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A Recursive Schur-Based Solution of the Four-Block Problem

Tiberiu Constantinescu, Ali H. Sayed, and Thomas Kailath

Abstract— **We describe a new solution to the four-block problem using the method of generalized Schur analysis. We first reduce the general problem to a simpler one by invoking a coprime factorization with a block-diagonal inner matrix. Then, using convenient spectral factorizations, we are able to parameterize the unknown entry in terms of a Schur-type matrix function, which is shown to satisfy a finite number of interpolation conditions of the Hermite–Féjer type. All possible interpolating functions are then determined via a simple recursive procedure that constructs a transmission-line (or lattice) cascade of elementary J-lossless sections. This also leads to a parameterization of all solutions of the four-block problem in terms of a linear fractional transformation.**

I. Introduction

A central problem in H^∞-optimal control is the design of stabilizing controllers that minimize or at least impose an upper bound on the H^∞-norm of the closed-loop transfer function. This problem has been widely studied in the literature and we may refer to the monograph of Francis [1] and the notes of Doyle [2] for more details and discussion. The existing approaches cover a wide range of settings and contexts. Doyle and Francis [2], [3] reduced the equivalent so-called model matching problem to a one-block (or Nehari) problem, which was then solved using the theory studied by Ball and Helton [4]. Foias and Tannenbaum [5] approached the four-block distance problem within the framework of skew Toeplitz operators and studied the associated spectral properties. Ball and Cohen [6] gave a parameterization of all suboptimal solutions based on J-spectral factorization theory, while Kimura and Kawatani [7] employed the notion of conjugation. Doyle *et al.* [8] provided state-space formulas for the stabilizing controllers by employing a separation argument and replacing the four-block problem by a pair of two-block problems. Most recently, Glover *et al.* [9] (see also Limebeer *et al.* [11], [10]) described a state-space procedure that yields an all-pass dilation of the original problem; part of this all-pass matrix was shown to generate all solutions.

We present a new solution that approaches the four-block problem within the framework of generalized Schur analysis and leads to a transmission-line (or lattice) structure that parameterizes all possible unknown entries. The derivation can be summarized as follows: we use a special factorization, with a block-diagonal inner factor, that reduces the original four-block problem with L^∞ functions to an equivalent problem with H^∞ functions. We then invoke convenient spectral factorizations and an inner dilation to express all possible choices of the unknown entry in terms of a Schur matrix function, which is shown to be characterized by a finite number

Manuscript received March 12, 1992; revised March 9, 1993. This research was supported in part by the Air Force Office of Scientific Research, Air Force Systems Command under Contract AFOSR91-0060, by the Army Research Office under Contract DAAL03-89-K-O109, and by a fellowship (for A. H. Sayed) from Fundação de Amparo à Pesquisa do Estado de São Paulo while on leave from the Department of Electrical Engineering, Escola Politećnica da Universidade de São Paulo, Brzail.

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IEEE Log Number 9401670.

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of interpolation conditions of the Hermite–Fejér type. The solution of the interpolation problem is then determined through a recursive (array) procedure recently proposed in[12]–[14], which exploits the so-called displacement structure to efficiently factor a structured matrix (implicitly defined by the interpolation data) via a generalized Schur algorithm. The factorization defines a transmission-line cascade of elementary J-lossless sections, which readily explains how the interpolation conditions are satisfied.

We should remark that the study of interpolation problems has had a renewed interest in control theory because of their role in H^∞-control, as first noted by Zames [15], [16] and then followed up and extended by several authors, especially Helton [17], [18], Kimura [19], and Limebeer *et al.* [11], [20].

The following notational conventions will be useful to remember. $RL^\infty_{p\times q}$ will denote the space of $p \times q$ matrices whose entries are rational functions without poles on the unit circle and $RH^\infty_{p\times q}$ will denote the space of $p \times q$ rational matrix-valued functions that are analytic and bounded inside the open unit disc. A matrix-valued function $K(z) \in RH^\infty_{p\times q}$ that is strictly bounded by unity ($\|K\|_\infty < 1$) will be referred to as a function of Schur-type. We shall write $A_*(z)$ to denote the para-Hermitian conjugate, $A_*(z) = [A((1/z^*))]^*$, where * stands for Hermitian conjugation (complex conjugation for scalars). We shall also use the notation $\mathcal{H}^k_A(z)$ to refer to the following block-Toeplitz upper-triangular matrix

$$\mathcal{H}^k_A(z)=\begin{bmatrix} A(z) & \frac{1}{1!}A^{(1)}(z) & \frac{1}{2!}A^{(2)}(z) & \cdots & \frac{1}{(k-1)!}A^{(k-1)}(z) \\ & A(z) & \frac{1}{1!}A^{(1)}(z) & \cdots & \frac{1}{(k-2)!}A^{(k-2)}(z) \\ & & & & \vdots \\ & \mathbf{0} & \ddots & & \frac{1}{1!}A^{(1)}(z) \\ & & & & A(z) \end{bmatrix}$$

where $A(z)$ is a rational matrix function analytic at z, $k \geq 1$ is a positive integer, and $A^{(i)}(z)$ denotes the ith derivative of $A(z)$ at z. Throughout this note we use, for convenience, and unlike the usual engineering convention, positive powers of z to denote time delays.

