} \left[-2b^T(A_1 + \delta A_1)A_1^T b\right] = -\rho < 0$$

for some $\rho > 0$. Then, $2b^T(A_1 + \delta A_1)A_1^T b \geq \rho > 0$ for all δA_1. Choose γ such that $\gamma[\|b\|^2\|A_1\|^2(\|A_1\| + \eta_1)^2] < \rho$. Then, it is easy to verify that this would lead to

$$\max_{\|\delta A_1\| \leq \eta_1} M_1 < 0$$

since

$$\max_{\|\delta A_1\| \leq \eta_1} M_1 < \gamma^2\|b\|^2\|A_1\|^2(\|A_1\| + \eta_1)^2 - \gamma\rho < 0$$

where we replaced the first term inside the $\max$ by its maximum value and the second term by its smallest value. The result contradicts (33) so that (34) must hold. The maximum the expression between parenthesis can be is $-2b^T A_1 A_1^T b + 2\eta_1 \|b\| \cdot \|A_1^T b\|$, which is achievable if we choose $\delta A_1 = -\eta_1 (bb^T A_1 / (\|b\| \|b^T A_1\|))$. This δA_1 is a valid perturbation since $\|\delta A_1\| \leq \eta_1$. Therefore, $-2\|A_1^T b\|^2 + 2\eta_1 \|b\| \|A_1^T b\| \geq 0$, which leads to the desired conclusion that $\eta_1 \geq \|A_1^T b\| / \|b\|$.

We can now repeat the argument by choosing x of the form $x = \gamma \text{col}\{0, A_2^T, 0\} b$ to conclude that $\eta_2 \geq \|A_2^T b\| / \|b\|$, and so on, until all η_i are proven to satisfy (12). $\square$

APPENDIX B
PROOF OF LEMMA 2

We first identify the perturbations $\{\delta A_i\}$ that maximize the residual norm in (5). Thus, in view of the triangle inequality of norms, it holds that for any δA and for any x

$$\begin{aligned}\|(A+\delta A)x - b)\| \leq \|Ax - b\| &+ \|\delta A_1 x_1\| \\ &+ \|\delta A_2 x_2\| + \cdots + \|\delta A_K x_K\|\end{aligned}$$

with equality if and only if the perturbations $\{\delta A_i\}$ are such that the vectors $\{\delta A_i x_i\}$ are collinear with the vector $(Ax - b)$, i.e.,

$$\delta A_i x_i = \beta_i (Ax - b) \tag{35}$$

for some scalars $\{\beta_i \geq 0\}$. Moreover, it holds that $\|\delta A_i x_i\| \leq \eta_i \|x_i\|$ with equality if and only if the perturbation δA_i is also such that

$$\|\delta A_i x_i\| = \eta_i \|x_i\|. \tag{36}$$

Combining (36) with (35), we see that (36) will hold only if

$$\beta_i = \frac{\eta_i \|x_i\|}{\|Ax - b\|}.$$

This expression for β_i is well defined since $\|Ax - b\| \neq 0$, in view of our earlier assumption that b does not lie in the column span of A.

The above discussion shows that if for a vector x there exists a perturbation δA in the valid domain $\{\|\delta A_i\| \leq \eta_i\}$ that satisfies, for each δA_i

$$\delta A_i x_i = \frac{\eta_i \|x_i\|}{\|Ax - b\|} (Ax - b) \tag{37}$$

then

$$\begin{aligned}&\max_{\|\delta A_i\| \leq \eta_i} \|(A+\delta A)x - b\| \\ &\quad = \|Ax - b\| + \eta_1 \|x_1\| + \cdots + \eta_K \|x_K\|.\end{aligned}$$

It is easy to verify that the following (rank-one) choice for δA_i satisfies (37)

$$\delta A_i^o(x) \triangleq \begin{cases} \eta_i \frac{(Ax-b)}{\|Ax-b\|} \frac{x_i^T}{\|x_i\|}, & \text{if } x_i \neq 0 \\ 0, & \text{if } x_i = 0 \end{cases} \tag{38}$$

and is a valid perturbation (since $\|\delta A_i^o(x)\| \leq \eta_i$). Therefore, the maximum residual in (14) is attainable. We remark that, in general, there can exist many other δA_i's that satisfy (37) for any given x (see, e.g., [7]). It is enough for our purposes, however, to identify one set of perturbations $\{\delta A_i\}$ that achieves the maximum residual, e.g., the $\{\delta A_i^o(x)\}$ above.

