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DISPLACEMENT STRUCTURE AND COMPLETION PROBLEMS *

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Abstract. We prove a general result concerning time-variant displacement equations with positive solutions in a general operatorial setting. We then show that the solutions of several completion problems, recently considered in connection with classical interpolation and moment theory, follow as special cases of the main result. The main purpose of this paper is to show that under supplementary finite-dimensionality conditions, a so-called generalized Schur algorithm, which naturally arises in connection with displacement equations, can be used to prove the above mentioned result. We also discuss the associated transmission-line interpretation in terms of a cascade of elementary sections with intrinsic blocking properties.

Key words. Displacement structure, generalized Schur algorithm, interpolation, completion problems.

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1. Introduction. We prove a general result concerning time-variant displacement equations with positive solutions. More specifically, we show that a contractive upper-triangular operator S can always be associated with a Pick solution $R(t)$ of a time-variant Lyapunov (or displacement) equation. This result is actually a variation of the commutant lifting theorem of Sarason-Sz.Nagy-Foias and many other formulations have been considered in the literature (see, *e.g.,* [2, 13, 16, 20, 21, 22]). Under supplementary finite-dimensionality and nondegeneracy conditions, we further derive a so-called generalized Schur algorithm and discuss an associated system-theoretic interpretation in terms of a cascade of elementary sections with intrinsic blocking properties. These considerations lead to a constructive proof of the previous result about displacement equations. Several classical algorithms proposed in the literature for the solution of interpolation problems, such as Schur, Nevanlinna-Pick, and extensions thereof, follow as special cases of the general framework presented here. We also extend the content of our companion paper [26] where several other applications of the algorithm are presented.

The paper is organized as follows: In Section 2 we introduce our notation and define the class of time-variant structured matrices. We also prove the main result concerning the existence of an upper-triangular operator S in connection with Pick solutions of time-variant Lyapunov equations. In Section 3 we show that several moment, interpolation, and completion problems, and extensions thereof, follow as special cases of the main theorem of Section 2. In Section 4 we derive a computationally-oriented

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recursive procedure that leads to a constructive realization of all possible solutions S in terms of a cascade of elementary sections with certain intrinsic blocking properties. In Section 5 we further elaborate on possible simplifications and describe the associated Schur parameters.

[An early account of the results of this paper was announced in [9]. We further remark that after submitting this paper, a closely related result to Theorem 2.2 was independently derived in [11]].

2. Displacement Structure and Abstract Interpolation. We first introduce our notation. The symbol $\mathbb{Z}$ denotes the set of integers, and for two Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$ we write $\mathcal{L}(\mathcal{H},\mathcal{H}')$ to denote the set of bounded linear operators acting from $\mathcal{H}$ into $\mathcal{H}'$. We further consider three families $\{\mathcal{U}(t),\mathcal{V}(t),\mathcal{R}(t)\}_{t\in\mathbb{Z}}$ of Hilbert spaces depending on the parameter $t\in\mathbb{Z}$, two families of bounded linear operators $G(t)\in\mathcal{L}(\mathcal{U}(t)\oplus\mathcal{V}(t),\mathcal{R}(t))$ and $F(t)\in\mathcal{L}(\mathcal{R}(t-1),\mathcal{R}(t))$, and we define the symmetry $J(t)=(I_{\mathcal{U}(t)}\oplus -I_{\mathcal{V}(t)})$ acting on $\mathcal{U}(t)\oplus\mathcal{V}(t)$, where $I_{\mathcal{U}(t)}$ denotes the identity operator on the space $\mathcal{U}(t)$. We partition $G(t)=\begin{bmatrix} U(t) & V(t)\end{bmatrix}$, where $U(t)\in\mathcal{L}(\mathcal{U}(t),\mathcal{R}(t))$ and $V(t)\in\mathcal{L}(\mathcal{V}(t),\mathcal{R}(t))$. We also use the symbol $*$ to denote the adjoint operator and we write $F^*(t)=(F(t))^*$.