II. A General Four-Block Problem

We consider the following general four-block problem.

Problem 2.1: Given $p_i \times q_j$, $(i, j = 1, 2)$ matrix functions $L_{ij}(z) \in RL^\infty_{p_i\times q_j}$, describe all $Q(z) \in RH^\infty_{p_2\times q_2}$ such that

$$\left\| \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} - Q \end{bmatrix} \right\|_\infty < 1. \tag{1}$$

■

Notice that we have stated the problem in function domain and in rather general terms. The entries $\{L_{ij}(z)\}$ are not assumed to have any type of relation to each other, in contrast to standard statements of the problem in the literature (see, e.g., [9, (1.5)]).

The first step in the solution is to reduce the original problem with RL^∞ entries to an equivalent problem with RH^∞ entries. For this purpose, we invoke a special factorization of the form

$$\begin{bmatrix} L_{11}(z) & L_{12}(z) \\ L_{21}(z) & L_{22}(z) \end{bmatrix} = \begin{bmatrix} A_{11}(z) & \mathbf{0} \\ \mathbf{0} & A_{22}(z) \end{bmatrix} \begin{bmatrix} B_{11}(z) & B_{12}(z) \\ B_{21}(z) & B_{22}(z) \end{bmatrix} \tag{2}$$

where $A_{11}(z)$ and $A_{22}(z)$ are, respectively, $p_1 \times p_1$ and $p_2 \times p_2$ all-pass rational matrix functions with poles inside the open unit disc, and $B_{ij}(z) \in RH^\infty_{p_i\times q_j}$, $i, j = 1, 2$. It is clear that (2) consists of two (decoupled) factorizations. Moreover, the block-diagonal inner matrix, $A(z) = A_{11}(z) \oplus A_{22}(z)$, can be chosen with an even simpler structure. We can, for instance, choose $A(z)$ to be diagonal or of the form $A(z) = a(z)I_{p_1+p_2}$, where $a(z)$ is a Blaschke factor and I_p denotes the $p \times p$ identity matrix. For the sake of generality, we shall assume throughout that $A(z)$ is block diagonal. Using (2), and the fact that (the para-Hermitian conjugate) $A_*(z) = A^{-1}(z)$ for the points z where the functions are defined, we conclude that (1) is equivalent to

$$\left\| \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} - A_{*22}Q \end{bmatrix} \right\|_\infty < 1. \tag{3}$$

Assume there exists a solution $Q(z)$ to the four-block problem. It then follows that we already have a necessary condition for its solvability, viz.,

$$[B_{11}(z) \quad B_{12}(z)] \quad \text{and} \quad \begin{bmatrix} B_{11}(z) \\ B_{21}(z) \end{bmatrix} \quad \text{must be Schur-type matrix functions.} \tag{4}$$

Notice that, due to the block-diagonal structure of the inner matrix $A(z)$, the unknown matrix $Q(z)$ appears only in the entry $B_{22}(z) - A_{*22}(z)Q(z)$. Our next step is to parameterize $Q(z)$ in terms of a Schur-type matrix function $K(z)$. We shall derive the following result in the next section.

Theorem 2.1 (All Possible Solutions Q): All possible choices of $Q(z)$ can be expressed as

$$\varphi(z)A_{*22}(z)Q(z) = \varphi(z)B_{22}(z) - K_{21}(z)P(z)K_{12}(z) - L_{K_{21}}(z)K(z)R_{K_{12}}(z) \tag{5}$$

where $K(z)$ is an arbitrary Schur function to be determined and all other quantities are given in terms of the entries $\{B_{ij}(z)\}$ and the spectral factors of $B_{11}(z)$. ■

The above result thus allows us to reduce the problem of finding $Q(z)$ to that of finding a suitable $K(z)$, since all the other quantities in (5) are known, as described ahead. We shall later show that $K(z)$ must satisfy interpolation conditions of the Hermite–Fejér type.

We should remark here that several earlier solutions of the four-block problem (or the closely related model matching problem) have been based on reducing the original problem to the so-called Nehari problem (see, e.g., [1], [21]). Our approach does not assume any *a priori* knowledge of the Nehari Problem or its solution. We instead use insights from the study of Schur-type matrix functions to parameterize all possible solutions $Q(z)$ as given in expression (5) and as detailed in the next section.

III. Schur Function Parameterization of the Solution

We first recall the notion of left and right spectral factors (see, e.g., [22], [23]), as well as the notion of an inner dilation [24]. Consider a $k \times l$ Schur-type rational matrix function $N(z)$. The right spectral factor of $N(z)$ is the $l \times l$ outer matrix function $R_N(z)$, uniquely determined up to a left unitary constant factor, such that $I - N^*(\zeta)N(\zeta) = R^*_N(\zeta)R_N(\zeta)$ for all $|\zeta| = 1$, where by outer function we mean $R_N(z)$ and $R^{-1}_N(z)$ are both in $RH^\infty_{l\times l}$. The left spectral factor of $N(z)$ is the $k \times k$ *-outer matrix function $L_N(z)$, uniquely determined up to a right unitary constant factor, such that $I - N(\zeta)N^*(\zeta) = L_N(\zeta)L^*_N(\zeta)$ for all $|\zeta| = 1$, where by *-outer function we mean $\tilde{L}_N(z)$ is outer, where $\tilde{L}_N(z) = [L_N(z^*)]^*$.

