From Theorem 1, we can then deduce that it is also inseparable. A direct check of inseparability agrees with this conclusion. In fact, we have the min–max function

$$\begin{aligned} F(X) &= A \otimes (B \odot X) \\ &= (\alpha_1 + \beta_3 + x_3, \alpha_2 + \beta_1 + x_1, \alpha_3 + \beta_2 + x_2)'. \end{aligned}$$

The corresponding monotone Boolean equation $X = F(X; \mathbf{0}))$ is $X = (x_3, x_1, x_2)'$, which has only trivial solutions.

Next, let us present an example of reducible min–max system.

Example 3: Consider the min–max system Σ defined by

$$X(k+1) = F(X(k)) = (x_1(k) \vee x_2(k), x_1(k) \wedge x_2(k))'.$$

It is straightforward to verify that Σ is not inseparable since $X = (1,0)'$ is a nontrivial solution to the Boolean function $F(X) = X$. Let us find a reducible pair for it along the line of the first half of the proof of Theorem 1. Note $F(X)$ is given in DNF form. Following the construction of (13), we introduce a vector $Y = (y_1, y_2, y_3)'$ as the collection of the minimal terms in the DNF of F, namely, $(y_1, y_2, y_3) = (x_1, x_2, x_1 \wedge x_2)'$ or in matrix form

$$Y = B \odot X = \begin{pmatrix} 0 & +\infty \\ +\infty & 0 \\ 0 & 0 \end{pmatrix} \odot X.$$

As a result, $F(X)$ can be expressed in terms of Y as

$$F(X) = A \otimes Y = \begin{pmatrix} 0 & 0 & -\infty \\ -\infty & -\infty & 0 \end{pmatrix} \otimes Y.$$

For $X = (1,0)'$, we have in $Y = (1,0,0)'$ in (15). We can then decide that $n_1 = |I_1| = 1, n_2 = |I_0| = 1$ and $m_1 = |J_1| = 1, m_2 = |J_0| = 2$. The first part of the proof claims that the pair (A, B) obtained in this way is reducible with $A_{21} = \mathcal{E}$ being an $n_2 \times m_1 = 1 \times 1$ matrix, $B_{12} = \mathcal{J}$ being an $m_1 \times n_2 = 1 \times 1$ matrix, and both permutations σ and τ being identity permutations over $\underline{n} = \{1, 2\}$ and $m = \{1, 2, 3\}$. This agrees with the direct observations on A and B.

Remark: From computational point of view, Theorem 1 has the following implication. Whenever it is easy to find a structural eigenvalue and the corresponding structural eigenvector in the (min, max,+)-algebra sense, as shown in [7, Th. 2], we can decide the irreducibility of a given min–max system and as a result establish the inseparability property. This will save us from the work of solving Boolean equations. On the other hand, if a min–max system is given directly in the general form of (7), it might be difficult to decide the irreducibility by rewriting the system in bipartite form and finding its structural eigenvalue. One potential difficulty is that the conversion to bipartite form may introduce an exponential many auxiliary variables, i.e., $m = O(2^n)$. For some of such cases, it might be easier to test inseparability.

Example 1: (continued) According to Theorem 1 and the inseparability we already established, we can deduce that the min–max system in Example 1 is irreducible. Note, we establish the irreducibility of this system without explicitly identifying a structural eigenvalue which seems a nontrivial task for this example. It should be made clear that the irreducibility means that there is no reducible pair (A, B) such that $F(X) = A \otimes (B \odot X)$.

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REFERENCES

[1] J. Gunawardena, "Timing analysis of digital circuits and the theory of min–max functions," in *Digest of Technical Papers of the ACM International Workshop on Timing Issues in the Specification and Synthesis of Digital Systems*. New York: ACM, 1993.

[2] ——, "Min–max functions," *Discrete Event Dyna. Syst.*, vol. 4, pp. 377–406, 1994.

[3] J. Cochet-Terrasson, S. Gaubert, and J. Gunawardena, "Dynamics of Min–Max Functions," HP Lab, Tech. Rep. HPL-BRIMS-97-13, 1997.

[4] F. Baccelli, G. Cohen, J. Olsder, and J.-P. Quadrat, *Synchronization and Linearity*. New York: Wiley, 1992.

[5] Q. Zhao, D.-Z. Zheng, and X. Zhu, "Structure properties of min–max systems and existence of global cycle time," *IEEE Trans. Automat. Contr.*, vol. 46, pp. 148–151, Jan. 2001.

[6] Q. Zhao and D.-Z. Zheng, "On stabilization of min–max systems," *Automatica*, vol. 39, no. 4, pp. 751–756, 2003.

[7] J. van der Woude and J. Subiono, "Conditions for the structural existence of an eigenvalue of a bipartite (min,max,+)-system," *Theoretical Comput. Sci.*, vol. 293, no. 1, pp. 13–24, 2003.

[8] S. Burns, "Performance analysis and optimization of asynchronous circuits," Ph.D. dissertation, California Inst. Technol., Pasadena, CA, 1991.

[9] S. Chakraborty, K. Yun, and D. Dill, "Timing analysis of asynchronous systems using time separation of events," *IEEE Trans. Computer-Aided Design Int. Circuits Syst.*, vol. 18, pp. 1061–1076, Aug. 1999.

[10] K. Yang and Q. Zhao, "Balance problem of min–max system is co-np hard," , 2004, submitted for publication.

[11] E. Clarke, J. Grumberg, and D. Peled, *Model Checking*. Cambridge, MA: MIT Press, 1999.

[12] G. Olsder, "On structural properties of min–max systems," Faculty Tech. Math. Inform., Delft Univ. Technol., Delft, The Netherlands, Tech. Rep. TWI Report 93-95, 1993.

Regularized Robust Filters for Time-Varying Uncertain Discrete-Time Systems

Ananth Subramanian and Ali H. Sayed

***Abstract*—This note develops robust filters for time-varying uncertain discrete-time systems. The developed filters are based on a data regularization solution and they enforce a minimum state-error variance propoerty. Simulation results confirm their superior performance over other robust filter designs.**

***Index Terms*—Convex optimization, least-squares, parametric uncertainty, regularization, robust filter.**

I. INTRODUCTION

The Kalman filter is the optimal linear least-mean squares estimator for systems that are described by linear state-space models [1]. However, when the model is not accurately known, the performance of the filter can deteriorate appreciably. This filter sensitivity to modeling errors has led to several works in the literature on the development of robust filters; robust in the sense that they limit the effect of model uncertainties on filter performance. Some known approaches to robust state-space estimation are $\mathcal{H}_\infty$ filtering, mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filtering,

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The authors are with the Electrical Engineering Department, University of California, Los Angeles, CA 90095 USA (e-mail: msananth@ee.ucla.edu; sayed@ee.ucla.edu).

