we obtain

$$B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C_3 = [3 \quad 1] \quad A_3 = [1 \quad 4].$$

A_2 results from

$$\begin{bmatrix} 1 & 4 \\ 3 & 1 \end{bmatrix} A_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

as

$$A_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The resulting periodic realization has state dimensions $n_1 = 1$, $n_2 = 1$ and $n_3 = 2$ and is minimal. In contrast, the realization obtained in [7] has constant order $n = 2$ and is not minimal.

V. Conclusion

We proposed a numerically sound and computationally efficient approach to compute minimal periodic realizations of transfer-funtion matrices. The resulting periodic representations have in general time-varying dimensions. The proposed approach relies exclusively on numerically stable algorithms, the key computations being $N-1$ rank revealing orthogonal decompositions. The proposed approach is straightforward to implement as robust numerical software. Numerical examples computed with a MATLAB-based implementation show the applicability of this method to high order periodic systems.

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Multiobjective Filter Design for Uncertain Stochastic Time-Delay Systems

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***Abstract*—This note addresses the problem of robust mutliobjective filtering for discrete time-delay systems with mixed stochastic and deterministic uncertainties, in addition to unmodeled nonlinearities. A procedure is developed for the design of linear and exponentially stable filters with a bounded error variance, exponential rate of decay, and robust performance for the error system.**

***Index Terms*—Exponential stability, Kalman filter, linear matrix inequality, robust filter, stochastic uncertainties, time-delay systems.**

I. Introduction

The Kalman filter is the optimal linear least-mean-squares-estimator for systems that are described by linear state–space Markov models (see, e.g., [1]). However, when the model is not accurately known, the performance of the filter can deteriorate appreciably. This filter sensitivity to modeling errors has led to several works in the literature on the development of robust state–space filters; robust in the sense that they attempt to limit, in certain ways, the effect of model uncertainties on the overall filter performance. Some known approaches to robust state–space estimation are $\mathcal{H}_\infty$ filtering, set-valued estimation, guaranteed-cost designs, minimum variance filtering and regularization-based methods (see, e.g., [2]–[15]). In this note, we show how to design robust filters that ensure a minimum bounded error variance for models with *mixed* stochastic and deterministic uncertainties, as well as with time delays and nonlinearities. We also show how to design filters that simultaneously guarantee an exponential rate of decay and meet a robust performance level. Thus, consider the following n-dimensional state–space model:

$$x_{k+1} = (A + \Delta A_k)x_k + A_d x_{k-\tau} + Bu_k + Df(x_k) \qquad (1)$$

$$y_k = Cx_k + v_k$$

$$z_k = Lx_k, \qquad k \geq 0 \qquad (2)$$

where $\{u_k, v_k\}$ are uncorrelated zero-mean random variables with unknown but bounded covariance matrices, say $Eu_k u_k^T < \rho_u I$ and $Ev_k v_k^T < \rho_v I$. The initial state x_0 is also a zero-mean random variable that is uncorrelated with $\{u_k, v_k\}$ for all k. The state matrices $\{A, A_d\}$, and the output matrix C, are *unknown* but lie inside a convex polytopic set. That is $(A, A_d, C) \in \mathcal{K}$,where $\mathcal{K}$ is a convex bounded polyhedral domain described by p vertices as follows:

$$\mathcal{K} = \left\{ (A, A_d, C) = \sum_{i=1}^{i=p} \alpha_i (A_i, A_{di}, C_i),\ \alpha_i \geq 0,\ \sum_{i=1}^{i=p} \alpha_i = 1 \right\}. \qquad (3)$$

Note that although the matrices $\{A, A_i\}$ are constant, the coefficient matrix in (1) is time variant due to the presence of the uncertainties ΔA_k. These uncertainties are assumed to be random in nature and are modeled as $\Delta A_k = E\Delta_k G$, where E and G are known matrices, while Δ_k is a random matrix whose entries have zero mean and are

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uncorrelated with each other. The variances of the entries of Δ_k are assumed unknown but bounded by ρ_Δ, $E\Delta_k\Delta_k^T \leq \rho_\Delta I$. The function $f(\cdot)$ in (1) accounts for unmodeled nonlinearities and it is assumed to satisfy $\|f(x_k)\| \leq \|Ux_k\|$, for some matrix U. Observe that model (1) and (2) incorporates both stochastic uncertainties (due to the presence of ΔA_k) and deterministic uncertainties (represented by the polyhedral domain $\mathcal{K}$). In this note, we investigate the design of a linear estimator for $\{x_k, z_k\}$ of the form

$$\hat{x}_{k+1} = A_f\hat{x}_k + B_f y_k \quad \hat{z}_k = L_f\hat{x}_k, \qquad k \geq 0 \tag{4}$$

where the constant matrices $\{A_f, B_f, L_f\}$ are filter parameters to be determined in order to meet certain performance criteria, including robustness, exponential stability, and bounded state error variance. In (4), the notation $\hat{x}_k$ and $\hat{z}_k$ denote the estimates of x_k and z_k, respectively, that are based on $\{y_0, y_1, \ldots y_{k-1}\}$. Let $\tilde{x}_k = x_k - \hat{x}_k$, denote the state error vector. It follows from (1) and (4) that the extended state vector $\eta_k = \mathrm{col}\{x_k, \tilde{x}_k\}$ satisfies

