An Invariant Matrix Structure in Multiantenna Communications

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***Abstract*—This letter shows that the matrix structure with 2×2 Alamouti sub-blocks remains invariant under several nontrivial matrix operations, including matrix inversion, Schur complementation, Riccati recursion, triangular factorization, and QR factorization.**

***Index Terms*—Alamouti code, invariant matrix structure, matrix structure, matrix inversion, QR factorization, Riccati recursion, Schur complementation, triangular factorization.**

I. Introduction

A BASIC building block in multiantenna space–time-coded communications is the Alamouti structure [1]–[3]. A 2×2 Alamouti matrix is defined by [2]

$$A = \begin{pmatrix} a_1 & a_2 \\ -a_2^* & a_1^* \end{pmatrix} \tag{1}$$

for some possibly complex scalars $\{a_1, a_2\}$. The purpose of this letter is to highlight some interesting properties of block matrices involving 2×2 Alamouti sub-blocks. It turns out that the Alamouti structure remains invariant under several nontrivial matrix operations, including matrix inversion, Schur complementation, Riccati recursion, and even triangular and QR factorizations. These properties are useful because they can be exploited to derive efficient receivers for multiantenna communications (see, e.g., [3], [5], and [6]).

A. Some Basic Properties

To begin with, every matrix of the form (1) satisfies the well-known relation

$$AA^* = (|a_1|^2 + |a_2|^2)\mathbf{I}_2$$

where $\mathbf{I}_2$ denotes the 2×2 identity matrix. It follows that the inverse of an Alamouti matrix is another Alamouti matrix since

$$A^{-1} = (|a_1|^2 + |a_2|^2)^{-1} A^*.$$

Note also that the sum, difference, or product of two Alamouti matrices is another Alamouti matrix. Moreover, the sum of an Alamouti matrix and its complex conjugate is a scaled multiple of the identity, i.e., $A + A^* = 2\mathrm{Re}\{a_1\}\mathbf{I}_2$, where the identity matrix can be seen as a special case of the Alamouti structure (corresponding to $a_1 = 1, a_2 = 0$).

B. Block Matrices

More generally, let C be a block row vector, and let D be a block matrix with entries

$$C = (C_1 \quad \cdots \quad C_M), \quad D = \begin{pmatrix} D_{11} & \cdots & D_{1M} \\ \vdots & \ddots & \vdots \\ D_{M1} & \cdots & D_{MM} \end{pmatrix}$$

where the individual sub-blocks $\{C_i\}$ and $\{D_{ij}\}$ are each 2×2 Alamouti. Then, the following properties are trivial to establish in view of the basic properties of Alamouti matrices.

1) The matrix CD consists of 2×2 Alamouti sub-blocks. The first block entry of CD is given by

$$C_1 D_{11} + C_2 D_{21} + C_3 D_{31} + \cdots + C_M D_{M1}$$

which consists of the sum of M products of 2×2 Alamouti matrices. Hence, the result is again 2×2 Alamouti. A similar argument applies to the other block entries of CD.

2) CC^* is a scaled multiple of the identity.

3) The matrices C^*C, D^*D, and DD^* are Hermitian with diagonal 2×2 sub-blocks that are scaled multiples of the identity matrix and with off-diagonal sub-blocks that are 2×2 Alamouti matrices, e.g., for $M = 3$

$$D = \begin{pmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{pmatrix}$$

then

$$D^*D = \begin{pmatrix} \gamma_1 \mathbf{I}_2 & A_1 & A_2 \\ A_1^* & \gamma_2 \mathbf{I}_2 & A_3 \\ A_2^* & A_3^* & \gamma_3 \mathbf{I}_2 \end{pmatrix}$$

where the individual A_i are 2×2 Alamouti, and the scalars γ_i are non-negative.

II. Invariance Under Inversion and Schur Complementation

A less obvious property is the fact that the Alamouti structure for block matrices is preserved under matrix inversion, as the following statement specifies.

Lemma 1 (Invariance Under Inversion): The inverse of a block square matrix with 2×2 Alamouti sub-blocks is another block matrix with 2×2 Alamouti sub-blocks.