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set-valued estimation, guaranteed-cost designs, and minimum variance filtering (see [2]–[8]). In [9], a robust filter design framework was proposed that performs data regularization as opposed to data de-regularization; a property that is useful for real-time operation. The design in [9] involved choosing a certain Ricatti variable so as to enforce local optimality and robustness properties. In this note, we pursue the design of such regularized robust filters further and show how to enforce certain minimum error variance property. We also consider two general classes of uncertain state-space models. One class involves stochastic uncertainties and another class involves polytopic uncertainties. For each class, we design robust filters that bound the state error covariance matrix. The robustness criterion used is different from prior robust designs (e.g., $\mathcal{H}_\infty$, guaranteed-cost, or set-valued estimation) in that it is based on robust regularization [9]. Simulation results are included to illustrate the superior performance of the proposed filters over other robust designs.

II. Least-Squares With Uncertainties

Let $J(x,y)$ denote a cost function of the form $J(x,y) = x^T \Pi x + R(x,y)$ with

$$R(x,y) = ((A+\delta A)x - (b+\delta b))^T\, W\, ((A+\delta A)x - (b+\delta b)) \quad (1)$$

where δA denotes an $N \times n$ perturbation matrix to A, δb denotes an $N \times 1$ perturbation vector to b, and $\{\delta A, \delta b\}$ are assumed to satisfy a model of the form

$$[\delta A \quad \delta b] = H\Delta[E_a \quad E_b] \quad (2)$$

where Δ is an arbitrary contraction, $\|\Delta\| \le 1$, and $\{H, E_a, E_b\}$ are known quantities of appropriate dimensions. Here, the notation $\|.\|$ denotes the two-induced norm of its matrix argument. Moreover, $\Pi > 0$ and $W > 0$. Consider then the constrained two-player game problem

$$\hat{x} = \arg\min_{x} \max_{\{\delta A, \delta b\}} J(x,y) \quad (3)$$

subject to (2). The following result is proven in [10].

Theorem 1: The problem (2), (3) has a unique solution $\hat{x}$ that is given by

$$\hat{x} = [\widehat{\Pi} + A^T \widehat{W} A]^{-1} \left[A^T \widehat{W} b + \widehat{\beta} E_a^T E_b\right] \quad (4)$$

where $\widehat{\Pi}$ and $\widehat{W}$ are modifications to Π and W

$$\widehat{\Pi} = \Pi + \widehat{\beta} E_a^T E_a \quad \widehat{W} = W + WH(\widehat{\beta} I - H^T W H)^\dagger H^T W \quad (5)$$

and where the positive scalar $\widehat{\beta}$ is determined from the optimization

$$\widehat{\beta} = \arg \min_{\beta \ge \|H^T W H\|} G(\beta) \quad (6)$$

where the function $G(\beta)$ is defined as follows:

$$G(\beta) = x^T(\beta)\Pi x(\beta) + \beta \|E_a x(\beta) - E_b\|^2 + [Ax(\beta) - b]^T\, W(\beta)\, [Ax(\beta) - b] \quad (7)$$

with

$$W(\beta) = W + WH(\beta I - H^T W H)^\dagger H^T W \quad \Pi(\beta) = \Pi + \beta E_a^T E_a$$

and

$$x(\beta) = \left[\Pi(\beta) + A^T W(\beta) A\right]^{-1} \left[A^T W(\beta) b + \beta E_a^T E_b\right]. \quad (8)$$

[The notation $X^\dagger$ denotes the pseudoinverse of X.] ◊

It was shown in [10], [11] that the function $G(\beta)$ has a unique global minimum (and no local minima) over the interval $\beta \ge \|H^T W H\|$, which means that the determination of $\widehat{\beta}$ can be pursued via search procedures without worrying about convergence to undesired local minima. It was argued in [9] that a reasonable approximation for $\hat{\beta}$ is to choose it as $\hat{\beta} = (1+\alpha)\beta_l$, for some $\alpha > 0$ and where $\beta_l = \|H^T W H\|$.

III. State-Space Models

We shall show how to use Theorem 1 to design robust filters. Each filter will be applicable to a particular uncertainty model. Thus, consider an n-dimensional state-space model of the form

$$\boxed{x_{k+1} = F_k x_k + G_k u_k \quad y_k = (H_k + \Delta H_k)x_k + v_k \quad k \ge 0} \quad (9)$$

where $\{u_k, v_k\}$ are uncorrelated white zero-mean random processes with covariance matrices

$$\boxed{\mathcal{E} u_k u_k^T = Q_k \quad \mathcal{E} v_k v_k^T = R_k}$$

and x_0 is a zero-mean random variable that is uncorrelated with $\{u_k, v_k\}$ for all k. Here, the symbol $\mathcal{E}$ denotes expectation. The uncertainties ΔH_k are modeled as

$$\boxed{\Delta H_k = M_k \Delta_k E_k} \quad (10)$$

where M_k and E_k are known matrices, while Δ_k is an arbitrary contraction, $\|\Delta_k\| < 1$. We shall consider two types of uncertainty descriptions for the state matrices F_k. One type is in terms of polytopic uncertainties and the other is in terms of stochastic uncertainties. In the first case, we assume that F_k lies inside a convex bounded polyhedral domain $\mathcal{K}_k$ that is described by m vertices as follows:

$$\mathcal{K}_k = \left\{ F_k = \sum_{i=1}^{m} \alpha_{i,k} F_{i,k} \quad \alpha_{i,k} \ge 0 \quad \sum_{i=1}^{m} \alpha_{i,k} = 1 \right\} \quad \text{(Polytopic uncertainties)} \quad (11)$$

Observe that $\mathcal{K}_k$ is allowed to vary with k. In the second case, we assume that F_k is instead described by

$$\begin{aligned} F_k &= F_{k,c} + \Delta F_k \\ \Delta F_k &= N_k \bar{\Delta}_k J_k \quad \text{(Stochastic uncertainties)} \end{aligned} \quad (12)$$

for some known $\{F_{k,c}, N_k, J_k\}$ and where $\bar{\Delta}_k$ is a random matrix whose entries are zero mean and uncorrelated with each other, and such that

$$\mathcal{E} \bar{\Delta}_k \bar{\Delta}_k^T \le \rho_{\bar{\Delta}} I \quad (13)$$

for some known positive scalar $\rho_{\bar{\Delta}}$.