$$\eta_{k+1} = (\bar{A} + \Delta\bar{A}_k)\eta_k + \bar{B}w_k + \bar{A}_d\eta_{k-\tau} + \bar{D}f(M\eta_k) \tag{5}$$

while the output error is given by $\tilde{z}_k = z_k - \hat{z}_k = [L - L_f \; L_f]\eta_k$ and where we are defining the extended quantities

$$\begin{aligned}
\eta_k &= \begin{pmatrix} x_k \\ \tilde{x}_k \end{pmatrix}\\
w_k &= \begin{pmatrix} w_k \\ v_k \end{pmatrix}\\
\Delta\bar{A}_k &= \bar{E}\triangle_k\bar{G}\\
\bar{E} &= \begin{pmatrix} E \\ E \end{pmatrix}\\
\bar{G} &= (G \quad 0)\\
M &= (I \quad 0)\\
\bar{D} &= \begin{pmatrix} D \\ D \end{pmatrix}\\
\bar{A}_d &= \begin{pmatrix} A_d & 0 \\ A_d & 0 \end{pmatrix}\\
\bar{A} &= \begin{pmatrix} A & 0 \\ A - A_f - B_fC & A_f \end{pmatrix}\\
\bar{B} &= \begin{pmatrix} B & 0 \\ B & -Bf \end{pmatrix}.
\end{aligned}$$

Definition 1 [Stability With Probability 1]: The stochastic process η_k of (5) will be said to be stable with probability 1 if and only if, for any $\delta > 0$ and $\varepsilon > 0$, there exists a $\sigma(\delta, \varepsilon) > 0$ such that if $\|\eta_0\| \leq \sigma(\delta, \varepsilon)$, then $P[\sup\|\eta_k\| \geq \varepsilon] \leq \delta$. If $P[\sup\|\eta_k\| \geq \varepsilon] \leq \delta$ holds for all η_0, then we say that the system is stable at large. ◊

Definition 2 [Asymptotic Stability]: The stochastic process η_k of (5) will be said to be asymptotically stable with probability 1 if and only if it is stable at large and $\|\eta(k)\| \to 0$ with probability 1 as $k \to \infty$ for any η_0. ◊

Definition 3 [Exponential Stability]: The stochastic process η_k of (5) will be said to be exponentially stable with level $0 < \varsigma < 1$, if there are real numbers $\mu > 0$, and $\nu > 0$ such that $E\|\eta_k\|^2 \leq \mu\|\eta_0\|^2\varsigma^k + \nu$ for any η_0. ◊

Our objective is to determine filter parameters $\{A_f, B_f, L_f\}$ in (4) such that for all admissible uncertainties in the model (1), (2), the augmented system (5) is asymptotically stable in the absence of noises and, when noises are present, the state estimation error $\tilde{x}_k$ is exponentially stable, independent of the unknown time-delay τ. We shall also minimize a bound on $E\|\tilde{x}_k\|^2$ and simultaneously ensure a robust performance level as will be explained in the sequel. We address first the requirement of asymptotic stability.

II. Asymptotic Stability

Assume that the noise w_k is absent from (5) so that

$$\eta_{k+1} = (\bar{A} + \Delta\bar{A}_k)\eta_k + \bar{A}_d\eta_{k-\tau} + \bar{D}f(M\eta_k), \qquad k \geq 0. \tag{6}$$

Introduce the vector $\phi_k = [\eta_k^T, \eta_{k-1}^T, \ldots, \eta_{k-r}^T]^T$. We shall seek a Lyapunov Krasovskii functional $V(.)$ of the form $V(\phi_k) = \eta_k^T P\eta_k + \sum_{i=k-r}^{i=k-1}\eta_i^T R\eta_i$ for some positive-definite matrices P and R to be chosen. Assume, for the moment, that the triplet (A, A_d, C) in (1) and (2) is fixed, i.e, ignore the polytopic set (3).

Theorem 1 (Asymptotic Stability): Given scalars $\epsilon_1 > 0$, $\epsilon_2 > 0$, and $0 < \alpha < 1$, if there exist matrices $\{A_f, B_f, P > 0, R > 0\}$, and a scalar $\beta > 0$, such that

$$\begin{pmatrix} \beta I & \bar{A}^T \\ \bar{A} & I \end{pmatrix} > 0 \tag{7}$$

and

$$W \triangleq \begin{pmatrix} H & -\bar{A}^T P\bar{A}_d \\ -\bar{A}_d^T P\bar{A} & R - \beta\epsilon_2^{-1}I - \bar{A}_d^T P\bar{A}_d \end{pmatrix} > \alpha I \tag{8}$$

where

$$\begin{aligned}
H = P - R - \rho_\Delta\bar{G}^T\bar{E}^T P\bar{E}\bar{G} - \bar{A}^T P\bar{A} - \lambda_{\max}(\bar{D}^T P\bar{D})\bar{U}^T\bar{U}\\
-\beta\epsilon_1^{-1}I - (\epsilon_1 + \epsilon_2)\lambda_{\max}(\bar{D}^T P^2\bar{D})\bar{U}^T\bar{U}
\end{aligned}$$

and $\bar{U} = UM$, then the process $\{\eta_k\}$ of (5), with fixed (A, A_d, C), will be asymptotically stable in the absence of noise for this choice of $\{A_f, B_f\}$.