Moreover, there exists [24] a unitary inner matrix function, $W_N(z) \in RH^\infty_{(k+l)\times(k+l)}$ (called the inner dilation of $N(z)$), viz., $W_N(\zeta)W^*_N(\zeta) = W^*_N(\zeta)W_N(\zeta) = I_{k+l}$ for all $|\zeta| = 1$, which has the form (up to a unitary factor)

$$W_N(z) = \begin{bmatrix} N(z) & L_N(z) \\ \varphi_N(z)R_N(z) & P_N(z) \end{bmatrix}$$

where $P_N(z) = \varphi_N(z)\overline{P}_N(z)$ with

$$\overline{P}_N(\zeta) = -R_N(\zeta)N^*(\zeta)[I - N(\zeta)N^*(\zeta)]^{-1}L_N(\zeta), \quad \forall\, |\zeta| = 1$$

and $\varphi_N(z)$ is a rational inner function (Blaschke) chosen such that $P_N(z) \in RH^\infty_{l \times k}$ (i.e., $\varphi_N(z)$ guarantees the analyticity of $P_N(z)$). We remark that the spectral factors $L_N(z)$ and $R_N(z)$ can be computed, among other ways, by applying the celebrated Schur algorithm to $N(z)$ (see the Appendix for a brief discussion).

We now use the above definitions to further refine the structure of the entries in (3) and to express $Q(z)$ in terms of a rational Schur matrix function. We first use (4) to express $B_{12}(z)$ and $B_{21}(z)$ in terms of the spectral factors of $B_{11}(z)$.

Lemma 3.1: There exist unique Schur-type rational matrix functions $K_{12}(z)$ and $K_{21}(z)$ such that $B_{12}(z) = L_{B_{11}}(z)K_{12}(z)$ and $B_{21}(z) = K_{21}(z)R_{B_{11}}(z)$.

Proof: It follows from (4) that $I - B_{11}(\zeta)B_{11}^*(\zeta) - B_{12}(\zeta)B_{12}^*(\zeta) > 0$ for all $|\zeta| = 1$, or equivalently, $L_{B_{11}}(\zeta)L_{B_{11}}^*(\zeta) > B_{12}(\zeta)B_{12}^*(\zeta)$. But $L_{B_{11}}(z)$ is *-outer. This implies [22] that there exists a unique Schur-type rational matrix function $K_{12}(z)$ such that $B_{12}(z) = L_{B_{11}}(z)K_{12}(z)$ for all $|z| < 1$. A similar argument holds for the existence of $K_{21}(z)$.

Let $W_{B_{11}}(z)$ be the inner dilation of $B_{11}(z)$. For notational simplicity, we write $\varphi(z)$ and $P(z)$ instead of $\varphi_{B_{11}}(z)$ and $P_{B_{11}}(z)$

$$W_{B_{11}}(z) = \begin{bmatrix} B_{11}(z) & L_{B_{11}}(z) \\ \varphi(z)R_{B_{11}}(z) & P(z) \end{bmatrix}.$$

Using the structures of $B_{21}(z)$ and $B_{12}(z)$, and the above inner dilation of $B_{11}(z)$, we can easily compute the left spectral factor of the rational matrix function

$$\begin{bmatrix} B_{11}(z) \\ \varphi(z)B_{21}(z) \end{bmatrix}. \tag{6}$$

Lemma 3.2: The left spectral factor of the matrix function (6) is given by

$$\boldsymbol{\Psi}(z) = \begin{bmatrix} L_{B_{11}}(z) & \mathbf{0} \\ K_{21}(z)P(z) & L_{K_{21}}(z) \end{bmatrix} \tag{7}$$

where $L_{K_{21}}(z)$ is the left spectral factor of the Schur-type rational matrix function $K_{21}(z)$.

Proof: The proof is a direct verification based on the innerness of $W_{B_{11}}(z)$. ■

In other words, we have shown that the left spectral factor of the 2×1 block matrix (6) can be specified in terms of the spectral factors of the smaller entry $B_{11}(z)$ and the Schur function $K_{21}(z)$ (compare with [21, pp. 215–219]).

We now use the results of Lemmas 3.1 and 3.2 and express all the entries in (3) in terms of the spectral factors of $B_{11}(z)$ and a Schur-type matrix function $K(z)$. For this purpose, observe that due to the innerness of $\varphi(z)$ we have that the matrix function

$$\begin{bmatrix} B_{11}(z) & B_{12}(z) \\ \varphi(z)B_{21}(z) & \varphi(z)B_{22}(z) - \varphi(z)A_{*22}(z)Q(z) \end{bmatrix} \tag{8}$$

is also in Schur class. Using an argument similar to the proof of Lemma 3.1, we readily verify that the second block-column of (8) can be expressed in terms of the left spectral factor of the first block-column of (8). That is

$$\begin{bmatrix} B_{12}(z) \\ \varphi(z)B_{22}(z) - \varphi(z)A_{*22}(z)Q(z) \end{bmatrix} = \boldsymbol{\Psi}(z)\overline{K}(z)$$

for some Schur matrix function $\overline{K}(z)$. It then follows from (7) that the right-hand side of the above expression has to be of the form