DEFINITION 2.1. *A family of operators $\{R(t)\in\mathcal{L}(\mathcal{R}(t))\}_{t\in\mathbb{Z}}$ is said to have a time-variant displacement structure with respect to $\{F(t),G(t)\}_{t\in\mathbb{Z}}$ if $\{R(t)\}_{t\in\mathbb{Z}}$ is uniformly bounded, viz., there exists $r>0$ such that $\|R(t)\|\le r$ for all $t\in\mathbb{Z}$, and $R(t)$ satisfies the time-variant Lyapunov (or displacement) equation*

$$R(t)-F(t)R(t-1)F^*(t)=G(t)J(t)G^*(t). \tag{2.1}$$

The cardinal number $r(t)=\dim\mathcal{U}(t)+\dim\mathcal{V}(t)$ is called the displacement rank of $R(t)$ in (2.1). We say that (2.1) has a Pick solution if $R(t)$ is positive-semidefinite for every $t\in\mathbb{Z}$.

[For more discussion on the application of time-variant structured matrices in adaptive filtering, matrix factorization, and interpolation problems, the reader is referred to the companion papers [25, 26, 29]].

We further introduce some assumptions that will guarantee the existence of a unique family with time-variant displacement structure with respect to a given set of generators $\{F(t),G(t)\}_{t\in\mathbb{Z}}$. To this effect, we consider the infinite (block) matrices

$$\mathbf{U}(t)=\begin{bmatrix}\dots & F(t)F(t-1)U(t-2) & F(t)U(t-1) & U(t)\end{bmatrix},$$
$$\mathbf{V}(t)=\begin{bmatrix}\dots & F(t)F(t-1)V(t-2) & F(t)V(t-1) & V(t)\end{bmatrix},$$

and assume that for each $t\in\mathbb{Z}$ and $h\in\mathcal{R}(t)$ we have

$$F^*(t-n)F^*(t-n+1)\dots F^*(t-1)F^*(t)h\to 0 \quad as\ n\to\infty, \tag{2.2a}$$

(2.2b) $\mathbf{U}(t)$ *and* $\mathbf{V}(t)$ *are well-defined bounded linear operators,*

$$\mathbf{U}(t)\in\mathcal{L}(\bigoplus_{j\le t}\mathcal{U}(j),\mathcal{R}(t)),\quad \mathbf{V}(t)\in\mathcal{L}(\bigoplus_{j\le t}\mathcal{V}(j),\mathcal{R}(t)).$$

(2.2c) $\{\mathbf{U}(t),\mathbf{V}(t)\}_{t\in\mathbb{Z}}$ *are uniformly bounded families:*

$$\exists\, c_u>0 \text{ and } c_v>0 \text{ such that } \|\mathbf{U}(t)\|\le c_u \text{ and } \|\mathbf{V}(t)\|\le c_v \text{ for all } t\in\mathbb{Z}.$$

The above assumptions imply that equation (2.1) has a unique uniformly bounded solution given by

$$R(t) = \mathbf{U}(t)\mathbf{U}^*(t) - \mathbf{V}(t)\mathbf{V}^*(t). \tag{2.3}$$

We should remark that assumptions (2.2a)-(2.2c) are automatically satisfied in some special, though frequent, cases such as:

1. $\{G(t)\}_{t\in\mathbf{Z}}$ is a uniformly bounded family, viz., $\exists\ c_g > 0$ such that $\|G(t)\| \leq c_g$, and $F(t) = 0$ for $|t|$ sufficiently large.
2. $\{G(t)\}_{t\in\mathbf{Z}}$ is a uniformly bounded family and $\{F(t)\}_{t\in\mathbf{Z}}$ is a *stable* family, *i.e.*, $\exists\ c_f > 0$ such that $\|F(t)\| \leq c_f < 1$, $\forall t$.
3. $\{F(t)\}_{t\in\mathbf{Z}}$ is a uniformly bounded family and $G(t) = 0$ for $|t|$ sufficiently large.

The following result shows that the existence of a Pick solution of (2.1) is equivalent to the existence of an upper-triangular contraction relating $\mathbf{U}(t)$ and $\mathbf{V}(t)$, which will play a fundamental role in subsequent sections.

THEOREM 2.2. *The displacement equation (2.1) has a Pick solution $R(t)$ if, and only if, there exists an upper-triangular contraction $S \in \mathcal{L}(\underset{t\in\mathbf{Z}}{\oplus}\mathcal{V}(t),\ \underset{t\in\mathbf{Z}}{\oplus}\mathcal{U}(t))$, ($\|S\| \leq 1$), such that*

$$\mathbf{V}(t) = \mathbf{U}(t)P_{\mathcal{U}}(t)S/\underset{j\leq t}{\oplus}\mathcal{V}(j) \ \textit{ for every } t \in \mathbf{Z}\ , \tag{2.4}$$

where $P_{\mathcal{U}}(t)$ denotes the orthogonal projection of $\underset{t\in\mathbf{Z}}{\oplus}\mathcal{U}(t)$ onto $\underset{j\leq t}{\oplus}\mathcal{U}(j)$.