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Proof: We establish the result by induction. Consider first a matrix D with 4 sub-blocks, i.e., let $M = 2$ so that D is 4×4 and given by

$$D = \left(\begin{array}{c|c} D_{11} & D_{12} \\ \hline D_{21} & D_{22} \end{array}\right) = \left(\begin{array}{cc|cc} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \hline \times & \times & \times & \times \\ \times & \times & \times & \times \end{array}\right)$$

where D_{11}, D_{12}, D_{21}, and D_{22} are all 2×2 Alamouti matrices. Then, D^{-1} can be found in terms of the sub-blocks of D by using the block matrix inversion formula [4]

$$D^{-1} = \left(\begin{array}{c|c} D_{11}^{-1} + D_{11}^{-1} D_{12} \Sigma^{-1} D_{21} D_{11}^{-1} & -D_{11}^{-1} D_{12} \Sigma^{-1} \\ \hline -\Sigma^{-1} D_{21} D_{11}^{-1} & \Sigma^{-1} \end{array}\right) \tag{2}$$

where

$$\Sigma = D_{22} - D_{21} D_{11}^{-1} D_{12}$$

denotes the 2×2 Schur complement of D with respect to D_{11}. Clearly, all the sub-blocks in D^{-1} are 2×2 Alamouti because they are obtained via sum or product or inversion combinations of 2×2 Alamouti matrices. Therefore, the result of the lemma holds for $M = 2$.

More generally, consider a $2(M+1) \times 2(M+1)$ matrix D with 2×2 Alamouti blocks, and assume it has already been established that the *inverse* of its leading $2M \times 2M$ block matrix D_{11} has the desired structure (i.e., with 2×2 Alamouti blocks). We partition D as

$$D = \left(\begin{array}{c|c} D_{11} & D_{12} \\ \hline D_{21} & D_{22} \end{array}\right) = \left(\begin{array}{c|c} (2M \times 2M) & (2M \times 2) \\ \hline (2 \times 2M) & (2 \times 2) \end{array}\right)$$

with D_{11}, D_{12}, D_{21}, and D_{22} now being the $2M \times 2M$ upper-left, $2M \times 2$ upper-right, $2 \times 2M$ lower-left, and 2×2 lower-right matrices, respectively. Then, we apply the block matrix inversion expansion (2) again to conclude that D^{-1} has Alamouti sub-blocks. ■

Lemma 2 (Invariance Under Schur Complementation): The Schur complement of D with respect to any leading $2K \times 2K$ block has a similar structure with 2×2 Alamouti sub-blocks.

Proof: Partition D as

$$\begin{aligned} D &= \left(\begin{array}{c|c} D_{11} & D_{12} \\ \hline D_{21} & D_{22} \end{array}\right) \\ &= \left(\begin{array}{c|c} (2K \times 2K) & (2K \times (M-2K)) \\ \hline ((M-2K) \times 2M) & ((M-2K) \times (M-2K)) \end{array}\right) \end{aligned} \tag{3}$$

with D_{11}, D_{12}, D_{21}, and D_{22} now being the $2K \times 2K$ upper-left, $2K \times (M-2K)$ upper-right, $(M-2K) \times 2K$ lower-left, and $(M-2K) \times (M-2K)$ lower-right matrices, respectively. The Schur complement with respect to D_{11} is given by $\Sigma = D_{22} - D_{21} D_{11}^{-1} D_{12}$. Clearly, from the previous lemma, D_{11}^{-1} has 2×2 Alamouti sub-blocks. Then, Σ is obtained via the sum or product or inversion combinations of matrices with 2×2 Alamouti sub-blocks, and we conclude that Σ has 2×2 Alamouti sub-blocks as well. ■

III. Invariance Under Matrix Factorizations

Another interesting property of block matrices with 2×2 Alamouti sub-blocks is that their triangular and QR factors will exhibit a similar structure. We establish the result first for block triangular factorizations.

Lemma 3 (Invariance Under Triangular Factorization): Consider a block matrix A with 2×2 Alamouti sub-blocks. Assume A can be factored as $A = \mathcal{L}\mathcal{D}\mathcal{U}$, where $\mathcal{D}$ is block-diagonal with 2×2 sub-blocks and $\mathcal{L}(\mathcal{U})$ is lower (upper) triangular with 2×2 identity matrices along its diagonal. Then, the sub-blocks of $\mathcal{D}$ are 2×2 Alamouti, and the matrices $\mathcal{L}$ and $\mathcal{U}$ have 2×2 Alamouti sub-blocks in their lower (upper) triangular parts, e.g.,

$$\mathcal{L} = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline \times & \times & 1 & 0 \\ \times & \times & 0 & 1 \end{array}\right) \quad \mathcal{D} = \left(\begin{array}{cc|cc} \times & \times & 0 & 0 \\ \times & \times & 0 & 0 \\ \hline 0 & 0 & \times & \times \\ 0 & 0 & \times & \times \end{array}\right)$$

where all 2×2 sub-blocks are Alamouti (similarly for $\mathcal{U}$).