IV. Robust State Space Filtering

When uncertainties are not present in the model (9), it is known that the optimal linear estimator for the state variable x_k is given by the Kalman filter [1]. This filter admits a deterministic interpretation as the solution to a regularized least-squares problem as follows [12]. Let[1]

$$\hat{x}_{k|k-1} \triangleq \text{an estimate of } x_k \text{ given } \{y_0, y_1, \ldots, y_{k-1}\}$$
$$\hat{x}_{k|k} \triangleq \text{an estimate of } x_k \text{ given } \{y_0, y_1, \ldots, y_{k-1}, y_k\}.$$

[1]When uncertainties are not present, the qualification “estimate” refers to the linear-least-mean-squares estimate.

Given the predicted estimate $\hat{x}_{k|k-1}$ and an observation y_k, the filtered estimate $\hat{x}_{k|k}$ that is computed by the Kalman filter is the solution of

$$\min_{x}\left[\|x-\hat{x}_{k|k-1}\|^2_{P_k^{-1}}+\|y_k-H_kx\|^2_{R_k^{-1}}\right] \tag{14}$$

where P_k and R_k are the state error covariance and the measurement noise covariance matrices, respectively, i.e., $P_k = \mathcal{E}(x_k - \hat{x}_{k|k-1})(x_k - \hat{x}_{k|k-1})^T$ and $R_k = \mathcal{E}v_kv_k^T$. When uncertainties are present in $\{H_k, F_k\}$, we could formulate a robust version of (14) by solving instead a min–max problem of the form

$$\min_{x}\max_{\delta H_k,\delta F_k}\left(\|x-\hat{x}_{k|k-1}\|^2_{P_k^{-1}}+\|y_k-(H_k+\delta H_k)x\|^2_{T_k^{-1}}\right) \tag{15}$$

for some matrices $\{P_k, T_k\}$ to be chosen. This formulation was proposed in [9], where T_k was chosen as $T_k = R_k$ while P_k was chosen via a Ricatti recursion so as to enforce a local robustness property. In this note, we shall determine P_k so as to minimize the state error covariance matrix as well. We do so by showing how to reparametrize the problem in terms of a single parameter W_k over which the global minimization of the state error covariance matrix reduces to a linear convex problem.

A. Polytopic Uncertainties in F_k Alone

We consider first the case of polytopic uncertainties in F_k alone as in (11) with no uncertainties in H_k. Our objective is to design a robust linear estimator for the state variable x_k of the form

$$\hat{x}_{k|k}=F_{p,k}\hat{x}_{k|k-1}+K_{p,k}y_k \quad \hat{x}_{k+1|k}=F_{k,c}\hat{x}_{k|k} \tag{16}$$

for some matrices $F_{p,k}$ and $K_{p,k}$ to be determined in order to minimize the state error covariance matrix and where $F_{k,c}$ denotes the centroid of the polytope $\mathcal{K}_k$

$$F_{k,c}=\frac{1}{m}\sum_{i=1}^{m}F_{i,k}. \tag{17}$$

Referring to problem (14), its solution $\hat{x}_{k|k}$ is given by

$$\hat{x}_{k|k}=\hat{x}_{k|k-1}+\left(P_k^{-1}+H_k^TR_k^{-1}H_k\right)^{-1} \times\left[H_k^TR_k^{-1}(y_k-H_k\hat{x}_{k|k-1})\right]. \tag{18}$$

If we introduce the matrix

$$W_k \triangleq \left(P_k^{-1}+H_k^TR_k^{-1}H_k\right)^{-1} \tag{19}$$

then (18) for $\hat{x}_{k|k}$ becomes

$$\hat{x}_{k|k}=\left(I-W_kH_k^TR_k^{-1}H_k\right)\hat{x}_{k|k-1}+W_kH_k^TR_k^{-1}y_k \tag{20}$$

in terms of the parameter W_k. Note that, in the absence of uncertainties, W_k would be the Ricatti variable $P_{k|k}$ of the time and measurement form of the Kalman filter [1]. Noting that u_k is a zero-mean white random process, we let the following be an estimate for x_{k+1}:

$$\hat{x}_{k+1|k} \triangleq F_{k,c}\hat{x}_{k|k} \tag{21}$$

where $F_{k,c}$ is given by (17). We then get

$$\hat{x}_{k+1|k}=F_{p,k}\hat{x}_{k|k-1}+K_{p,k}y_k \tag{22}$$

where $F_{p,k}$ and $K_{p,k}$ are defined in terms of W_k as

$$\boxed{F_{p,k}=F_{k,c}\left(I-W_kH_k^TR_k^{-1}H_k\right),\quad K_{p,k}=F_{k,c}W_kH_k^TR_k^{-1}}. \tag{23}$$

Denoting $\tilde{x}_k = x_k - \hat{x}_{k|k-1}$, we define the extended weight vector $\eta_k \triangleq \binom{x_k}{\tilde{x}_k}$. Then, ignoring the uncertainties in F_k, we find that η_k satisfies

$$\boxed{\eta_{k+1}=\bar{F}_k\eta_k+\bar{G}_kw_k} \tag{24}$$

where

$$w_k=\begin{pmatrix}u_k\\ v_k\end{pmatrix}$$
$$\bar{F}_k=\begin{pmatrix}F_k & 0\\ F_k-F_{p,k}-K_{p,k}H_k & F_{p,k}\end{pmatrix}$$
$$\bar{G}_k=\begin{pmatrix}G_k & 0\\ G_k & -K_{p,k}\end{pmatrix} \tag{25}$$

and the covariance matrix of η_k satisfies

$$\Sigma_{k+1}=\bar{F}_k\Sigma_k\bar{F}_k^T+\bar{G}_kS_k\bar{G}_k^T \quad \text{where} \quad S_k=\begin{pmatrix}Q_k & 0\\ 0 & R_k\end{pmatrix} \tag{26}$$

and Σ_0 is the covariance matrix of η_0. Now, observe that the expressions for $\{F_{p,k}, K_{p,k}\}$ are parametrized linearly in terms of the parameter W_k. We then choose W_k so as to minimize the covariance matrix of η_k. Specifically, we choose $W_k > 0$ so as to minimize Σ_{k+1}. This can be obtained by solving

$$\min_{W_k>0}\mathrm{Tr}(\Sigma_{k+1}) \tag{27}$$

subject to

$$\Sigma_{k+1}\geq\bar{F}_k\Sigma_k\bar{F}_k^T+\bar{G}_kS_k\bar{G}_k^T \tag{28}$$

or, equivalently

$$\begin{pmatrix}-\Sigma_{k+1} & \bar{F}_k\Sigma_k & \bar{G}_kS_k^{\frac{1}{2}}\\ \Sigma_k\bar{F}_k^T & -\Sigma_k & 0\\ S_k^{\frac{T}{2}}\bar{G}_k^T & 0 & -I\end{pmatrix}\leq 0. \tag{29}$$

So far we have ignored uncertainties in F_k. In order to incorporate the polytopic uncertainties in the F_k, as defined by the sets $\mathcal{K}_k$ in (11), we need to solve the above optimization problem with F_k taking values at the m vertices of the convex polytope $\mathcal{K}_k$, i.e., from the set $\{F_{1,k}, F_{2,k}, \ldots\ldots, F_{m,k}\}$. Since the inequality (29) is affine in F_k, the W_k thus found will ensure minimum error covariance Σ_k over all possible F_k in $\mathcal{K}_k$. Therefore, the desired time-varying robust filter is given by (22) and (23), where W_k is the positive definite solution of (27)–(29) with F_k taking values on the vertices of the convex polytope $\mathcal{K}_k$, and initializing $\Sigma_0 = \mathrm{diag}\{P_o, \epsilon I\}$ for some positive–definite P_o and scalar $\epsilon > 0$. The resulting filter is listed in Table I.