Proof: Note that

$$\begin{aligned}
E&\left[V(\phi_{k+1})|\phi_k, \phi_{k-1}, \ldots \phi_0\right] - V(\phi_k)\\
&\leq \eta_k^T\bar{A}P\bar{A}\eta_k - \eta_k^T P\eta_k\\
&\quad + \rho_\Delta\eta_k^T\bar{G}^T\bar{E}^T P\bar{E}\bar{G}\eta_k + \eta_k^T\bar{A}^T P\bar{A}_d\eta_{k-\tau}\\
&\quad + \eta_{k-\tau}^T\bar{A}_d^T P\bar{A}\eta_k + \eta_k^T\bar{A}^T P\bar{D}f(M\eta_k)\\
&\quad + f^T(M\eta_k)\bar{D}^T P\bar{A}\eta_k + f^T(M\eta_k)\bar{D}^T P\bar{D}f(M\eta_k)\\
&\quad + \eta_{k-\tau}^T\bar{A}_d^T P\bar{A}_d\eta_{k-\tau} + \eta_{k-\tau}^T\bar{A}_d^T P\bar{D}f(M\eta_k)\\
&\quad + f^T(M\eta_k)\bar{D}^T P\bar{A}_d\eta_{k-\tau} + \eta_k^T R\eta_k - \eta_{k-\tau}^T R\eta_{k-\tau}.
\end{aligned} \tag{9}$$

Now, it is a well-known result [17] that for any real matrices $\{X, Y, J\}$ with $JJ^T \leq \mu I$, it holds for any scalar $\varepsilon > 0$ that $XJY + Y^TJ^TX^T \leq \varepsilon^{-1}\mu XX^T + \varepsilon Y^TY$. From (7), we have $\bar{A}^T\bar{A} < \beta I$. Choosing $J = \bar{A}^T$, we can write

$$\begin{aligned}
\eta_k^T\bar{A}^T P\bar{D}f(M\eta_k) &+ f^T(M\eta_k)\bar{D}^T P\bar{A}\eta_k\\
&\leq \beta\epsilon_1^{-1}\eta_k^T\eta_k + \epsilon_1\lambda_{\max}(\bar{D}^T P^2\bar{D})\eta_k^T\bar{U}^T\bar{U}\eta_k.
\end{aligned}$$

Similarly

$$\begin{aligned}
\eta_{k-\tau}^T\bar{A}_d^T P\bar{D}f(M\eta_k) &+ f^T(M\eta_k)\bar{D}^T P\bar{A}_d\eta_{k-\tau}\\
&\leq \beta\epsilon_2^{-1}\eta_{k-\tau}^T\eta_{k-\tau} + \epsilon_2\lambda_{\max}(\bar{D}^T P^2\bar{D})\eta_k^T\bar{U}^T\bar{U}\eta_k
\end{aligned}$$

for some ϵ_1, $\epsilon_2 > 0$ and, moreover, $f^T(M\eta_k)\bar{D}^T P\bar{D}f(M\eta_k) \leq \lambda_{\max}(\bar{D}^T P\bar{D})\eta_k^T\bar{U}^T\bar{U}\eta_k$. Then, we have

$$\begin{aligned}
E&\left[V(\phi_{k+1})|\phi_k, \phi_{k-1}, \ldots \phi_0\right] - V(\phi_k)\\
&\leq -\left[\eta_k^T \quad \eta_{k-\tau}^T\right] W \left[\eta_k^T \quad \eta_{k-\tau}^T\right]^T.
\end{aligned} \tag{10}$$

From (8), we get

$$E\left[V(\phi_{k+1})|\phi_k, \phi_{k-1}, \ldots, \phi_0\right] - V(\phi_k) \leq -\alpha\|\eta_k\|^2 < 0$$

which implies that the process $\{\phi_k\}$, and consequently $\{\eta_k\}$, is asymptotically stable [16]. ◊

Now, assume that we restrict our choice of P to block diagonal positive-definite matrices, and partition $\{P, R\}$ in conformity with η_k, and define Q_1 and Q_2, respectively, as

$$P=\begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} \quad R=\begin{pmatrix} R_1 & R_3 \\ R_3^T & R_2 \end{pmatrix} \quad Q_1=A_f^T P_2,\ Q_2=B_f^T P_2.$$

We can then see that the requirement (8) is satisfied if

$$\begin{pmatrix} R_1 & R_3 \\ R_3^T & R_2 \end{pmatrix} > 0 \begin{pmatrix} \sigma_{1I} & D^T P_1 & D^T P_2 \\ P_{1D} & P_1 & 0 \\ P_2 D & 0 & P_2 \end{pmatrix} > 0 \qquad (11)$$

$$\begin{pmatrix} \sigma_2 I & D^T P_1 & D^T P_2 \\ P_1 D & I & 0 \\ P_2 D & 0 & I \end{pmatrix} > 0 \qquad (12)$$

and (13)–(15), as shown at the bottom of the page, hold.