$$\begin{bmatrix} L_{B_{11}}(z)\overline{K}_1(z) \\ K_{21}(z)P(z)\overline{K}_1(z) + L_{K_{21}}(z)\overline{K}_2(z) \end{bmatrix}$$

where we partitioned $\overline{K}(z)$ as $\overline{K}(z) = [\overline{K}_1^T(z) \quad \overline{K}_2^T(z)]^T$. Invoking the uniqueness of $K_{12}(z)$ (as stated in Lemma 3.1) we conclude that we must have $\overline{K}_1(z) = K_{12}(z)$. But $\overline{K}(z)$ is a Schur function and hence (as in the proof of Lemma 3.1), $\overline{K}_2(z)$ can be expressed in terms of the right spectral factor of $\overline{K}_1(z)$ $(= K_{12}(z))$, viz., $\overline{K}_2(z) = K(z)R_{K_{12}}(z)$, for some rational Schur-type matrix function $K(z)$. Therefore, we conclude that the entries of (8) can be written in the form as shown in (9) at the bottom of the page. Comparing (8) and (9) we readily verify that (5) is indeed satisfied.

IV. The Hermite-Fejér Interpolation Problem

We first review the definition of transmission zeros of a $k \times k$ matrix function $T(z)$ with $\det T(z)$ not vanishing identically (see [25, ch. 1] and [26, pp. 446–451]). Let $\{z_i,\ 0 \le i \le m-1\}$ denote the zeros of $\det T(z)$ [$T(z)$ is assumed analytic at the $z_i's$]. A left-null chain of order $\overline{r}$ at z_i is a set of $\overline{r}$ row vectors $\{x_i,\ 1 \le i \le \overline{r}\}$ (of dimension $1 \times k$ each), such that

$$[x_1 \quad x_2 \quad \cdots \quad x_{\overline{r}}]\mathcal{H}^{\overline{r}}{}_T(z_i) = \mathbf{0}.$$

Each z_i, however, can have more than one left-null chain and those can be of different orders.

Definition 4.1 (Canonical Sets [25]): A canonical set of left-null chains at z_i is an ordered set of left-null chains $\{\boldsymbol{y}_1^{(i)}, \boldsymbol{y}_2^{(i)}, \cdots, \boldsymbol{y}_{t_i}^{(i)}\}$, of orders $\{r_1^{(i)}, r_2^{(i)}, \cdots, r_{t_i}^{(i)}\}$, where each $\boldsymbol{y}_j^{(i)}$ is composed of $r_j^{(i)}$ row vectors, viz., $[x_{j1}^{(i)} \cdots x_{j,r_j^{(i)}}^{(i)}]$, such that $\{x_{11}^{(i)}, x_{21}^{(i)}, \cdots, x_{t_i,1}^{(i)}\}$ are linearly independent and form a basis for the nullspace of $T(z_i)$, and $r_1^{(i)} \ge r_2^{(i)} \ge \cdots \ge r_{t_i}^{(i)}$. ■

Hence, if z_i is a zero with t_i left-null chains, then it satisfies t_i conditions of the form

$$\boldsymbol{y}_j^{(i)}\mathcal{H}_T^{r_j^{(i)}}(z_i) = \mathbf{0} \quad \text{for } j = 1, \cdots, t_i \tag{10}$$

and it is said to be a transmission zero of order $m_i = \sum_{j=1}^{t_i} r_j^{(i)}$. We shall also refer to the $\{r_j^{(i)}\}$ as the partial multiplicities of z_i.

We now go back to (5) and consider the zero structure of the inner matrix function $\varphi(z)A_{*22}(z)$. Assume $\det\varphi(z)A_{*22}(z)$ has m zeros $\{z_i,\ |z_i| < 1\}$ and let $\{r_j^{(i)}, \boldsymbol{y}_j^{(i)},\ 1 \le j \le t_i\}$ designate the corresponding partial multiplicities and canonical set of left-null chains. It follows from (5) that

$$\begin{aligned} \boldsymbol{y}_j^{(i)}\mathcal{H}_{\varphi B_{22}}^{r_j^{(i)}}(z_i) = \boldsymbol{y}_j^{(i)}[&\mathcal{H}_{K_{21}PK_{12}}^{r_j^{(i)}}(z_i) \\ &+ (\mathcal{H}_{L_{K_{21}}}^{r_j^{(i)}}\mathcal{H}_K^{r_j^{(i)}}\mathcal{H}_{R_{K_{12}}}^{r_j^{(i)}})(z_i)]. \end{aligned} \tag{11}$$

Recall that $R_{K_{12}}(z)$ is outer and hence, $\mathcal{H}_{R_{K_{12}}}^{r_j^{(i)}}(z_i)$ is invertible. For notational convenience, we introduce the row vectors $\boldsymbol{a}_j^{(i)}$ and $\boldsymbol{b}_j^{(i)}$ defined by $\boldsymbol{a}_j^{(i)} = \boldsymbol{y}_j^{(i)}\mathcal{H}_{L_{K_{21}}}^{r_j^{(i)}}(z_i)$ and

$$\boldsymbol{b}_j^{(i)} = \boldsymbol{y}_j^{(i)}[\mathcal{H}_{\varphi B_{22}}^{r_j^{(i)}}(z_i) - \mathcal{H}_{K_{21}PK_{12}}^{r_j^{(i)}}(z_i)]\{\mathcal{H}_{R_{K_{12}}}^{r_j^{(i)}}(z_i)\}^{-1}.$$