B. Polytopic Uncertainties in F_k and Bounded Uncertainties in H_k

We now incorporate uncertainties in to H_k. That is, we consider polytopic uncertainties in F_k as in (11) and bounded uncertainties in H_k as in (10). Again, our objective is to design a robust linear estimator for the state variable x_k of the form

$$\hat{x}_{k|k}=F_{p,k}\hat{x}_{k|k-1}+K_{p,k}y_k \quad \hat{x}_{k+1|k}=F_{k,c}\hat{x}_{k|k} \tag{30}$$

for some matrices $F_{p,k}$ and $K_{p,k}$ to be determined. We will design $F_{p,k}$ and $K_{p,k}$ by following a two-step procedure. Assume first that there are

TABLE I
MINIMUM VARIANCE FILTER FOR POLYTOPIC UNCERTAINTIES IN F_k ALONE

Assumed uncertain model. $x_{k+1} = F_k x_k + G_k u_k$; $y_k = H_k x_k + v_k$ where $F_k \in \mathcal{K}_k$ as in (11).
Initial conditions: $\hat{x}_0 = 0, \Sigma_0 = \text{diag}\{P_o, \epsilon I\}$, where $P_o > 0$ and $\epsilon > 0$.
Step 1. Using Σ_k, compute $\{W_k, \Sigma_{k+1}\}$ by solving (27) subject to the inequality (29) where $\{\bar{F}_k, \bar{G}_k, S_k\}$ are defined by (25) and (26).
Step 2. Update $\hat{x}_{k\|k-1}$ to $\hat{x}_{k+1\|k}$ as $\hat{x}_{k+1\|k} = F_{p,k}\hat{x}_{k\|k-1} + K_{p,k} y_k$, where $F_{p,k}$, $K_{p,k}$ and $F_{k,c}$ are defined by (17) and (23).

no uncertainties in F_k; we will incorporate the uncertainties in F_k later. With uncertainties in the output matrices H_k alone, we consider (15), which becomes

$$\min_{x} \max_{\delta H_k} \left(\|x - \hat{x}_{k|k-1}\|^2_{P_k^{-1}} + \|y_k - (H_k + \delta H_k)x\|^2_{T_k^{-1}} \right) \quad (31)$$

for some matrices P_k and T_k to be determined. Problem (31) can be written more compactly in the form of (1)–(3) with the identifications

$$\begin{gathered} x \longleftarrow \{x_k - \hat{x}_{k|k-1}\} \quad b \longleftarrow y_k - H_k \hat{x}_{k|k-1} \\ \delta A \longleftarrow M_k \Delta_k E_k \quad \delta b \longleftarrow -M_k \Delta_k E_k \hat{x}_{k|k-1} \\ \Pi \longleftarrow P_k^{-1} W \longleftarrow T_k^{-1} \quad H \longleftarrow M_k \quad E_a \longleftarrow E_k \\ E_b \longleftarrow -E_k \hat{x}_{k|k-1} \quad \Delta \longleftarrow \Delta_k \quad A \longleftarrow H_k. \end{gathered}$$

From Theorem 1, the solution $\hat{x}_{k|k}$ of (31) is given by

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + \left(P_k^{-1} + \hat{\beta} E_k^T E_k + H_k^T \hat{R}_k^{-1} H_k\right)^{-1} \times \left[H_k^T \hat{R}_k^{-1}(y_k - H_k \hat{x}_{k|k-1}) - \hat{\beta} E_k^T E_k \hat{x}_{k|k-1}\right]. \quad (32)$$

We are going to select $\hat{\beta}$ approximately as $\hat{\beta} = (1+\alpha)\beta_{l,k}$ where $\beta_{l,k} = \|M_k^T T_k^{-1} M_k\|$. Moreover

$$\hat{R}_k^{-1} = \left(T_k - \hat{\beta}^{-1} M_k M_k^T\right)^{-1}. \quad (33)$$

With a new definition of W_k as

$$W_k \triangleq \left(P_k^{-1} + \hat{\beta} E_k^T E_k + H_k^T \hat{R}_k^{-1} H_k\right)^{-1} \quad (34)$$

expression (32) for $\hat{x}_{k|k}$ becomes

$$\hat{x}_{k|k} = \left(I - \hat{\beta} W_k E_k^T E_k - W_k H_k^T \hat{R}_k^{-1} H_k\right)\hat{x}_{k|k-1} + W_k H_k^T \hat{R}_k^{-1} y_k \quad (35)$$

in terms of the parameter W_k. We again let

$$\hat{x}_{k+1|k} \triangleq F_{k,c} \hat{x}_{k|k} \quad (36)$$

where $F_{k,c}$ is given by (17). We then get

$$\hat{x}_{k+1|k} = F_{p,k} \hat{x}_{k|k-1} + K_{p,k} y_k \quad (37)$$

where $F_{p,k}$ and $K_{p,k}$ are now defined in terms of W_k as

$$\boxed{\begin{aligned} F_{p,k} &= F_{k,c}\left(I - \hat{\beta} W_k E_k^T E_k - W_k H_k^T \hat{R}_k^{-1} H_k\right) \\ K_{p,k} &= F_{k,c} W_k H_k^T \hat{R}_k^{-1} \end{aligned}}. \quad (38)$$