Proof. One implication is immediate. If an upper-triangular contraction S exists such that (2.4) holds then the solution given by (2.3) is a Pick solution. The converse is a consequence of a commutant lifting theorem. Thus assume equation (2.1) has a Pick solution. Then $R(t) = \mathbf{U}(t)\mathbf{U}^*(t) - \mathbf{V}(t)\mathbf{V}^*(t)$ are positive operators for all $t \in \mathbf{Z}$. Hence, there exist contractive operators $\bar{S}(t)$ (*i.e.*, $\|\bar{S}(t)\| \leq 1$),

$$\bar{S}(t) \in \mathcal{L}(\underset{j\leq t}{\oplus}\mathcal{V}(j), \overline{\mathcal{R}(\mathbf{U}^*(t))})\ ,$$

such that $\mathbf{V}(t) = \mathbf{U}(t)\bar{S}(t)$ for all $t \in \mathbf{Z}$, where $\overline{\mathcal{R}(\mathbf{U}^*(t))}$ denotes the closure of the range of $\mathbf{U}^*(t)$. Let us define, for every $t \in \mathbf{Z}$, the shift (or marking) operator $M_{\mathcal{U}}(t) : \underset{j\leq t-1}{\oplus}\mathcal{U}(j) \longrightarrow \underset{j\leq t}{\oplus}\mathcal{U}(j)$,

$$M_{\mathcal{U}}(t) = \begin{bmatrix} \ddots & \ddots & & \\ & \mathbf{0} & I & \\ & & \mathbf{0} & I \\ & & & \mathbf{0} \end{bmatrix}.$$

It is easy to check that for all $t \in \mathbf{Z}$, $\mathbf{U}(t)M_{\mathcal{U}}(t) = F(t)\mathbf{U}(t-1)$ and $\mathbf{V}(t)M_{\mathcal{V}}(t) = F(t)\mathbf{V}(t-1)$. Hence,

$$\begin{aligned} M_{\mathcal{V}}^*(t)\bar{S}^*(t)\mathbf{U}^*(t) &= M_{\mathcal{V}}^*(t)\mathbf{V}^*(t) = \mathbf{V}^*(t-1)F^*(t) \\ &= \bar{S}^*(t-1)\mathbf{U}^*(t-1)F^*(t) = \bar{S}^*(t-1)M_{\mathcal{U}}^*(t)\mathbf{U}^*(t). \end{aligned}$$

We now use the comutant lifting theorem of Sarason-Sz.Nagy-Foias [13, 23] in its "time-variant" formulation in [4] – actually we use a slight variation in [8] – in order to conclude that there exists a family $\{\hat{S}(t) \in \mathcal{L}(\underset{j\leq t}{\oplus} \mathcal{V}(j), \underset{j\leq t}{\oplus} \mathcal{U}(j))\}_{t\in\mathbf{Z}}$ of contractions, with the properties: $\bar{S}^*(t) = \hat{S}^*(t)/\overline{\mathcal{R}(\mathbf{U}^*(t))}$, and $\hat{S}(t)M_{\mathcal{V}}(t) = M_{\mathcal{U}}(t)\hat{S}(t-1)$. This is a rather standard argument by now – see [21], the proof of Theorem VIII-2.2 in [13], or Theorem 5.C.4 in [16]. We then conclude from the last equality that there exists an upper-triangular contraction $S \in \mathcal{L}(\underset{t\in\mathbf{Z}}{\oplus} \mathcal{V}(t), \underset{t\in\mathbf{Z}}{\oplus} \mathcal{U}(t))$ such that $\bar{S}(t) = P_{\mathcal{U}}(t)S/\underset{j\leq t}{\oplus} \mathcal{V}(j)$. This can be viewed as a time-variant version of Lemma V-3.5 in [31]. Consequently, S satisfies (2.4) and the proof is complete. $\square$

We have thus shown that an upper-triangular contraction S can always be associated with a Pick solution of time-variant displacement equations of the form (2.1). This is a general result that includes, as special cases, solutions of several interpolation, completion, and moment problems considered in the literature. In fact, it will become clear throughout our discussion that the solutions of these problems correspond to determining the appropriate contraction S that is associated with the Pick solution $R(t)$ of (2.1) for specific choices of $F(t), G(t)$, and $J(t)$. Some examples to this effect are discussed in the next section. It should be noted though that the argument used in the above proof only assures the existence of S. It does not show how to construct such an S. We shall, however, describe later in Section 4 a recursive algorithm that, under suitable finite-dimensionality conditions, leads to a constructive proof of Theorem 2.2.