Proof: Decompose the matrix A as follows [4]:

$$\begin{aligned} A &= \left(\begin{array}{c|c} A_0 & B_0 \\ \hline C_0 & D_0 \end{array}\right) \\ &= \underbrace{\begin{pmatrix} \mathbf{I}_2 & \mathbf{0} \\ C_0 A_0^{-1} & \mathbf{I} \end{pmatrix}}_{\mathcal{L}_0} \begin{pmatrix} A_0 & \mathbf{0} \\ \mathbf{0} & \Delta_0 \end{pmatrix} \underbrace{\begin{pmatrix} \mathbf{I}_2 & A_0^{-1} B_0 \\ \mathbf{0} & \mathbf{I} \end{pmatrix}}_{\mathcal{U}_0} \end{aligned} \tag{4}$$

with A_0 denoting its 2×2 Alamouti upper-left block of A, and Δ_0 denoting the Schur complement of A with respect to A_0, namely, $\Delta_0 = D_0 - C_0 A_0^{-1} B_0$. From Lemma 2, we know that Δ_0 has 2×2 Alamouti sub-blocks, and $\mathcal{L}_0$ and $\mathcal{U}_0$ have 2×2 Alamouti sub-blocks in their triangular parts with identity sub-blocks on the main diagonal. We proceed by decomposing Δ_0 similarly (where A_1 is 2×2 Alamouti)

$$\begin{aligned} \Delta_0 &= \left(\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right) \\ &= \underbrace{\begin{pmatrix} \mathbf{I}_2 & \mathbf{0} \\ C_1 A_1^{-1} & \mathbf{I} \end{pmatrix}}_{L_1} \begin{pmatrix} A_1 & \mathbf{0} \\ \mathbf{0} & \Delta_1 \end{pmatrix} \underbrace{\begin{pmatrix} \mathbf{I}_2 & A_1^{-1} B_1 \\ \mathbf{0} & \mathbf{I} \end{pmatrix}}_{U_1} \end{aligned} \tag{5}$$

where L_1 and U_1 have 2×2 Alamouti sub-blocks in their triangular parts, and A_1 is assumed to be invertible. Likewise, Δ_1 has a block structure with 2×2 Alamouti sub-blocks. Then

$$A = \mathcal{L}_0 \left(\begin{array}{c|c} A_0 & \mathbf{0} \\ \hline \mathbf{0} & L_1 \begin{pmatrix} A_1 & \mathbf{0} \\ \mathbf{0} & \Delta_1 \end{pmatrix} U_1 \end{array}\right) \mathcal{U}_0. \tag{6}$$

Equation (6) can be rewritten as

$$A = \underbrace{\mathcal{L}_0 \begin{pmatrix} \mathbf{I}_2 & \\ & L_1 \end{pmatrix}}_{\mathcal{L}_1} \begin{pmatrix} A_0 & & \\ & A_1 & \\ & & \Delta_1 \end{pmatrix} \underbrace{\begin{pmatrix} \mathbf{I}_2 & \\ & U_1 \end{pmatrix} \mathcal{U}_0}_{\mathcal{U}_1}. \tag{7}$$

If we continue the factorization procedure in the same fashion, we arrive at

$$A = \mathcal{L}_{M-1} \mathcal{D} \mathcal{U}_{M-1} \equiv \mathcal{L}\mathcal{D}\mathcal{U} \tag{8}$$

where $\mathcal{L}(\mathcal{U})$ are lower (upper) triangular block matrices with the identity matrix $\mathbf{I}_2$ along their diagonals and 2×2 Alamouti sub-blocks in their lower (upper) triangular parts. Likewise, $\mathcal{D}$ is a block diagonal matrix with 2×2 Alamouti sub-blocks. ■

Lemma (Invariance Under QR Factorization): Consider a block matrix A with 2×2 Alamouti sub-blocks. Assume A is factored as $A = \mathcal{Q}\mathcal{R}$, where $\mathcal{Q}$ is square unitary, and $\mathcal{R}$ is upper triangular with positive diagonal entries. Then, $\mathcal{Q}$ is a block matrix with 2×2 Alamouti sub-blocks. Likewise, $\mathcal{R}$ is a block matrix with multiples of $\mathbf{I}_2$ along its diagonal and with 2×2 Alamouti sub-blocks in its upper triangular part, e.g.,

$$\mathcal{Q} = \left(\begin{array}{cc|cc} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \hline \times & \times & \times & \times \\ \times & \times & \times & \times \end{array}\right) \quad \mathcal{R} = \left(\begin{array}{cc|cc} \times & 0 & \times & \times \\ 0 & \times & \times & \times \\ \hline 0 & 0 & \times & 0 \\ 0 & 0 & 0 & \times \end{array}\right)$$

where all 2×2 sub-blocks are Alamouti.