These expressions for $F_{p,k}$ and $K_{p,k}$ have been determined by assuming uncertainties in H_k alone. We now move on to select the parameter W_k by assuming uncertainties in F_k alone. By doing so, we will arrive at a filter that minimizes a bound on the state error covariance matrix when there are uncertainties in F_k alone and one that meets the robustness criterion (31) when there are uncertainties in H_k. With no uncertainties in F_k and H_k, the covariance matrix of η_k again satisfies

$$\Sigma_{k+1} = \bar{F}_k \Sigma_k \bar{F}_k^T + \bar{G}_k S_k \bar{G}_k^T \quad \text{where} \quad S_k = \begin{pmatrix} Q_k & 0 \\ 0 & R_k \end{pmatrix} \quad (39)$$

where Σ_0 is the covariance matrix of η_0. Now, observe again that the expressions for $\{F_{p,k}, K_{p,k}\}$ are parametrized linearly in terms of the parameter W_k. We will then choose W_k so as to minimize the covariance matrix of η_k. Specifically, we shall again choose $W_k > 0$ so as to minimize Σ_{k+1} of (39). This can be obtained by solving

$$\min_{W_k > 0} \text{Tr}(\Sigma_{k+1}) \quad (40)$$

subject to

$$\Sigma_{k+1} \geq \bar{F}_k \Sigma_k \bar{F}_k^T + \bar{G}_k S_k \bar{G}_k^T \quad (41)$$

or, equivalently

$$\begin{pmatrix} -\Sigma_{k+1} & \bar{F}_k \Sigma_k & \bar{G}_k S_k^{\frac{1}{2}} \\ \Sigma_k \bar{F}_k^T & -\Sigma_k & 0 \\ S_k^{\frac{T}{2}} \bar{G}_k^T & 0 & -I \end{pmatrix} \leq 0. \quad (42)$$

In order to incorporate the polytopic uncertainties in the F_k, as defined by the sets $\mathcal{K}_k$ in (11), we solve the aforementioned optimization problem with F_k taking values at the m vertices of the convex polytope $\mathcal{K}_k$, i.e., from the set $\{F_{1,k}, F_{2,k}, \ldots\ldots, F_{m,k}\}$. Therefore, the desired time-varying robust filter is given by (37) and (38), where W_k is the positive–definite solution of (40)–(42) with F_k taking values on the vertices of the convex polytope $\mathcal{K}_k$, and initializing $\Sigma_0 = \text{diag}\{P_o, \epsilon I\}$ for some positive definite P_o. Note that there always exists a solution to (40)–(42). The resulting filter is listed in Table II.

TABLE II
REGULARIZED ROBUST FILTER FOR THE MODEL (9)–(11)

Assumed uncertain model. $x_{k+1} = F_k x_k + G_k u_k$; $y_k = (H_k + \Delta H_k)x_k + v_k$ where $F_k \in \mathcal{K}_k$ as in (11).
Initial conditions: $\hat{x}_0 = 0$, $\Sigma_0 = \text{diag}\{P_o, \epsilon I\}$, where $P_o > 0$ and $\epsilon > 0$.

Step 1. Select T_k (usually, $T_k = R_k$). If $M_k = 0$, then set $\widehat{\beta}_k = 0$. Otherwise, set $\widehat{\beta}_k = (1+\alpha)\beta_{l,k}$ where $\beta_{l,k} = \|M_k^T T_k^{-1} M_k\|$.

Step 2. Using Σ_k, compute $\{W_k, \Sigma_{k+1}\}$ by solving (40) subject to the inequality (42).

Step 3. Update $\hat{x}_{k|k-1}$ to $\hat{x}_{k+1|k}$ as

$$\hat{x}_{k+1|k} = F_{p,k}\hat{x}_{k|k-1} + K_{p,k} y_k$$

where $F_{p,k}$, $K_{p,k}$ and $F_{k,c}$ defined by (17), (33) and (38) .

TABLE III
REGULARIZED ROBUST FILTER FOR STOCHASTIC UNCERTAINTIES IN F_k

Assumed uncertain model. $x_{k+1} = (F_k + \Delta F_k)x_k + G_k u_k$; $y_k = (H_k + \Delta H_k)x_k + v_k$ where $\Delta F_k = N_k \bar{\Delta}_k J_k$, and $\Delta H_k = M_k \Delta_k E_k$.
Initial conditions: $\hat{x}_0 = 0$, $\Sigma_0 = \text{diag}\{P_o, \epsilon I\}$, where $P_o > 0$ and $\epsilon > 0$.

Step 1. Select T_k (usually, $T_k = R_k$). If $M_k = 0$, then set $\widehat{\beta}_k = 0$. Otherwise, set $\widehat{\beta}_k = (1+\alpha)\beta_{l,k}$ where $\beta_{l,k} = \|M_k^T T_k^{-1} M_k\|$.

Step 2. Using Σ_k, compute $\{W_k, \Sigma_{k+1}\}$ by solving (47) subject to the inequality (49).

Step 3. Update $\hat{x}_{k|k-1}$ to $\hat{x}_{k+1|k}$ as in Table 2.

C. *Stochastic Uncertainties in F_k and Bounded Uncertainties in H_k*

We now consider the case of stochastic uncertainties in F_k as in (12) as well as uncertainties in H_k as in (10). Here again, our objective is to design a robust linear estimator for the state variable x_k of the form

$$\hat{x}_{k|k} = F_{p,k}\hat{x}_{k|k-1} + K_{p,k} y_k \quad \hat{x}_{k+1|k} = F_{k,c}\hat{x}_{k|k} \tag{43}$$

for some matrices $F_{p,k}$ and $K_{p,k}$ to be determined, and where $F_{k,c}$ denotes the nominal state matrix from (12). Proceeding in the same manner as in the previous section from the robustness condition (31), we know that the expressions for $\{F_{p,k}, K_{p,k}\}$ can be parametrized linearly in terms of a parameter W_k. We shall choose W_k so as to minimize an upper bound on the covariance of η_k in the absence of uncertainties in H_k. Here, η_k satisfies

$$\eta_{k+1} = (\bar{F}_{k,c} + \bar{N}_k \bar{\Delta} \bar{J}_k)\eta_k + \bar{G}_k w_k \tag{44}$$

where

$$w_k = \begin{pmatrix} u_k \\ v_k \end{pmatrix} \quad \bar{F}_{k,c} = \begin{pmatrix} F_{k,c} & 0 \\ F_{k,c} - F_{p,k} - K_{p,k}H_k & F_{p,k} \end{pmatrix}$$
$$\bar{N}_k = \begin{pmatrix} N_k & 0 \\ N_k & 0 \end{pmatrix} \quad \bar{J}_k = \begin{pmatrix} J_k & 0 \\ 0 & 0 \end{pmatrix} \tag{45}$$

and the covariance matrix of η_k then satisfies

$$\Sigma_{k+1} = \mathcal{E}\left\{(\bar{F}_{k,c} + \bar{N}_k \bar{\Delta} \bar{J}_k)\Sigma_k (\bar{F}_{k,c} + \bar{N}_k \bar{\Delta} \bar{J}_k)^T\right\} + \bar{G}_k S_k \bar{G}_k^T. \tag{46}$$