3. Connections with Completion Problems. In this section we illustrate the application of Theorem 2.2 to the solution of some moment and completion problems (and extensions thereof).

3.1. A Positive Completion Problem. We begin by considering the following moment-type problem. We fix a positive integer p and a family $\{\mathcal{E}(n)\}_{n\in\mathbf{Z}}$ of Hilbert spaces.

PROBLEM 3.1. *Given a family $\{\tilde{Q}_{ij}/\ i,j \in \mathbf{Z},\ \ |j-i| \leq p\}$ of operators such that $\tilde{Q}_{ij} = \tilde{Q}^*_{ji}$ and $\tilde{Q}_{ij} \in \mathcal{L}(\mathcal{E}(j), \mathcal{E}(i))$, it is required to find conditions for the existence of a positive definite kernel $M = [Q_{ij}]_{i,j\in\mathbf{Z}}$ such that for $i,j \in \mathbf{Z}$ and $|j-i| \leq p$ we have $Q_{ij} = \tilde{Q}_{ij}$.*

By a positive-definite kernel we mean an application $M = [Q_{ij}]_{i,j\in\mathbf{Z}}$ on $\mathbf{Z}\times\mathbf{Z}$ such that for $i,j \in \mathbf{Z}$ we have $Q_{ij} \in \mathcal{L}(\mathcal{E}(j), \mathcal{E}(i))$ and $\sum_{i,j=-n}^{n} < Q_{ij}h_j, h_i > \geq 0$, for every integer $n > 0$ and every set of vectors $\{h_{-n}, h_{-n+1}, \ldots, h_n\}$, $h_k \in \mathcal{E}(k)$, $|k| \leq n$. We show here how to solve the above problem by using Theorem 2.2 and connections with displacement structure theory.

We can assume, without loss of generality, that $\tilde{Q}_{ii} = I$ for all $i \in \mathbf{Z}$. We also define the Hilbert spaces $\mathcal{R}(t) = \overset{p}{\underset{k=0}{\oplus}} \mathcal{E}(-t+k), \mathcal{U}(t) = \mathcal{V}(t) = \mathcal{E}(-t)$, and the operators

$$U(t) = \begin{bmatrix} I \\ \tilde{Q}_{-t+1,-t} \\ \tilde{Q}_{-t+2,-t} \\ \vdots \\ \tilde{Q}_{-t+p,-t} \end{bmatrix}, \quad V(t) = \begin{bmatrix} \mathbf{0} \\ \tilde{Q}_{-t+1,-t} \\ \tilde{Q}_{-t+2,-t} \\ \vdots \\ \tilde{Q}_{-t+p,-t} \end{bmatrix}.$$

We further consider the operators $J(t) = (I_{\mathcal{U}(t)} \oplus -I_{\mathcal{V}(t)})$, $G(t) = \begin{bmatrix} U(t) & V(t) \end{bmatrix}$, and

$$F(t) = \begin{bmatrix} \mathbf{0} & & & \\ I & \mathbf{0} & & \\ & \ddots & \ddots & \\ & & I & \mathbf{0} \end{bmatrix}. \tag{3.1}$$

The elements $\{F(t), G(t), J(t)\}_{t\in\mathbf{Z}}$, as defined above, specify a displacement structure of the form (2.1). Using the result of Theorem 2.2 we conclude the following.

THEOREM 3.2. *Problem 3.1 has solutions if, and only if, the displacement equation (2.1), associated with the data $\{F(t), G(t), J(t)\}_{t\in\mathbf{Z}}$ defined above, has a Pick solution.*

Proof. Using the defined operators $\{F(t), G(t), J(t)\}_{t\in\mathbf{Z}}$ we readily check that the solution $R(t)$ of the corresponding (2.1) can be written as

$$R(t) = \mathbf{U}(t)\mathbf{U}^*(t) - \mathbf{V}(t)\mathbf{V}^*(t) = \begin{bmatrix} I & Q_{-t,-t+1} & & Q_{-t,-t+p} \\ Q_{-t+1,-t} & I & & Q_{-t+1,-t+p} \\ \vdots & & \ddots & \vdots \\ Q_{-t+p,-t} & Q_{-t+p,-t+1} & \cdots & I \end{bmatrix}.$$