The second inequality in (11) guarantees $\lambda_{\max}(\bar{D}^T P \bar{D}) < \sigma_1$, while (12) guarantees $\lambda_{\max}(\bar{D}^T P^2 \bar{D}) < \sigma_2$. These inequalities do not require the $\lambda_{\max}(.)$ operations for H and S and, therefore, they are linear inequalities in the unknowns.

III. Exponential Performance

We now show that the process η_k is also exponentially mean square-stable, as well as almost surely stable in norm. To begin with, in the presence of measurement and process noises, and with $\{A_f, B_f\}$ chosen from the feasible solution of (7) and (11)–(13), we obtain the following inequality by repeating the argument of Theorem 1:

$$E\left[V(\phi_{k+1})|\phi_k, \phi_{k-1}, \dots \phi_0\right] - V(\phi_k) < -\alpha\|\eta_k\|^2 + \rho_u Tr\left(B^T(P_1+P_2)B\right) + \rho_v Tr\left(B_f^T P_2 B_f\right).$$

Now, note that $V(\phi_k) = \phi_k^T \Gamma \phi_k$, where $\Gamma \triangleq \mathrm{diag}\{P, R, \dots R\}$ It follows that

$$E\left[V(\phi_{k+1})|\phi_k, \phi_{k-1}, \dots \phi_0\right] - V(\phi_k) < -\psi_k V(\phi_k) + \rho_u Tr\left(B^T(P_1+P_2)B\right) + \rho_v Tr\left(B_f^T P_2 B_f\right) \qquad (16)$$

where $\psi_k \triangleq (\alpha\|\eta_k\|^2/\lambda_{\max}(\Gamma)\|\phi_k\|^2)$. If $\{P, R\}$ are further chosen such that $\Gamma > I$, then $0 < \psi_k < 1 - \delta$, for some $\delta > 0$.

Consequently

$$E\left[V(\phi_{k+1}|\phi_k, \phi_{k-1}, \dots, \phi_0)\right] - \frac{V(\phi_k)}{\theta} < \rho_u Tr\left(B^T(P_1+P_2)B\right) + \rho_v Tr\left(B_f^T P_2 B_f\right) \qquad (17)$$

where $\theta = \inf_k(1/1-\psi_k)$. Let $\psi = \sup_k \psi_k$. Then, $0 < \psi < 1$ and

$$E\left[V(\phi_{k+1}|\phi_k, \phi_{k-1}, \dots, \phi_0)\right] - V(\phi_k) \le \rho_u Tr\left(B^T(P_1+P_2)B\right) + \rho_v Tr\left(B_f^T P_2 B_f\right) - \psi V(\eta_k). \qquad (18)$$

Inequality (18) allows us to establish that the process $\{\eta_k\}$ is exponentially mean-square stable. In order to arrive at this conclusion, we call upon the following auxiliary results.

Lemma 1: If there exist positive real numbers λ, μ, ν, and $0 < \psi < 1$ such that

$$\mu\|\phi_k\|^2 \le V(\phi_k) \le \nu\|\phi_k\|^2 \qquad (19)$$

and

$$E\left[V(\phi_{k+1})|\phi_k, \phi_{k-1}, \dots, \phi_0\right] - V(\phi_k) \le \lambda - \psi V(\phi_k) \qquad (20)$$

then the process ϕ_k is exponentially stable. Moreover, it holds that

$$E\|\phi_k\|^2 \le \frac{\nu}{\mu} E\|\phi_0\|^2 (1-\psi)^k + \frac{\lambda}{\mu\psi}. \qquad (21)$$

Proof: This result is a combination of Lemma 3 and [19, Th. 2].◇

Lemma 2: If $V(\phi_k)$ satisfies

$$E\left[V(\phi_{k+1})|\phi_k, \phi_{k-1}, \dots, \phi_0\right] - \frac{V(\phi_k)}{\theta} - L < 0 \quad \text{a.s.} \qquad (22)$$

for some $\theta > 1$, $L > 0$, then $V(\phi_k)$ is bounded with probability 1 and, moreover, $EV(\phi_k)$ remains bounded for all k with

$$E\left[V(\phi_k)\right] < \frac{V(\phi_0)}{\theta^k} + L\frac{\theta}{\theta-1}\left\{1 - \frac{1}{\theta^{k+1}}\right\}. \qquad (23)$$