Clearly, $\boldsymbol{a}_j^{(i)}$ and $\boldsymbol{b}_j^{(i)}$ can be partitioned as

$$\boldsymbol{a}_j^{(i)} = \begin{bmatrix} u_{j1}^{(i)} & u_{j2}^{(i)} & \cdots & u_{j,r_j^{(i)}}^{(i)} \end{bmatrix} \quad \text{and}$$

$$\boldsymbol{b}_j^{(i)} = \begin{bmatrix} v_{j1}^{(i)} & v_{j2}^{(i)} & \cdots & v_{j,r_j^{(i)}}^{(i)} \end{bmatrix}.$$

$$\begin{bmatrix} B_{11}(z) & L_{B_{11}}(z)K_{12}(z) \\ \varphi(z)K_{21}(z)R_{B_{11}}(z) & K_{21}(z)P(z)K_{12}(z) + L_{K_{21}}(z)K(z)R_{K_{12}}(z) \end{bmatrix}. \tag{9}$$

Equation (11) can then be compactly rewritten as

$$\boldsymbol{b}_j^{(i)} = \boldsymbol{a}_j^{(i)} \mathcal{H}_K^{r_j^{(i)}}(z_i), \qquad 0 \le i \le m-1, \quad 1 \le j \le t_i, \qquad (12)$$

which shows that $K(z)$ is indeed the solution of a tangential Hermite–Fejér problem.

Several remarks are due here. To begin with, observe that to obtain the interpolation problem we need to compute the transmission zeros of a single matrix function, $\varphi(z)A_{*22}(z)$. More importantly, this is an inner function, and its zeros are simply reflections of the poles. Secondly, our derivation leads to one-sided interpolation conditions only and avoids some technical difficulties that may arise in the case of two-sided conditions at the same point (see, e.g., [11], [20]). Finally, note from (11) that it is sufficient to compute the values of the involved matrix functions at the interpolation points $\{z_i\}$.

V. Recursive Solution of the Interpolation Problem

We now derive necessary and sufficient conditions for the solvability of the four-block problem. We first follow [12]–[14] and describe conditions for the solvability of the interpolation problem (12), as well as a simple recursive array procedure for the construction of a cascade (lattice) structure with the desired interpolation properties. The first step in the solution consists of constructing three matrices F, G, and J directly from the interpolation data: F contains the information relative to the points $\{z_i\}$ and the dimensions $\{r_j^{(i)}\}$, G contains the information relative to the direction vectors $\{\boldsymbol{a}_j^{(i)}\}$ and $\{\boldsymbol{b}_j^{(i)}\}$, and $J = \text{diagonal}\{I_{p_2}, -I_{q_2}\}$ is a signature matrix. The matrices F and G are constructed as follows: we associate t_i Jordan blocks with each z_i. That is, we define a block-diagonal matrix $F_{z_i} = F_{1i} \oplus F_{2i} \oplus \cdots \oplus F_{t_i,i}$, where F_{ji} is an $r_j^{(i)} \times r_j^{(i)}$ Jordan block with eigenvalue at z_i and ones on the first subdiagonal. Then F is a block-diagonal matrix of the form $F = F_{z_0} \oplus F_{z_1} \oplus \cdots \oplus F_{z_{m-1}}$. We also define t_i matrices $U_j^{(i)}$ ($r_j^{(i)} \times p_2$) and $V_j^{(i)}$ ($r_j^{(i)} \times q_2$) that are composed of the row vectors $\boldsymbol{a}_j^{(i)}$ and $\boldsymbol{b}_j^{(i)}$ associated with z_i

$$U_j^{(i)} = \begin{bmatrix} u_{j1}^{(i)} \\ \vdots \\ u_{j,r_j^{(i)}}^{(i)} \end{bmatrix} \quad \text{and} \quad V_j^{(i)} = \begin{bmatrix} v_{j1}^{(i)} \\ \vdots \\ v_{j,r_j^{(i)}}^{(i)} \end{bmatrix}.$$

If we write

$$U_{z_i} = \begin{bmatrix} U_1^{(i)} \\ \vdots \\ U_{t_i}^{(i)} \end{bmatrix}, \qquad V_{z_i} = \begin{bmatrix} V_1^{(i)} \\ \vdots \\ V_{t_i}^{(i)} \end{bmatrix}.$$

Then we construct

$$G = \begin{bmatrix} U_{z_0} & V_{z_0} \\ \vdots & \vdots \\ U_{z_{m-1}} & V_{z_{m-1}} \end{bmatrix}.$$

Let $n = \sum_{i=0}^{m-1} m_i$ and $r = (p_2 + q_2)$, then F and G are $n \times n$ and $n \times r$ matrices respectively, and we consider the following so-called displacement equation

$$R - FRF^* = GJG^* \qquad (13)$$

where F and G are as defined above. Clearly, R is unique since F is a stable matrix (i.e., $|f_i| < 1\ \forall i$, where we denote the diagonal entries of F by $\{f_i\}_{i=0}^{n-1}$). Moreover, the tangential Hermite–Fejér problem has a solution if, and only if, R is positive-definite (see, e.g., [27, pp. 294–298] and [12]-[14]). This discussion leads to the following result.