Let $\hat{\alpha}_k$ be a scalar such that $\hat{\alpha}_k I - \bar{J}_k \Sigma_k \bar{J}_k^T > 0$. Expanding (46), we can see that the error covariance matrix is bounded by

$$\Sigma_{k+1} \geq \bar{F}_{k,c}\Sigma_k \bar{F}_{k,c}^T + \bar{G}_k S_k \bar{G}_k^T + \rho_{\bar{\Delta}} \hat{\alpha}_k \bar{N}_k \bar{N}_k^T.$$

Hence, we shall choose $W_k > 0$ by solving

$$\min_{W_k > 0} \text{Tr}(\Sigma_{k+1}) \tag{47}$$

subject to

$$\Sigma_{k+1} \geq \bar{F}_k \Sigma_k \bar{F}_k^T + \bar{G}_k S_k \bar{G}_k^T + \rho_{\bar{\Delta}} \hat{\alpha}_k \bar{N}_k \bar{N}_k^T \tag{48}$$

or, equivalently

$$\begin{pmatrix} -\Sigma_{k+1} + \rho_{\bar{\Delta}} \hat{\alpha}_k \bar{N}_k \bar{N}_k^T & \bar{F}_k \Sigma_k & \bar{G}_k S_k^{\frac{1}{2}} \\ \Sigma_k \bar{F}_k^T & -\Sigma_k & 0 \\ S_k^{\frac{T}{2}} \bar{G}_k^T & 0 & -I \end{pmatrix} \leq 0. \tag{49}$$

The resulting filter is listed in Table III. The only difference relative to the filter of Table II is the term $\rho_{\bar{\Delta}} \hat{\alpha}_k \bar{N}_k \bar{N}_k^T$. The filters of Tables I–III have complexity $\mathcal{O}(n^4)$ per iteration, where n is the state dimension. In the appendix, we describe a filter that helps reduce the computational complexity to $\mathcal{O}(n^3)$ per iteration.

V. SIMULATIONS

To illustrate the filters developed in Sections IV-A and B for polytopic uncertainties in F_k, we choose an implementation of order 2. The uncertain state matrices F_k are assumed to lie inside the convex polytope

$$F_k \in \left\{\begin{pmatrix} .68 & -.5 \\ 1 & .7 + .016\delta \end{pmatrix}\right\} \tag{50}$$

TABLE IV

MSE PERFORMANCE OF THE PROPOSED ROBUST FILTERS OF SECTIONS IV-A– IV-C AND APPENDIX IN COMPARISON TO OTHER ROBUST FILTERS FOR FOUR CASES: (a) STOCHASTIC UNCERTAINTIES IN F_k FOR ($\rho_{\bar{\Delta}} \leq 1$) AND WITH NO UNCERTAINTIES IN H_k; b) STOCHASTIC UNCERTAINTIES IN F_k FOR ($\rho_{\bar{\Delta}} \leq 0.0001$) AND WITH BOUNDED UNCERTAINTIES IN H_k; (c) POLYTOPIC UNCERTAINTIES IN F_k AND NO UNCERTAINTIES IN H_k; AND (d) POLYTOPIC UNCERTAINTIES IN F_k AND BOUNDED UNCERTAINTIES IN H_k

Filters	MSE (dB) - (a)	MSE (dB) - (b)	MSE (dB) - (c)	MSE (dB) - (d)
Proposed filter from Sec. 4.1	–	–	**27.79**	28.5
Proposed filter from Sec. 4.2	–	–	**27.79**	**28.1**
Proposed filter of Sec. 4.3	**22.5**	18.0	–	–
Proposed filter from appendix	23.1	18.3	–	–
Regularized robust filter of [9]	24.1	**17.9**	29.19	31.2
Deconvoluted filter [13]	25.91	19.1	–	–
Guaranteed-cost filter [2]	27.4	23.1	30.91	32.2
Kalman filter with nominal model	35.4	23.5	36.98	37.9
Set-valued filter [3]	45.5	27.5	38.67	39.1

with $|\delta| \leq 1$. The vertices of the polytope are

$$F_1 = \begin{pmatrix} .68 & -.5 \\ 1 & .716 \end{pmatrix} \quad F_2 = \begin{pmatrix} .68 & -.5 \\ 1 & .684 \end{pmatrix}.$$

We choose $H_k = [10\ 1]$ and $G_k = [6\ 3]^T$. Table IV shows the steady-state mean square state-error (MSE) values i.e., $\mathrm{Tr}[\mathcal{E}(x_k - \hat{x}_{k|k-1})(x_k - \hat{x}_{k|k-1})^T]$, obtained by averaging over 50 experiments for the proposed filters in comparison to other filters. It is seen that the filters of Sections IV-A and IV-B result in smaller MSE, albeit at increased computational cost $\mathcal{O}(n^4)$ versus $\mathcal{O}(n^3)$ operations per iteration. The filter of [9] is also seen to result in similar MSE values at the reduced computational cost of $O(n^3)$ operations per iteration. To illustrate the filter developed for stochastic uncertainties in Section IV-C, we choose an implementation of order 2 with $E_k = [3.6\ 0.6]$, $M_k = 1$ for all k. The uncertain state matrices F_k are assumed to be

$$F_k = \begin{pmatrix} .68 & -.5 \\ 1 & .7 + 0.016\bar{\Delta} \end{pmatrix} \tag{51}$$

for the choice of $\rho_{\bar{\Delta}} = 1$, $N_k = 0.4I$ and $J_k = 0.04 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Table IV shows the resulting MSE values.

VI. CONCLUSION

In this note, we developed regularized robust filters for state-space estimation. The design procedure is through the solution of a regularized weighted recursive least squares problem and it enforces a minimum state error variance property.