Proof: See [20] ◇

Theorem 2 (Exponential Stability): Given scalars $\epsilon_1 > 0$, $\epsilon_2 > 0$, and $0 < \alpha < 1$, let $\{A_f, B_f, P_1, P_2, R_1, R_2, R_3, \sigma_1, \sigma_2\}$ be a solution to (7) and (11)–(13) with $\Gamma > I$. Then the resulting processes $\{\phi_k, \eta_k\}$ are exponentially stable in the presence of measurement and process noises and for fixed (A, A_d, and C). Moreover, the variance of ϕ_k is bounded as follows:

$$\boxed{E\|\phi_k\|^2 < \frac{1}{\lambda_{\min}(\Gamma)}\left\{\frac{V(\phi_0)}{\theta^k} + L\frac{\theta}{\theta-1}\left(1 - \frac{1}{\theta^{k+1}}\right)\right\}} \qquad (24)$$

where

$$L \triangleq \rho_u \mathrm{Tr}\left(B^T(P_1+P_2)B\right) + \rho_v \mathrm{Tr}\left(B_f^T P_2 B_f\right). \qquad (25)$$

Proof: The result follows from (17), (18), and Lemmas 1 and 2.◇

$$\begin{pmatrix} S'-\alpha I & -R_3 & \tilde{J} & 0 & A^T P_1 & \hat{J} & 0 & 0 \\ -R_3^T & P_2 - R_2 - \beta\epsilon_1^{-1} I - \alpha I & -Q_1 A_d & 0 & 0 & Q_1 & 0 & 0 \\ \tilde{J}^T & -A_d^T Q_1^T & R_1 - \alpha I - \beta{\epsilon_2}^{-1} I & R_3 & 0 & 0 & A_d^T P_1 & A_d^T P_2 \\ 0 & 0 & R_3^T & R_2 - \alpha I - \beta\epsilon_2^{-1} I & 0 & 0 & 0 & 0 \\ P_1 A & 0 & 0 & 0 & P_1 & 0 & 0 & 0 \\ \hat{J}^T & Q_1^T & 0 & 0 & 0 & P_2 & 0 & 0 \\ 0 & 0 & P_1 A_d & 0 & 0 & 0 & P_1 & 0 \\ 0 & 0 & P_2 A_d & 0 & 0 & 0 & 0 & P_2 \end{pmatrix} > 0 \qquad (13)$$

where

$$S' = P_1 - \rho_\Delta G^T E^T (P_1+P_2) E G - R_1 - (\sigma_1 + (\epsilon_1+\epsilon_2)\sigma_2) U^T U - \beta\epsilon_1^{-1} I$$

$$\hat{J} \triangleq -C^T Q_2 - Q_1 + A^T P_2 \qquad (14)$$

$$\tilde{J} \triangleq -A^T(P_1+P_2)A_d + C^T Q_2 A_d + Q_1 A_d. \qquad (15)$$

Remark: Apart from the above result, we can also show almost sure exponential stability of (5) in norm in the absence of noises. Using Chebychev's inequality [21], in the absence of noises, we have

$$P\left\{\|\phi_k\| > \frac{1}{\lambda_{\min}(\Gamma)\theta^{\frac{k}{4}}}\right\} \leq \left(\lambda_{\min}(\Gamma)\theta^{\frac{k}{4}}\right)^2 E\left(\|\phi_k\|^2\right). \quad (26)$$

Summing over k, we get

$$\begin{aligned}\sum_{k=0}^{\infty} P\left\{\|\phi_k\| > \frac{1}{\lambda_{\min}(\Gamma)\theta^{\frac{k}{4}}}\right\} &\leq (\lambda_{\min}(\Gamma))^2 \sum_{k=0}^{\infty} \theta^{\frac{k}{2}} E\left(\|\phi_k\|^2\right) \\ &\leq (\lambda_{\min}(\Gamma))^2 \sum_{k=0}^{\infty} \frac{V(\phi_0)}{\theta^{\frac{k}{2}}} \\ &= \frac{(\lambda_{\min}(\Gamma))^2 V(\phi_0)}{1 - \frac{1}{\theta}^{\frac{1}{2}}} < \infty. \quad (27)\end{aligned}$$

Now, from the Borel Cantelli Lemma [21], we conclude that the event $\|\phi_k\| \geq (1/\lambda_{\min}(\Gamma)\theta^{k/4})$ cannot occur infinitely often, i.e,

$$P\left\{\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}\left\{\|\phi_k\| > \frac{1}{\lambda_{\min}(\Gamma)\theta^{\frac{k}{4}}}\right\}\right\} = 0. \quad (28)$$

Then, it holds that $\|\phi_k\| \leq (1/\lambda_{\min}(\Gamma)\theta^{k/4})$ as desired.

IV. Polytopic Uncertainties

We can now incorporate the fact that the matrices $\{A, A_d, C\}$ are not fixed but lie within the polytopic set $\mathcal{K}$ defined by (3).

Theorem 3 (Exponentially-Stable Filter): Given scalars $\epsilon_1 > 0$, $\epsilon_2 > 0$, $\epsilon_3 > 0$ and $0 < \alpha < 1$, any filter defined by the matrices

$$A_f = \left(Q_1 P_2^{-1}\right)^T \quad B_f = \left(Q_2 P_2^{-1}\right)^T \quad (29)$$

where Q_1, Q_2, P_2, and L_f are obtained from a feasible solution of the matrix inequalities (7) and (11)–(13) for all A taking values in $[A_1, \ldots, A_p]$, A_d taking values in $[A_{d1}, \ldots, A_{dp}]$, and C taking values in $[C_1, \ldots, C_p]$, ensures the following:

i) $E\|\phi_k\|^2$ is bounded as stated in Theorem 2;
ii) exponential and asymptotic stability of (5) for all admissible parameters A, A_d, and C.