Theorem 5.1: The four-block problem is solvable if, and only if, the two conditions in (4) are satisfied and R is positive-definite.

Proof: The discussion in the previous sections clearly shows that if the four-block problem has a solution $Q(z)$ then (4) is satisfied and (12) is solvable. Conversely, assume that (4) holds true and that (12) has a solution $K(z)$. Then the matrix function (see (14) at the bottom of the page) is in Schur class. Define the rational matrix function (analytic in the unit disc)

$$H(z) = \varphi(z)B_{11}(z) - K_{21}(z)P(z)K_{12}(z) - L_{K_{21}}(z)K(z)R_{K_{12}}(z)$$

and observe that the left-null structure of $\varphi(z)A_{*22}(z)$ is a co-restriction of the left-null structure of $H(z)$ [25]. Using Proposition 12.1.1 and Theorem 12.3.1 in [25] (or even a more direct argument if $A_{*22}(z)$ is chosen to be diagonal), we deduce that there exists a bounded rational matrix function $Q(z) \in RH^\infty_{p_2 \times q_2}$ relating $H(z)$ and $\varphi(z)A_{*22}(z)$, viz., $H(z) = \varphi(z)A_{*22}(z)Q(z)$ in $|z| < 1$. It then follows from (14) that

$$\begin{bmatrix} B_{11}(z) & B_{12}(z) \\ \varphi(z)B_{21}(z) & \varphi(z)B_{22}(z) - \varphi(z)A_{*22}(z)Q(z) \end{bmatrix}$$

is in Schur class and hence, the four-block problem has a solution $Q(z)$. ■

A. A Recursive Solution

There is a vast literature on the solution of interpolation problems of various kinds. We refer only to the books and monographs [25], [27], [28] and the references therein. The Hermite–Fejér problem is often not addressed, with most attention to the Nevanlinna–Pick and Schur problems. While (different) recursive solutions are classical for these special problems, for more general cases and to achieve a unified solution, global expressions for the solution are given involving the inverse of the matrix R (see expression (16) ahead). In [12]–[14] we presented a new recursive solution procedure using only the matrices F and G. It is interesting that the procedure is actually the so-called generalized Schur algorithm for the fast triangular factorization of the matrix R implicitly defined by the (less complex) matrices F and G via (13). We now apply this recursive algorithm to obtain a cascade (or a lattice) structure that solves the desired interpolation problem. The algorithm is a recursive procedure that uses only the data available in the matrices $\{F, G, J\}$, without explicitly computing the matrix R itself or its inverse. The determination of an interpolating function $K(z)$ in (12) then reduces to the following recursive scheme: start with $G_0 = G$, $F_0 = F$, and apply the following procedure to compute G_i, $i \ge 1$ (the diagonal entries of F and the first rows of the successive G_i will be used to construct the solution):

1) At step i we have G_i and F_i. Let g_i denote the first row of G_i.
2) Choose a J-unitary matrix Θ_i (i.e., $\Theta_i J \Theta_i^* = J$) such that $g_i\Theta_i$ is reduced to the form $g_i\Theta_i = [\delta_i 0 \cdots 0]$, where δ_i is a scalar. The matrix Θ_i can be implemented in a variety of ways, e.g., as a sequence of elementary unitary and hyperbolic rotations.

$$\begin{bmatrix} B_{11}(z) & L_{B_{11}}(z)K_{12}(z) \\ \varphi(z)K_{21}(z)R_{B_{11}}(z) & K_{21}(z)P(z)K_{12}(z) + L_{K_{21}}(z)K(z)R_{K_{12}}(z) \end{bmatrix} \qquad (14)$$

3) Compute G_{i+1} as

$$\begin{bmatrix} 0_{1\times r} \\ G_{i+1} \end{bmatrix} = \boldsymbol{\Phi}_i G_i \Theta_i \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + G_i \Theta_i \begin{bmatrix} 0 & 0 \\ 0 & I_{r-1} \end{bmatrix} \tag{15}$$

where $\boldsymbol{\Phi}_i = (F_i - f_i I_{n-i})(I_{n-i} - f_i^* F_i)^{-1}$.

4) Set F_{i+1} equal to the submatrix obtained after deleting the first row and column of F_i, and go back to the first step.

Expression (15) for G_{i+1} has a simple and interesting array interpretation: multiply G_i by Θ_i and keep the last $r-1$ columns, and then multiply the first column of $G_i\Theta_i$ by $\boldsymbol{\Phi}_i$. Notice that the (0, 0) entry of $\boldsymbol{\Phi}_i$ is zero and that $\boldsymbol{\Phi}_i$ has the form of a "Blaschke" matrix.