Proof: The result follows from the following two observations. Given 2×2 Alamouti matrices

$$B = \begin{pmatrix} b_1 & b_2 \\ -b_2^* & b_1^* \end{pmatrix} \quad C = \begin{pmatrix} c_1 & c_2 \\ -c_2^* & c_1^* \end{pmatrix}$$

then there exists a 4×4 unitary matrix Θ with 2×2 Alamouti sub-blocks such that

$$(B \quad C)\Theta = (\gamma \mathbf{I}_2 \quad \mathbf{0}) \tag{9}$$

where γ is a positive constant defined by $\gamma = \sqrt{\alpha_B + \alpha_C}, \alpha_B = |b_1|^2 + |b_2|^2$, and $\alpha_C = |c_1|^2 + |c_2|^2$. Indeed, set

$$\Theta = \frac{1}{\sqrt{\alpha_B + \alpha_C}} \begin{pmatrix} B^* & -C \\ C^* & B \end{pmatrix} \tag{10}$$

which consists of 2×2 Alamouti sub-blocks. Then it can be verified that Θ is unitary and that it achieves the required transformation (9).

Likewise, given B and C as above, there exists a 4×4 unitary matrix Θ with 2×2 Alamouti sub-blocks such that

$$(B \quad C)\Theta = (B \quad \sqrt{\alpha_C}\mathbf{I}_2). \tag{11}$$

Indeed, set

$$\Theta = \begin{pmatrix} \mathbf{I}_2 & \\ & C^*/\sqrt{\alpha_C} \end{pmatrix}. \tag{12}$$

Now, the QR factorization of A can be achieved by successive application of the above results. We illustrate the procedure as follows. Without loss of generality, we assume the matrix A has nine Alamouti sub-blocks. Each 2×2 sub-block is denoted by the capital letter X. The first step would involve using a unitary matrix Θ_1 to achieve the transformation

$$\underbrace{\left(\begin{array}{c|cc} \mathsf{X} & \mathsf{X} & \mathsf{X} \\ \mathsf{X} & \mathsf{X} & \mathsf{X} \\ \mathsf{X} & \mathsf{X} & \mathsf{X} \end{array}\right)}_{A^*} \Theta_1 = \left(\begin{array}{c|cc} \gamma\mathbf{I}_2 & \mathbf{0} & \mathsf{X} \\ \hline \mathsf{X}' & \mathsf{X}' & \mathsf{X} \\ \mathsf{X}' & \mathsf{X}' & \mathsf{X} \end{array}\right)$$

where the prearray is A^*, and Θ_1 has the form

$$\Theta_1 = \begin{pmatrix} \mathsf{X} & \mathsf{X} & \\ \mathsf{X} & \mathsf{X} & \\ & & \mathbf{I}_2 \end{pmatrix}$$

with 2×2 Alamouti sub-blocks. All sub-blocks $\{\mathsf{X}, \mathsf{X}'\}$ in the post-array that results from the application of Θ_1 will be 2×2 Alamouti. The next step involves using a second unitary matrix Θ_2 to achieve the transformation

$$\left(\begin{array}{c|cc} \gamma\mathbf{I}_2 & \mathbf{0} & \mathsf{X} \\ \hline \mathsf{X}' & \mathsf{X}' & \mathsf{X} \\ \mathsf{X}' & \mathsf{X}' & \mathsf{X} \end{array}\right) \Theta_2 = \left(\begin{array}{c|cc} \gamma_1\mathbf{I}_2 & \mathbf{0} & \mathbf{0} \\ \hline \mathsf{X}'' & \mathsf{X}' & \mathsf{X}' \\ \mathsf{X}'' & \mathsf{X}' & \mathsf{X}' \end{array}\right)$$

where Θ_2 has the form

$$\Theta_2 = \begin{pmatrix} \mathsf{X} & & \mathsf{X} \\ & \mathbf{I}_2 & \\ \mathsf{X} & & \mathsf{X} \end{pmatrix}$$