APPENDIX
ANOTHER ROBUST FILTER

Consider again (46) in the *absence* of uncertainties in H_k. We now show how to generate a sequence of matrices $\tilde{\Sigma}_k$ and $\hat{\Sigma}_k$ such that $\Sigma_k \leq \tilde{\Sigma}_k \leq \hat{\Sigma}_k$. We will seek matrices $\tilde{\Sigma}_k$ and $\hat{\Sigma}_k$ of the special form

$$\tilde{\Sigma}_{k+1} = \begin{pmatrix} \tilde{Y}_{k+1} & \tilde{X}_{k+1} \\ \tilde{X}_{k+1}^T & \tilde{Y}_{k+1} - \tilde{Z}_{k+1} \end{pmatrix} \quad \hat{\Sigma}_k = \begin{pmatrix} \hat{Y}_k & \hat{Y}_k - \hat{Z}_k \\ \hat{Y}_k - \hat{Z}_k & \hat{Y}_k - \hat{Z}_k \end{pmatrix}, \tag{52}$$

This construction will enable us to avoid the solution of the optimization problem (47) at each iteration thus reducing the computational complexity of the algorithm from $\mathcal{O}(n^4)$ to $\mathcal{O}(n^3)$ per step. At every iteration, we will find a suboptimal W_k that minimizes the bound $\hat{\Sigma}_k$ on Σ_k. At time instant $k+1$, assuming we have $\Sigma_k \leq \tilde{\Sigma}_k \leq \hat{\Sigma}_k$, then the state error covariance matrix is bounded by the (2,2) block element of the matrix $\tilde{\Sigma}_{k+1}$ defined by

$$\tilde{\Sigma}_{k+1} = \bar{F}_{k,c}\hat{\Sigma}_k\bar{F}_{k,c}^T + \bar{G}_k S_k \bar{G}_k^T + \rho_{\bar{\Delta}}\hat{\alpha}_k \bar{N}_k \bar{N}_k^T \ \geq \ \Sigma_{k+1} \tag{53}$$

where $\hat{\alpha}_k$ is also such that $\hat{\alpha}_k I - \bar{J}_k \hat{\Sigma}_k \bar{J}_k^T > 0$. Using (52) and (53), we get the relations

$$\begin{aligned}
\tilde{Y}_{k+1} &= F_{k,c}\hat{Y}_k F_{k,c}^T + \rho_{\bar{\Delta}}\hat{\alpha}_k N_k N_k^T + G_k Q_k G_k^T \\
\tilde{Z}_{k+1} &= F_{k,c}\hat{Z}_k F_{k,c}^T - F_{k,c}W_k H_k^T \hat{R}_k^{-1} R_k \hat{R}_k^{-1} H_k W_k F_{k,c}^T \\
&\quad + F_{k,c}(\hat{Y}_k - \hat{Z}_k)H_k^T \hat{R}_k^{-1} H_k W_k F_{k,c}^T \\
&\quad + F_{k,c}W_k H_k^T \hat{R}_k^{-1} H_k(\hat{Y}_k - \hat{Z}_k)F_{k,c}^T \\
&\quad - F_{k,c}W_k H_k^T \hat{R}_k^{-1} H_k(\hat{Y}_k - \hat{Z}_k)H_k^T \hat{R}_k^{-1} H_k W_k F_{k,c}^T \\
\tilde{X}_{k+1} &= F_{k,c}(\hat{Y}_k - \hat{Z}_k)F_{p,k}^T + \rho_{\bar{\Delta}}\hat{\alpha}_k N_k N_k^T + G_k Q_k G_k^T.
\end{aligned}$$

Also, the (2,2) block element of $\tilde{\Sigma}_{k+1}$ is given by $\tilde{\Sigma}_{k+1}^{2,2} = \tilde{Y}_{k+1} - \tilde{Z}_{k+1} = C_{1,k}$ where

$$\begin{aligned}
C_{1,k} &= F_{k,c}(\hat{Y}_k - \hat{Z}_k)F_{k,c}^T + \rho_{\bar{\Delta}}\hat{\alpha}_k N_k N_k^T + G_k Q_k G_k^T \\
&\quad + F_{k,c}W_k H_k^T \hat{R}_k^{-1} R_k \hat{R}_k^{-1} H_k W_k F_{k,c}^T \\
&\quad - F_{k,c}(\hat{Y}_k - \hat{Z}_k)H_k^T \hat{R}_k^{-1} H_k W_k F_{k,c}^T \\
&\quad - F_{k,c}W_k H_k^T \hat{R}_k^{-1} H_k(\hat{Y}_k - \hat{Z}_k)F_{k,c}^T \\
&\quad + F_{k,c}W_k H_k^T \hat{R}_k^{-1} H_k(\hat{Y}_k - \hat{Z}_k)H_k^T \hat{R}_k^{-1} H_k W_k F_{k,c}^T.
\end{aligned}$$

We will choose the weighing matrix T_k such that $\hat{R}_k = R_k$. We now find a lower bound for $C_{1,k}$. After some algebra, we can show that for

$$W_{k,\mathrm{opt}} = (\hat{Y}_k - \hat{Z}_k) - (\hat{Y}_k - \hat{Z}_k)H_k^T \bar{R}_{e,k}^{-1} H_k(\hat{Y}_k - \hat{Z}_k)$$

and $\bar{R}_{e,k} = R_k + H_k(\hat{Y}_k - \hat{Z}_k)H_k^T$, we have $(\partial \mathrm{Tr}(C_{1,k})/W_k) = 0$. That is, $W_{k,\mathrm{opt}}$ minimizes $\mathrm{Tr}(C_{1,k})$. It can be seen through the matrix inversion lemma that $W_{k,\mathrm{opt}}$ is positive definite. Now, we will derive an upper bound for $\hat{\Sigma}_{k+1}$ in the form [which is compatible with the form we started with in (52)]

$$\hat{\Sigma}_{k+1} = \begin{pmatrix} \hat{Y}_{k+1} & \hat{Y}_{k+1} - \hat{Z}_{k+1} \\ \hat{Y}_{k+1} - \hat{Z}_{k+1} & \hat{Y}_{k+1} - \hat{Z}_{k+1} \end{pmatrix} \tag{54}$$

for some matrices $\hat{Y}_{k+1}$ and $\hat{Z}_{k+1}$. Choose ψ_k as the maximum singular value of $I + B$ where

$$\begin{aligned}
B &= F_{k,c}W_k H_k^T \hat{R}_k^{-1} H_k(\hat{Y}_k - \hat{Z}_k)H_k^T \hat{R}_k^{-1} H_k W_k F_{k,c}^T \\
&\quad + F_{k,c}W_k H_k^T \hat{R}_k^{-1} H_k W_k F_{k,c}^T \\
&\quad - F_{k,c}W_k H_k^T \hat{R}_k^{-1} H_k(\hat{Y}_k - \hat{Z}_k)F_{k,c}^T.
\end{aligned}$$

Now, with $\hat{Y}_{k+1} = \psi_k^2 I + \tilde{Y}_{k+1}$ and $\hat{Z}_{k+1} = \tilde{Z}_{k+1} + \psi_k^2 I - I$, $\hat{\Sigma}_{k+1}$ is an upper bound of $\tilde{\Sigma}_{k+1}$. This is because

$$\hat{\Sigma}_{k+1} - \tilde{\Sigma}_{k+1} = \begin{pmatrix} \psi_k^2 I & I+B \\ (I+B)^T & I \end{pmatrix} > 0. \quad (55)$$

Hence, the filter is given by $\hat{x}_{k+1|k} = F_{p,k}\hat{x}_{k|k-1} + K_{p,k}y_k$ where $F_{p,k}$ and $K_{p,k}$ are defined in terms of W_k as

$$F_{p,k} = F_{k,c}\left(I - \hat{\beta} W_k E_k^T E_k - W_k H_k^T \hat{R}_k^{-1} H_k\right)$$
$$K_{p,k} = F_{k,c} W_k H_k^T \hat{R}_k^{-1} \quad (56)$$

and W_k is determined in terms of $\hat{Y}_k$ and $\hat{Z}_k$ at every k as explained before.