We further associate with each recursive step an elementary J-lossless section $\Theta_i(z)$, viz., $\boldsymbol{\Theta}_i(z)$ is analytic in $|z| < 1$, $\Theta_i(z)J\boldsymbol{\Theta}_i^*(z) = J$ on $|z| = 1$, and is given by

$$\boldsymbol{\Theta}_i(z) = \boldsymbol{\Theta}_i \begin{bmatrix} \frac{z-f_i}{1-zf_i^*} & 0 \\ 0 & I_{r-1} \end{bmatrix}.$$

Let us introduce the overall transfer matrix

$$\boldsymbol{\Theta}(z) = \boldsymbol{\Theta}_0(z)\boldsymbol{\Theta}_1(z)\cdots\boldsymbol{\Theta}_{n-1}(z)$$

and partition it accordingly with J. It can be shown (see [25] and the references therein or [12]–[14]) that all solutions $K(z)$ to the Hermite–Fejér problem can be written as

$$K(z) = -[\boldsymbol{\Theta}_{11}(z)S(z) + \boldsymbol{\Theta}_{12}(z)][\boldsymbol{\Theta}_{21}(z)S(z) + \boldsymbol{\Theta}_{22}(z)]^{-1}$$

where $S(z) \in RH^\infty_{p_2\times q_2}$ is an arbitrary Schur-type matrix function.

Theorem 5.2: All solutions $Q(z)$ to the four-block problem are given by the linear fractional transformation

$$\varphi(z)A_{*22}(z)Q(z) = [\boldsymbol{\Phi}_{11}(z)S(z) + \boldsymbol{\Phi}_{12}(z)] \cdot [\boldsymbol{\Phi}_{21}(z)S(z) + \boldsymbol{\Phi}_{22}(z)]^{-1}$$

where $S(z) \in RH^\infty_{p_2\times q_2}$ is an arbitrary Schur-type matrix function and

$$\begin{bmatrix} \boldsymbol{\Phi}_{11}(z) & \boldsymbol{\Phi}_{12}(z) \\ \boldsymbol{\Phi}_{21}(z) & \boldsymbol{\Phi}_{22}(z) \end{bmatrix} = \begin{bmatrix} L_{K_{21}}(z) & \varphi(z)B_{22}(z) - K_{21}(z)P(z)K_{12}(z) \\ 0 & I \end{bmatrix} \cdot \begin{bmatrix} I & 0 \\ 0 & R^{-1}_{K_{12}}(z) \end{bmatrix} \begin{bmatrix} \boldsymbol{\Theta}_{11}(z) & \boldsymbol{\Theta}_{12}(z) \\ \boldsymbol{\Theta}_{21}(z) & \boldsymbol{\Theta}_{22}(z) \end{bmatrix}. \quad \blacksquare$$

For completeness, we may present the previously mentioned global expression for $\boldsymbol{\Theta}(z)$ [12], [14], [25], viz.,

$$\boldsymbol{\Theta}(z) = \{I - (1-z)JG^*(I - zF^*)^{-1}R^{-1}(I-F)^{-1}G\}\boldsymbol{\Theta} \tag{16}$$

where $\boldsymbol{\Theta}$ is an arbitrary J-unitary matrix. To use this global representation requires the knowledge of R^{-1}, whereas the recursive solution described before does not require R or R^{-1}. We should also remark that one does not need to explicitly determine the solution R of (13) and check for its positive-definiteness. It can be shown that [13], [14] R is positive-definite if, and only if, $g_iJg_i^* > 0$ for $i = 0, \cdots, n-1$. This test can clearly be performed during the recursive construction since the rows g_i are available at each step and $g_iJg_i^* = |\delta_i|^2$.

VI. Concluding Remarks

In this note, we addressed a standard problem in H^∞-control from the viewpoint of Schur analysis. We stated the problem in the function domain and in more general form than usual: the entries $L_{ij}(z)$ in (2) were not assumed to have any relation to each other. We then derived an expression that parameterizes the unknown function $Q(z)$ in terms of known quantities and a Schur function that satisfies certain interpolation conditions of the Hermite-Fejér type. These known quantities were expressed in terms of certain RH^∞ functions $\{B_{ij}(z)\}$ in (2) that were obtained via two decoupled factorization problems. A recursive procedure was also described for the determination of a cascade structure that parameterizes all interpolants, and consequently, all $Q(z)$.

Here we have assumed that the problem data is given via rational (transfer) functions. More work is needed to translate our results to the case where the initial data is in state-space form. Nevertheless we might note that certain simplifications may be available in this case. The spectral factorizations called for in our solution can then be carried out via the solution of certain Riccati equations, which need to be compared with the Riccati equations of the state-space theory. We also note that the more efficient Chandrasekhar recursions can be used to carry out spectral factorization (see [14], [29], [30]); the point is that the Schur algorithm for spectral factorization can be simplified to the Chandreasekhar recursions when state-space structure is present.

Appendix
The Schur Algorithm and Inner Dilations/Factorization

We now briefly describe the relation of the Schur algorithm to inner dilations and spectral factorization. For more details and connections the reader is referred to [31]–[33], [29]. Let $N(z)$ be a Schur-type matrix function in $RH^\infty_{k\times l}$ and consider its Taylor series expansion around the origin, viz., $N(z) = N_0 + N_1z + N_2z^2 + \cdots$, where $\{N_i,\ i \geq 0\}$ are $k \times l$ matrices of norm less than one. We introduce the displacement equation

$$\mathcal{R} - \mathcal{Z}\mathcal{R}\mathcal{Z}^* = \mathcal{G}\mathcal{I}\mathcal{G}^* \tag{A.1}$$

where $\mathcal{I} = (I_k \oplus -I_l)$ is a signature matrix, $\mathcal{Z}$ is the (semi-infinite) lower triangular block-shift matrix with identities I_k on the first block-subdiagonal and zeros elsewhere, and $\mathcal{G}$ is the so-called generator matrix constructed from the Taylor series coefficients of $N(z)$ as follows

$$\mathcal{G} = \begin{bmatrix} I_k & N_0 \\ 0 & N_1 \\ 0 & N_2 \\ \vdots & \vdots \end{bmatrix}.$$