Proof: These properties follow from the definition of Q_1 and Q_2, and from the fact that the inequalities (7), (11), and (12) are linear in A, A_d, and C. ⋄

The result (24) in Thm. 2 further suggests that we can minimize an upper bound on the error variance, $E\|\tilde{x}_k\|^2$, by seeking filter coefficients $\{A_f, B_f\}$ that minimize the following function over the variables $\{A_f, B_f, P_1, P_2, R, \sigma_1, \sigma_2, \beta\}$:

$$\rho_u \mathrm{Tr}\left(B^T(P_1+P_2)B\right) + \rho_v \mathrm{Tr}\left(B_f^T P_2 B_f\right)$$

subject to conditions (7) and (11)–(13) and $I < \Gamma$. The last term in the previous cost function is nonlinear in (B_f, P_2). We can instead solve the following convex optimization problem over the variables $\{A_f, B_f, P_1, P_2, R, \Lambda, \sigma_1, \sigma_2, \beta\}$:

$$\min \ \mathrm{Tr}\left(\rho_u B^T(P_1+P_2)B + \rho_v \Lambda\right) \quad (30)$$

subject to conditions (7) and (11)–(13) and $\begin{pmatrix}\Lambda & Q_2 \\ Q_2^T & P_2\end{pmatrix} > 0$, with $\Gamma > I$. This last condition enforces a bound $B_f^T P_2 B_f < \Lambda$. Note that since $P > I$, (7) can be enforced by the inequality

$$\begin{pmatrix}\beta I & 0 & A^T P_1 & \hat{J} \\ 0 & \beta I & 0 & Q_1 \\ P_1 A & 0 & P_1 & 0 \\ \hat{J}^T & Q_1^T & 0 & P_2\end{pmatrix} > 0. \quad (31)$$

V. Robust Performance

In this section, we shall further assume that

$$E\sum_{k=0}^{\infty} u_k^T u_k < \infty \quad E\sum_{k=0}^{\infty} v_k^T v_k < \infty. \quad (32)$$

We shall also rely on the following definition.

Definition 4 [Robust Performance]: The error system (5) will be said to have a robust performance of level $\gamma > 0$ if for all nonzero u_k, v_k as in (32), it holds for some $\chi > 0$ that

$$E\left\{\sum_{k=0}^{\infty} \tilde{z}_k^T \tilde{z}_k\right\} < \chi\|\eta_0\|^2 + \gamma^2 E\left\{\sum_{k=0}^{\infty} u_k^T u_k + v_k^T v_k\right\}.$$

◇

Observe that contrary to a standard $\mathcal{H}_\infty$ formulation, we use the expectation operator on both sides of the above inequality in order to account for the presence of stochastic uncertainties. In addition to asymptotic and exponential stabilities, we can enforce a robust performance level by requiring (P, R) to satisfy, along with the feasibility conditions (7) and (11)–(13) with $\Gamma > I$, the following requirement:

$$EV(\phi_{k+1}) - EV(\phi_k) - \gamma^2 E\left(u_k^T u_k + v_k^T v_k\right) + E\tilde{z}_k^T \tilde{z}_k < 0 \quad (33)$$

for some given $\gamma > 0$. Indeed, if we sum (33) over k, and noting that the error system is exponentially mean-square stable, we get

$$E\left\{\sum_{k=0}^{\infty} \tilde{z}_k^T \tilde{z}_k\right\} < EV(\phi_0) + \gamma^2 E\left\{\sum_{k=0}^{\infty} u_k^T u_k + v_k^T v_k\right\} \quad (34)$$

which is consistent with criterion (33). Now, if β also satisfies

$$\begin{pmatrix}\beta I & \bar{B}^T \\ \bar{B} & I\end{pmatrix} > 0 \quad (35)$$

or, in other words, if $\bar{B}^T\bar{B} < \beta I$, then using the same methodology as in Section II, we can verify that (33) is satisfied if (36), as shown at the bottom of page, holds true for any given $\epsilon_3 > 0$, where

$$\hat{S} \triangleq P_1 - \rho_\Delta G^T E^T(P_1+P_2)EG - R_1 - \beta\epsilon_1^{-1} I - (\sigma_1 + (\epsilon_1+\epsilon_2+\epsilon_3)\sigma_2)U^T U.$$

$$\left(\begin{array}{cccccc|ccc}
\hat{S} & -R_3 & 0 & 0 & 0 & 0 & A^T P_1 & \hat{J} & L^T - L_f^T \\
-R_3^T & P_2 - R_2 - \beta\epsilon_1^{-1} I & 0 & 0 & 0 & 0 & 0 & Q_1 & L_f^T \\
0 & 0 & R_1 - \beta\epsilon_2^{-1} I & R_3 & 0 & 0 & A_d^T P_1 & A_d^T P_2 & 0 \\
0 & 0 & R_3^T & R_2 - \beta\epsilon_2^{-1} I & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \gamma^2 I - \beta\epsilon_3^{-1} I & 0 & B^T P_1 & B^T P_2 & 0 \\
0 & 0 & 0 & 0 & 0 & \gamma^2 I - \beta\epsilon_3^{-1} I & 0 & -Q_2 & 0 \\
\hline
P_1 A & 0 & P_1 A_d & 0 & P_1 B & 0 & P_1 & 0 & 0 \\
\hat{J}^T & Q_1^T & P_2 A_d & 0 & P_2 B & -Q_2^T & 0 & P_2 & 0 \\
L - L_f & L_f & 0 & 0 & 0 & 0 & 0 & 0 & I
\end{array}\right) > 0 \quad (36)$$