We thus conclude that if Problem 3.1 has positive-definite solutions M then $R(t)$ is a positive block matrix for all $t \in \mathbf{Z}$. Conversely, if (2.1) has a Pick solution $R(t)$ then, by Theorem 2.2, there exists an upper triangular contraction $S = [S_{ij}]_{i,j\in\mathbf{Z}} \in \mathcal{L}(\underset{t\in\mathbf{Z}}{\oplus}\mathcal{V}(t), \underset{t\in\mathbf{Z}}{\oplus}\mathcal{U}(t))$ such that $\mathbf{V}(t) = \mathbf{U}(t)P_{\mathcal{U}}(t)S/\underset{j\leq t}{\oplus}\mathcal{V}(j)$. If we take the structure of $\mathbf{U}(t)$ and $\mathbf{V}(t)$ into account we then conclude that $S_{tt} = \mathbf{0}$ for all $t \in \mathbf{Z}$. We define $Q_{ij} = \sum_{k=j+1}^{i-1} Q_{i,k}S_{-k,-j} + S_{-i,-j}$, for $i > j$, $|j-i| > p$, $Q_{ij} = Q_{ji}^*$ for $i < j$, $|j-i| > p$, and $Q_{ij} = \tilde{Q}_{ij}$ for $|j-i| \leq p$, and consider the kernel $M = [Q_{ij}]_{i,j\in\mathbf{Z}}$. We now check that M is indeed a positive definite solution of Problem 3.1. For this purpose, we consider a positive integer $N > p$ and define the operators

$$U_N(t) = \begin{bmatrix} I \\ Q_{-t+1,-t} \\ \vdots \\ Q_{-t+N,-t} \end{bmatrix}, \quad V_N(t) = \begin{bmatrix} 0 \\ Q_{-t+1,-t} \\ \vdots \\ Q_{-t+N,-t} \end{bmatrix}.$$

Then

$$\begin{aligned} \begin{bmatrix} I & Q_{-t,-t+1} & & Q_{-t,-t+N} \\ Q_{-t+1,-t} & I & & Q_{-t+1,-t+N} \\ \vdots & & & \vdots \\ Q_{-t+N,-t} & Q_{-t+N,-t+1} & \cdots & I \end{bmatrix} &= \mathbf{U}_N(t)\mathbf{U}_N^*(t) - \mathbf{V}_N(t)\mathbf{V}_N^*(t) \\ &= \mathbf{U}_N(t)\left[I - S_tS_t^*\right]\mathbf{U}_N^*(t) \ , \end{aligned}$$

where $S_t = P_{\mathcal{U}}(t)S/\underset{j\leq t}{\oplus}\mathcal{V}(j)$. Consequently, M is positive-definite. □

We further remark that the well-known trigonometric moment problem [1] corresponds to the special case $\tilde{Q}_{ij} = \tilde{Q}_{|j-i|}$ (*i.e.*, the entries of the specified band exhibit a Toeplitz structure). The case $\mathcal{E}(n) = \mathbf{0}$, for $|n|$ large enough, was considered and solved in [12].

3.2. The Hermite-Fejér Interpolation Problem. The solution of the positive completion problem corresponds to the special choice of the operator $F(t)$ as in (3.1). The statement of Theorem 2.2, however, allows for other choices of $F(t)$. Indeed, choosing $F(t)$ in a general Jordan form leads to the solution of a general interpolation problem of the Hermite-Fejér type as we now elaborate.

We first introduce some notation. We consider three families of Hilbert spaces $\{\mathcal{U}(t), \mathcal{V}(t), \mathcal{F}\}_{t\in\mathbf{Z}}$, and a bounded upper-triangular operator

$$\mathbf{T} \in \mathcal{L}(\bigoplus_{t\in\mathbf{Z}} \mathcal{U}(t), \bigoplus_{t\in\mathbf{Z}} \mathcal{V}(t)), \quad \mathbf{T} = [T_{ij}]_{i,j\in\mathbf{Z}}, \quad T_{ij} \in \mathcal{L}(\mathcal{U}(j), \mathcal{V}(i)),$$