We now apply the following array procedure to $\mathcal{G}$ (which is similar to (15) with F_i replaced by a shift matrix)

$$\begin{bmatrix} 0_{k\times k+l} \\ \mathcal{G}_{i+1} \end{bmatrix} = \mathcal{Z}\mathcal{G}_i\Theta_i \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} + \mathcal{G}_i\Theta_i \begin{bmatrix} 0 & 0 \\ 0 & I_l \end{bmatrix}, \quad \mathcal{G}_0 = \mathcal{G} \tag{A.2}$$

where $\boldsymbol{\Theta}_i$ is a $(k+l)\times(k+l)$ $\mathcal{I}$-unitary matrix chosen so as to reduce the first k rows of $\mathcal{G}_i$ (denoted by $\boldsymbol{g}_i$) to the form $\boldsymbol{g}_i\boldsymbol{\Theta}_i = [\Delta_i 0_{k\times l}]$. That is, a $k \times l$ block zero is introduced in the second-block column of $\boldsymbol{g}_i$. We can choose $\boldsymbol{\Theta}_i$ as

$$\boldsymbol{\Theta}_i = \begin{bmatrix} I_k & -\gamma_i \\ -\gamma_i^* & I_l \end{bmatrix} \begin{bmatrix} (I_k - \gamma_i\gamma_i^*)^{-1/2} & 0 \\ 0 & (I_l - \gamma_i^*\gamma_i)^{-1/2} \end{bmatrix}$$

where γ_i are usually referred to as the Schur or reflection coefficients of $N(z)$. Assume we start with $\mathcal{G}$ and apply (A.2) m times. This produces m reflection coefficients $\gamma_0, \gamma_1, \cdots, \gamma_{m-1}$, and m $\mathcal{I}$-lossless sections of the form (we use the superscript N to denote that the section is associated with the matrix function $N(z)$)

$$\boldsymbol{\Theta}_i^{(N)}(z) = \boldsymbol{\Theta}_i \begin{bmatrix} zI_k & 0 \\ 0 & I_l \end{bmatrix}.$$

Consider the cascade $\boldsymbol{\Theta}^{(N)}(z)$ of these m sections, $\boldsymbol{\Theta}^{(N)}(z) = \boldsymbol{\Theta}_0^{(N)}(z)\boldsymbol{\Theta}_1^{(N)}(z)\cdots\boldsymbol{\Theta}_{m-1}^{(N)}(z)$, and partition $\boldsymbol{\Theta}^{(N)}(z)$ accordingly with $\mathcal{I}$. Let $\boldsymbol{\Sigma}^{(N)}(z)$ be the associated scattering matrix

$$\boldsymbol{\Theta}^{(N)}(z) = \begin{bmatrix} \boldsymbol{\Theta}_{11}^{(N)} - \boldsymbol{\Theta}_{12}^{(N)}\boldsymbol{\Theta}_{22}^{-(N)}\boldsymbol{\Theta}_{21}^{(N)} & -\boldsymbol{\Theta}_{12}^{(N)}\boldsymbol{\Theta}_{22}^{-(N)} \\ \boldsymbol{\Theta}_{22}^{-(N)}\boldsymbol{\Theta}_{21}^{(N)} & \boldsymbol{\Theta}_{22}^{-(N)} \end{bmatrix}(z).$$

It is straightforward to verify that $\Sigma^{(N)}(z)$ is an inner dilation (since $\Theta^{(N)}(z)$ is $\mathcal{I}$-lossless). The point to stress here is that, for sufficiently large m, the Schur matrix function $\hat{N}(z)$ determined by the first Schur coefficients $\{\gamma_0, \gamma_1, \cdots, \gamma_{m-1}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \cdots\}$, is a good norm approximation of $N(z)$, uniformly on the compact subsets of the unit disc. Moreover, $R_N(z)$ is equally well approximated in norm by $R_{\hat{N}}(z)$ (see [32, pp. 79–83] and [33] for more details). More precisely, $\hat{N}(z) = -\Theta_{12}^{(N)}\Theta_{22}^{-(N)}(z)$, $R_{\hat{N}}(z) = \Theta_{22}^{-(N)}(z)$ and $z^m L_{\hat{N}}(z) = \Theta_{11}^{(N)} - \Theta_{12}^{(N)}\Theta_{22}^{-(N)}\Theta_{21}^{(N)}(z)$. That is, the entries of $\Sigma^{(N)}(z)$ are good norm approximations of the spectral factors of $N(z)$ for sufficiently large m. The computational complexity of the above procedure is $O(m^2)$ operations (multiplications and additions). In case an underlying n-dimensional state-space model is assumed for $N(z)$, then the Schur algorithm (A.2) reduces to the so-called Chandrasekhar recursions [14], [29], which requires $O(mn^2)$ operations. This represents great savings in computation since $n \ll m$ usually.

Acknowledgment

The authors would like to thank J. A. Ball and D. J. N. Limbeer for their valuable comments and corrections to the first version of this paper.

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