REFERENCES

[1] T. Kailath, A. H. Sayed, and B. Hassibi, *Linear Estimation*. Upper Saddle River, NJ: Prentice-Hall, 2000.

[2] I. R. Petersen and A. V. Savkin, *Robust Kalman Filtering for Signals and Systems with Large Uncertainties*. Boston, MA: Birkhauser, 1999.

[3] A. V. Savkin and I. R. Petersen, "Robust state estimation and model validation for discrete time uncertain systems with a deterministic description of noise and uncertainties," *Automatica*, vol. 34, no. 2, pp. 271–274, 1998.

[4] L. Xie, Y. C. Soh, and C. E. de Souza, "Robust Kalman filtering for uncertain discrete-time systems," *IEEE Trans. Automat. Contr.*, vol. 39, pp. 1310–1314, June 1994.

[5] M. Fu, C. E. de Souza, and Z. Luo, "Finite horizon robust kalman filter design," in *Proc. IEEE Conf. Decision Control*, Phoenix, AZ, 1999, pp. 4555–4560.

[6] Y. Theodor and U. Shaked, "Robust discrete time minimum variance filtering," *IEEE Trans. Signal Processing*, vol. 44, pp. 181–189, Feb. 1996.

[7] F. Yang, Z. Wang, and Y. S. Hung, "Robust Kalman filtering for discrete time-varying uncertain systems with multiplicative noises," *IEEE Trans. Automat. Contr.*, vol. 47, pp. 1179–1184, July 2002.

[8] P. P. Khargonekar and K. M. Nagpal, "Filtering and smoothing in an $\mathcal{H}_\infty$ setting," *IEEE Trans. Automat. Contr.*, vol. 36, pp. 151–166, Feb. 1991.

[9] A. H. Sayed, "A framework for state space estimation with uncertain models," *IEEE Trans. Automat. Contr.*, vol. 46, pp. 998–1013, July 2001.

[10] A. H. Sayed, V. H. Nascimento, and F. A. M. Cipparrone, "A regularized robust design criterion for uncertain data," *SIAM J. Matrix Anal. Appl.*, vol. 23, no. 4, pp. 1120–1142, 2002.

[11] A. H. Sayed and H. Chen, "A uniqueness result concerning a robust regularized least-squares solution," *Syst. Control Lett.*, vol. 46, no. 5, pp. 361–369, Aug. 2002.

[12] A. E. Bryson and Y.-C. Ho, *Applied Optimal Control: Optimization, Estimation, and Control*. New York: Taylor Francis, 1975.

[13] Y. L. Chen and B. S. Chen, "Minmax robust deconvoluted filter under stochastic parametric and noise uncertainties," *IEEE Trans. Signal Processing*, vol. 42, pp. 32–45, Jan. 1994.

[14] A. Garulli, A. Vicino, and G. Zappa, "Conditional central algorithms for worst case set-membership identification and filtering," *IEEE Trans. Automat. Contr.*, vol. 45, pp. 14–23, Jan. 2000.

[15] J. C. Geromel, "Optimal linear filtering under parameter uncertainty," *IEEE Trans. Signal Processing*, vol. 47, Jan. 1999.

Pricing and Congestion Management in a Network With Heterogeneous Users

Shaler Stidham, Jr.

***Abstract*—This note presents an economic model for a communication network with utility-maximizing elastic users who adapt to congestion by adjusting their flows. Users are heterogeneous with respect to both the utility they attach to different levels of flow and their sensitivity to delay. Following Kelly *et al.* (1998), we introduce dynamic rate-control algorithms, based on the users' utility functions and delay sensitivities, as well as tolls charged by the system, and examine the behavior of these algorithms. We show that allowing heterogeneity with respect to delay sensitivity introduces a *fundamental* nonconvexity into the congestion-cost functions. As a result, there are often multiple stationary points of the aggregate net utility function. Hence, marginal-cost pricing—equating users' marginal utilities to their marginal costs—may identify a local maximum or even a saddle point, rather than a global maximum. Moreover, the dynamic rate-control algorithm may converge to a local rather than global maximum, depending on the starting point. We present examples with different user utility functions, including some in which the only interior stationary point is a saddlepoint which is dominated by *all* the single-user optimal allocations. We also consider variants of the dynamic algorithm and their performance in a network with heterogeneous users. Our results suggest that applying a rate-control algorithm such as TCP (Transmission Control Protocol), even when augmented by some form of implicit or explicit pricing, may have unexpected and perhaps undesirable effects on the allocation of flows among heterogeneous delay-sensitive users.**

***Index Terms*—Communication network, congestion pricing, dynamic rate-control algorithm, elastic users, heterogeneous users.**

I. INTRODUCTION

We consider a variant of an economic model proposed by Kelly [2] and elaborated by Kelly, Maulloo, and Tan [1] (hereafter referred to as the KMT model) for a communication network with utility-maximizing elastic users who adapt to congestion by adjusting their flows. A distinctive feature of our model is that users are explicitly sensitive to delays as well as flows. Moreover, they not only differ in the utility they attach to different levels of flow, but are also heterogeneous with respect to the cost of delay. Following Kelly *et al.* [1], we introduce dynamic rate-control algorithms, based on the users' utility functions and delay sensitivities as well as tolls charged by the system, and examine the behavior of these algorithms. Algorithms of this type have been introduced as an aid to understanding the behavior of rate-control mechanisms such as TCP (Transmission Control Protocol) and its variants, which have been proposed for the Internet (see [3]–[5]).

Heterogeneous delay sensitivities may arise, for example, in networks (such as the current and future Internet) that handle diverse types of traffic, ranging from file transfers (with a low sensitivity to delay) to real-time traffic such as streaming audio and video, which can tolerate only minor delays. Many authors have suggested that such diversity of traffic will require differentiated services, in which some types of traffic are given priority (see, for example, [6] and the references therein). Others [7] have argued that a "self-managed Internet" may be able provide a diverse set of services with low levels of loss or delay,

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The author is with the Department of Statistics and Operations Research, University of North Carolina, Chapel Hill, NC 27599-3180 USA.

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