Theorem 4 (Robust Performance): Given scalars $\epsilon_1 > 0$, $\epsilon_2 > 0$, $\epsilon_3 > 0$, and $0 < \alpha < 1$, let $\{A_f, B_f, P_1, P_2, R_1, R_2, R_3\}$ be a solution to the linear matrix inequalities (7), (11)–(13), (35), and (36), with $\Gamma > I$. Then the error system in (5) is exponentially stable and has a robust performance level of γ for fixed (A, A_d, and C).

VI. Delayless Systems

If we assume a delayless system, i.e., if we set $A_d = 0$ in (1), and if we drop the robustness requirement of Section V, we can enforce a tighter upper bound on the variance of the error, $E\|\tilde{x}_k\|^2$. Thus, consider the system

$$x_{k+1} = (A + \Delta A_k)x_k + Bu_k + Df(x_k) \tag{37}$$

$$y_k = Cx_k + v_k, \qquad k \geq 0. \tag{38}$$

Then, we have the following result (see also [22]).

Theorem 5 (Exponential Stability): Given scalars $\epsilon_1 > 0$, $\epsilon_2 > 0$, and $0 < \alpha < 1$, let $\{A_f, B_f, P_1, P_2, \sigma_1, \sigma_2\}$ be a solution to the following inequalities:

$$\begin{pmatrix} \beta I & \bar{A}^T \\ \bar{A} & I \end{pmatrix} > 0, \ \begin{pmatrix} \sigma_1 I & D^T P_1 & D^T P_2 \\ P_1 D & P_1 & 0 \\ P_2 D & 0 & P_2 \end{pmatrix} > 0 \tag{39}$$

$$\begin{pmatrix} \sigma_2 I & D^T P_1 & D^T P_2 \\ P_1 D & I & 0 \\ P_2 D & 0 & I \end{pmatrix} > 0 \tag{40}$$

$$\left(\begin{array}{cc|cc} S' & 0 & A^T P_1 & \hat{J} \\ 0 & P_2 - \epsilon^{-1}\beta I - \alpha I & 0 & Q_1 \\ \hline P_1 A & 0 & P_1 & 0 \\ \hat{J}^T & Q_1^T & 0 & P_2 \end{array}\right) > 0 \tag{41}$$

with $P > I$, where

$$S' \triangleq P_1 - \alpha I - \rho_\Delta G^T E^T (P_1 + P_2) E G - (\sigma_1 + \epsilon\sigma_2) U^T U - \epsilon^{-1}\beta I.$$

Then, the resulting process $\{\eta_k\}$ is exponentially stable in the presence of measurement and process noises. Moreover, its variance is bounded as follows:

$$\boxed{E\|\eta_k\|^2 < \frac{1}{\lambda_{\min}(P)}\left\{\frac{V(\eta_0)}{\theta^k} + L\frac{\theta}{\theta - 1}\left(1 - \frac{1}{\theta^{k+1}}\right)\right\}} \tag{42}$$

where

$$L = \rho_u \mathrm{Tr}\left(B^T(P_1 + P_2)B\right) + \rho_v \mathrm{Tr}\left(B_f^T P_2 B_f\right) \tag{43}$$

and $\theta = (1)/(1 - \psi)$ with $\psi = (\alpha)/(\lambda_{\max}(P))$.

VII. Simulations

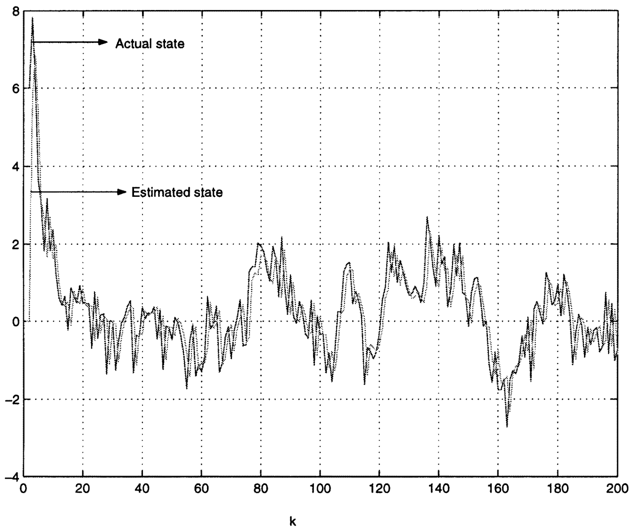
To illustrate the mutliobjective filter developed for state-delayed systems, we choose an implementation of order 2 for a nonlinear uncertain stochastic system (1) as follows:

$$A_1 = \begin{pmatrix} 0.62 & 0 \\ 0 & 0.61 \end{pmatrix}$$
$$A_2 = \begin{pmatrix} 0.5 & -1 \\ 0.2 & 0.5 \end{pmatrix}$$
$$A_3 = \begin{pmatrix} 0.54 & 1 \\ 0 & 0.56 \end{pmatrix}$$
$$C_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$C_2 = \begin{pmatrix} 0.2 & 0 \\ 0 & 0 \end{pmatrix}$$
$$B = \begin{pmatrix} 0.2 \\ 0.2 \end{pmatrix}$$
$$D = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}$$
$$A_d = \begin{pmatrix} 0.1 & 21 \\ 0.31 & 0.1 \end{pmatrix}$$
$$G = \begin{pmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \end{pmatrix}$$
$$E = \begin{pmatrix} 0.01 & 0.02 \\ 0.01 & 0.02 \end{pmatrix}$$
$$\begin{pmatrix} x_{1,k+1} \\ x_{2,k+1} \end{pmatrix} = (A + \Delta A_k)\begin{pmatrix} x_{1,k} \\ x_{2,k} \end{pmatrix} + A_d\begin{pmatrix} x_{1,k-\tau} \\ x_{2,k-\tau} \end{pmatrix} + Bw_k + D\begin{pmatrix} 0.1\sin(x_{1,k}) \\ 0.1\sin(x_{1,k}) \end{pmatrix}$$
$$y_k = C\begin{pmatrix} x_{1,k} \\ x_{2,k} \end{pmatrix} + v_k.$$

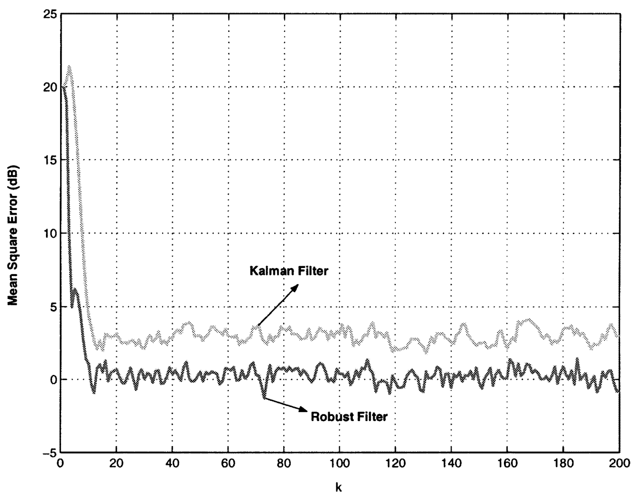
The delay τ in the example is chosen as 4. The values of ϵ_1 and ϵ_2 are chosen as 1.1. The value of β is 1. The robustness level is $\gamma = 8$. The performance of the filter is illustrated in Fig. 1(a), which shows its tracking capability. To illustrate the robust minimum variance filter developed in Section VI for delayless systems, we choose the following model:

$$A_1 = \begin{pmatrix} 0.62 & 0 \\ 0 & 0.61 \end{pmatrix}$$
$$A_2 = \begin{pmatrix} 0.5 & -1 \\ 0.2 & 0.5 \end{pmatrix}$$
$$A_3 = \begin{pmatrix} 0.54 & 1 \\ 0 & 0.56 \end{pmatrix}$$
$$E = \begin{pmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \end{pmatrix}$$
$$G = \begin{pmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \end{pmatrix}$$
$$C_1 = \begin{pmatrix} 100 & 0 \\ 50 & 10 \end{pmatrix}$$
$$C_2 = \begin{pmatrix} 90 & 0 \\ 50 & 10 \end{pmatrix}$$
$$D = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}$$
$$B = \begin{pmatrix} -6 \\ 1 \end{pmatrix}$$
$$\begin{pmatrix} x_{1,k+1} \\ x_{2,k+1} \end{pmatrix} = (A + \Delta A_k)\begin{pmatrix} x_{1,k} \\ x_{2,k} \end{pmatrix} + Bw_k + D\begin{pmatrix} 0.1\sin(x_{1,k}) \\ 0.1\sin(x_{1,k}) \end{pmatrix}$$
$$y_k = C\begin{pmatrix} x_{1,k} \\ x_{2,k} \end{pmatrix} + v_k.$$

Fig. 1(b) compares the mean-square-error $E\|\tilde{x}_k\|^2$ in dB when the actual state matrix is A_3, for both the Kalman filter operating at the centroid of the polytopic region and the robust filter. The noise variances are equal to 1.



(a)



(b)

Fig. 1. Performance of the robust filters. (a) Tracking performance of the mutliobjective robust filter of Theorem 3. (b) Mean square error behavior of the Kalman filter and the robust filter for delayless systems of Theorem 5.

VIII. CONCLUSION

In this note, we developed a mutliobjective robust state estimator for uncertain discrete time state-delay systems with mixed deterministic and stochastic uncertainties. The design guarantees almost-sure bounded error variance with exponential stability and robust performance.

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