with 2×2 Alamouti sub-blocks. Again, all sub-blocks $\{\mathsf{X}', \mathsf{X}''\}$ in the post-array that results from the application of Θ_2 will be 2×2 Alamouti. The next step involves using a third unitary matrix Θ_3 to achieve the transformation

$$\left(\begin{array}{c|cc} \gamma_1\mathbf{I}_2 & \mathbf{0} & \mathbf{0} \\ \hline \mathsf{X}'' & \mathsf{X}' & \mathsf{X}' \\ \mathsf{X}'' & \mathsf{X}' & \mathsf{X}' \end{array}\right) \Theta_3 = \left(\begin{array}{c|cc} \gamma_1\mathbf{I}_2 & \mathbf{0} & \mathbf{0} \\ \hline \mathsf{X}'' & \gamma_2\mathbf{I}_2 & \mathbf{0} \\ \mathsf{X}'' & \mathsf{X}'' & \mathsf{X}'' \end{array}\right)$$

where Θ_3 has the form

$$\Theta_3 = \begin{pmatrix} \mathbf{I}_2 & & \\ & \mathsf{X} & \mathsf{X} \\ & \mathsf{X} & \mathsf{X} \end{pmatrix}$$

with 2×2 Alamouti sub-blocks. The last step involves using a fourth unitary matrix Θ_4 to achieve the transformation

$$\left(\begin{array}{c|cc} \gamma_1\mathbf{I}_2 & \mathbf{0} & \mathbf{0} \\ \hline \mathsf{X}'' & \gamma_2\mathbf{I}_2 & \mathbf{0} \\ \mathsf{X}'' & \mathsf{X}'' & \mathsf{X}'' \end{array}\right) \Theta_4 = \left(\begin{array}{c|cc} \gamma_1\mathbf{I}_2 & \mathbf{0} & \mathbf{0} \\ \hline \mathsf{X}'' & \gamma_2\mathbf{I}_2 & \mathbf{0} \\ \mathsf{X}'' & \mathsf{X}'' & \gamma_3\mathbf{I}_2 \end{array}\right)$$

where Θ_4 has the form

$$\Theta_4 = \begin{pmatrix} \mathbf{I}_2 & & \\ & \mathbf{I}_2 & \\ & & (\mathsf{X}'')^*/\sqrt{\alpha_{\mathsf{X}''}} \end{pmatrix}$$

where X'' denotes the rightmost 2×2 Alamouti sub-block on the last block row of the prearray.

The combination $\Theta = \Theta_1\Theta_2\Theta_3\Theta_4$ would, therefore, result in the transformation

$$\underbrace{\left(\begin{array}{c|cc} \mathsf{X} & \mathsf{X} & \mathsf{X} \\ \mathsf{X} & \mathsf{X} & \mathsf{X} \\ \mathsf{X} & \mathsf{X} & \mathsf{X} \end{array}\right)}_{A^*} \Theta = \underbrace{\left(\begin{array}{c|cc} \gamma_1\mathbf{I}_2 & \mathbf{0} & \mathbf{0} \\ \hline \mathsf{X}'' & \gamma_2\mathbf{I}_2 & \mathbf{0} \\ \mathsf{X}'' & \mathsf{X}'' & \gamma_3\mathbf{I}_2 \end{array}\right)}_{\mathcal{L}}$$

or, equivalently, $A = \mathcal{Q}\mathcal{R}$, where $\mathcal{Q} = \Theta$ and $\mathcal{R} = \mathcal{L}^*$. It is easy to conclude from the structure of the individual Θ_i that the unitary matrix $\mathcal{Q}$ has 2×2 Alamouti sub-blocks. ■

IV. Inverting a Triangular Factor

Given a triangular factor with 2×2 Alamouti sub-blocks, its inversion can be carried out in a manner that involves only the product and inversion of 2×2 Alamouti sub-blocks. Consider first a 4×4 upper-triangular factor of the form

$$R_1 = \begin{pmatrix} \gamma_1\mathbf{I}_2 & D \\ \mathbf{0} & \gamma_2\mathbf{I}_2 \end{pmatrix} \tag{13}$$

where D is 2×2 Alamouti. Using the matrix inversion formula [4]

$$\begin{pmatrix} A & D \\ 0 & B \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}DB^{-1} \\ 0 & B^{-1} \end{pmatrix} \tag{14}$$

it follows that

$$R_1^{-1} = \begin{pmatrix} \frac{1}{\gamma_1}\mathbf{I}_2 & -\frac{1}{\gamma_1\gamma_2}D \\ \mathbf{0} & \frac{1}{\gamma_2}\mathbf{I}_2 \end{pmatrix}. \tag{15}$$