Let us now establish the two properties that are stated in Lemma 2. To prove the full-rank property, assume to the contrary that $A + \delta A^o(x)$ is rank deficient for some x. This means that there exists a nonzero vector p such that

$$\left\{A + \frac{(Ax - b)q^T(x)}{\|Ax - b\|}\right\} p = 0.$$

The vector p is necessarily not orthogonal to $q(x)$ since otherwise, we would obtain $Ap = 0$, which contradicts our assumption that A itself has full column rank. Define the scalar nonzero quantity $\kappa = q^T(x)p / \|Ax - b\|$. It then follows from the above equality that

$$A\left[x + \frac{1}{\kappa} p\right] = b.$$

This means that b should lie in the column span of A, which again contradicts our earlier assumption about b.

Finally, it also follows from (35) that any δA that attains the maximum residual in (14) leads to a residual vector $(A + \delta A)x - b$ that is necessarily collinear with $(Ax - b)$ since

$$(A + \delta A)x - b = (1 + \beta_1 + \cdots + \beta_K)(Ax - b). \tag{39}$$

$\square$

APPENDIX C
PROOF OF THEOREM 1

Let $\hat{x} \in \mathcal{X}$ be a regular vector that satisfies the orthogonality condition (21) or (22) $A^T(\hat{x})[A\hat{x} - b] = 0$. Let us now show that it has to be the unique global minimizer of the cost function in (18). Indeed, pick any other vector x (in $\mathcal{X}$ or otherwise). Then, we necessarily have

$$\|A(x)x - b\| \geq \|A(\hat{x})x - b\|.$$

This is because we already know from Lemma 2 that for a given x, $A(x)$ is a matrix that maximizes $\|(A + \delta A)x - b\|$ over δA. We now verify that because of the above orthogonality condition, it holds that

$$\|A(\hat{x})\hat{x} - b\| \leq \|A(\hat{x})x - b\|$$

in order to conclude that

$$\|A(\hat{x})\hat{x} - b\| \leq \|A(x)x - b\|$$

so that $\hat{x}$ is a minimizer. To establish this fact, we perform the following calculations:

$$\begin{aligned}\|A(\hat{x})x - b\|^2 &= \|A(\hat{x})(x + \hat{x} - \hat{x}) - b\|^2 \\ &= \|[A(\hat{x})\hat{x} - b] + A(\hat{x})(x - \hat{x})\|^2 \\ &= \|A(\hat{x})\hat{x} - b\|^2 + \|A(\hat{x})(x - \hat{x})\|^2 \\ &\geq \|A(\hat{x})\hat{x} - b\|^2\end{aligned}$$

where in the third step, we used the fact that $A^T(\hat{x})[A(\hat{x})\hat{x} - b] = 0$. We thus established that if $\hat{x}$ satisfies the orthogonality condition (22), then $\|A(\hat{x})\hat{x} - b\| \leq \|A(x)x - b\|$ for any nonzero x, and therefore, $\hat{x}$ is a minimizer. However, since the global minimum is unique, $\hat{x}$ is the unique minimizer.

Conversely, suppose that $\hat{x} \in \mathcal{X}$ is a nonzero minimizer of the cost function in (18) [or of $J(x)$ in (14)]. Then, the gradient of $J(x)$ at $x = \hat{x}$ must be zero. (Note that the gradient of $J(x)$ is defined at all $x \in \mathcal{X}$; the function is not differentiable only at points that have some x_i equal to zero.) Using the relations

$$\nabla_x \|Ax - b\| = \frac{A^T(Ax-b)}{\|Ax-b\|}, \quad \nabla_x \|x_1\| = \begin{bmatrix} \frac{x_1}{\|x_1\|} \\ 0 \end{bmatrix}$$

and

$$\nabla_x \|x_2\| = \begin{bmatrix} 0 \\ \frac{x_2}{\|x_2\|} \\ 0 \end{bmatrix}, \text{etc.}$$

we obtain that

$$\nabla_x J(x) = \frac{A^T(Ax-b)}{\|Ax-b\|} + q(x).$$

Therefore, $\nabla_x J(\hat{x}) = 0$ leads to

$$A^T(A\hat{x} - b) + q(\hat{x})\|A\hat{x} - b\| = 0$$

which is equivalent to

$$\left[A + \frac{(A\hat{x}-b)q^T(\hat{x})}{\|A\hat{x}-b\|}\right]^T (A\hat{x} - b) = 0$$

as desired.

This completes the proof of Theorem 1. $\square$

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