Now, the inversion of larger matrices R can be computed iteratively by reducing its inversion to smaller dimension blocks as above. Thus, consider an upper triangular matrix R, and partition it as follows:

$$R = \left(\begin{array}{c|c} R_1 & D_1 \\ \hline \mathbf{0} & \Delta_1 \end{array}\right) \tag{16}$$

where R_1 and Δ_1 have scaled multiples of $\mathbf{I}_2$ along their diagonals and 2×2 Alamouti sub-blocks in their upper triangular parts. Likewise, D_1 has 2×2 Alamouti sub-blocks. Moreover, R_1 denotes the leading 4×4 submatrix of R. We already know how to evaluate the inverse of R_1, as indicated by (15). Applying the matrix inversion formula (14) to R, we get

$$R^{-1} = \left(\begin{array}{c|c} R_1^{-1} & -R_1^{-1}D_1\Delta_1^{-1} \\ \hline \mathbf{0} & \Delta_1^{-1} \end{array}\right). \tag{17}$$

Thus, the inverse of R now requires knowledge of Δ_1^{-1}. In order to compute Δ_1^{-1}, we partition Δ_1 as

$$\Delta_1 = \left(\begin{array}{c|c} R_2 & D_2 \\ \hline \mathbf{0} & \Delta_2 \end{array}\right) \tag{18}$$

where R_2 denotes its leading 4×4 submatrix with scaled multiples of $\mathbf{I}_2$ along the diagonal. Note again that we already know how to evaluate the inverse of R_2, as indicated by (15). Applying the matrix inversion formula (14) to Δ_1 gives

$$\Delta_1^{-1} = \left(\begin{array}{c|c} R_2^{-1} & -R_2^{-1}D_2\Delta_2^{-1} \\ \hline \mathbf{0} & \Delta_2^{-1} \end{array}\right). \tag{19}$$

In other words, the inverse of Δ_1 requires knowledge of Δ_2^{-1}, and the procedure can be continued in this manner.

V. Invariance Under Riccati Recursion

We illustrate the application of some of the above properties in the context of block RLS filtering (e.g., for either channel estimation or channel equalization applications), which involves update equations of the form (see, e.g., [5] and [6])

$$\begin{aligned} \Gamma_i &= (\mathbf{I} + U_i P_{i-1} U_i^*)^{-1} \\ w_i &= w_{i-1} + P_{i-1}U_i^*\Gamma_i(d_i - U_i w_{i-1}) \\ P_i &= P_{i-1} - P_{i-1}U_i^*\Gamma_i U_i P_{i-1}^* \end{aligned} \tag{20}$$

where d_i is a vector, and U_i is a regression data matrix.

It is assumed that the data matrix U_i consists of 2×2 Alamouti sub-blocks, which is a common situation in multiantenna communications involving space–time-coded transmissions. In this case, it turns out that the Alamouti structure is preserved by the Riccati recursion for P_i, as the following statement indicates.

Lemma 5 (Invariance Under Riccati Recursion): Assume the initial condition P_{-1} is chosen as a matrix with 2×2 Alamouti sub-blocks. Then, all successive matrices P_i will have the same structure, with Alamouti sub-blocks, for $i \geq 0$.

Proof: The result follows by induction. It is easy to see that $\mathbf{I} + U_0 P_{-1} U_0$ will have 2×2 Alamouti sub-blocks since both U_0 and P_{-1} have this structure. Therefore, Γ_0 will have the same structure by Lemma 1. It follows that

$$P_0 = P_{-1} - P_{-1}U_0^*\Gamma_0 U_0 P_{-1}^*$$

will have the same structure since P_0 is obtained as the sum and product combination of matrices involving 2×2 Alamouti sub-blocks. The argument can now proceed by induction to establish the conclusion of the lemma for $i \geq 1$. ∎

VI. Conclusion

We showed that matrix structures with Alamouti sub-blocks remain invariant under matrix inversion, matrix Schur complementation, triangular and QR factorization, and Riccati recursions. We may remark that a 2×2 Alamouti matrix can be regarded as the matrix representation of a quaternion [7], which brings up useful connections with the study of matrices with quaternionic entries. Most of the earlier works in the literature on quaternionic matrices has been concerned with the characterization of the eigenspace of such matrices and their Schur decomposition (e.g., [7]–[10]). Such results can be useful for multiantenna communications as well [10].

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