$$\mathbf{T} = \begin{bmatrix} \ddots & \ddots & & & & \\ & T_{-1,-1} & T_{-1,0} & T_{-1,1} & \cdots & \\ & & \boxed{T_{00}} & T_{01} & T_{02} & \cdots \\ & O & & T_{11} & T_{12} & T_{13} \\ & & & & \ddots & \ddots \end{bmatrix},$$

where $\boxed{T_{00}}$ denotes the $(0,0)$ entry of $\mathbf{T}$. We further consider a *stable* family of operators $\{f(t) \in \mathcal{L}(\mathcal{F})\}_{t\in\mathbf{Z}}$, viz., $\exists\ c > 0$ such that $|f(t)| < c < 1$ for all $t \in \mathbf{Z}$. We also introduce the symmetric functions $s_k^{(n)}$ of n variables (taken k at a time). That is, $s_0^{(n)} = 1$ and

$$s_k^{(n)}(x_1, x_2, \ldots, x_n) = \sum_{1\le i_1<\ldots<i_k\le n} x_{i_1} x_{i_2} \ldots x_{i_k}.$$

For example, $n = 3$ corresponds to $s_0^{(3)} = 1$, $s_1^{(3)}(x_1, x_2, x_3) = x_1 + x_2 + x_3$, $s_2^{(3)}(x_1, x_2, x_3) = x_1x_2 + x_1x_3 + x_2x_3$, and $s_3^{(3)}(x_1, x_2, x_3) = x_1x_2x_3$.

For a uniformly bounded family of operators $\{u(t) \in \mathcal{L}(\mathcal{V}(t), \mathcal{F})\}_{t\in\mathbf{Z}}$, viz., $\exists\ \bar{c} > 0$ such that $\|u(t)\| < \bar{c}$ for all t, we define the operator $u(t) \bullet \mathbf{T}(f(t))$ as follows

$$u(t) \bullet \mathbf{T}(f(t)) = u(t)T_{tt} + f(t)u(t-1)T_{t-1,t} + f(t)f(t-1)u(t-2)T_{t-2,t} + \ldots.$$

This corresponds to a time-variant tangential evaluation along the "direction" defined by $u(t)$. More generally, we define the operators (for $p \geq 0$)

$$u(t) \bullet \frac{1}{p!}\mathbf{T}^{(p)}(f(t)) \equiv \sum_{m=0}^{\infty} s_m^{(m+p)}\left[f(t), f(t-1), \ldots, f(t-m-p+1)\right] u(t-m-p)T_{t-m-p,t}\,.$$

We shall also use the compact notation $\begin{bmatrix} u_1(t) & u_2(t) \end{bmatrix} \bullet H_{\mathbf{T}}^2(f(t))$ to denote the operator $\begin{bmatrix} u_1(t) \bullet \mathbf{T}(f(t)) & u_1(t) \bullet \frac{1}{1!}\mathbf{T}^{(1)}(f(t)) + u_2(t) \bullet \mathbf{T}(f(t)) \end{bmatrix}$, which we also write as

$$\begin{bmatrix} u_1(t) & u_2(t) \end{bmatrix} \bullet \begin{bmatrix} \mathbf{T}(f(t)) & \frac{1}{1!}\mathbf{T}^{(1)}(f(t)) \\ & \mathbf{T}(f(t)) \end{bmatrix}.$$

More generally, we write $\begin{bmatrix} u_1(t) & u_2(t) & \ldots & u_r(t) \end{bmatrix} \bullet H_{\mathbf{T}}^r(f(t)) =$

$$\begin{bmatrix} u_1(t) & u_2(t) & \ldots & u_r(t) \end{bmatrix} \bullet \begin{bmatrix} \mathbf{T}(f(t)) & \frac{1}{1!}\mathbf{T}^{(1)}(f(t)) & \cdots & \frac{1}{(r-1)!}\mathbf{T}^{(r-1)}(f(t)) \\ & \mathbf{T}(f(t)) & \cdots & \frac{1}{(r-2)!}\mathbf{T}^{(r-2)}(f(t)) \\ & \ddots & & \vdots \\ & O & \mathbf{T}(f(t)) & \frac{1}{1!}\mathbf{T}^{(1)}(f(t)) \\ & & & \mathbf{T}(f(t)) \end{bmatrix}.$$

To formulate the Hermite-Fejér problem we again consider three families of Hilbert spaces $\{\mathcal{U}(t), \mathcal{V}(t), \mathcal{F}_i\}_{t\in\mathbf{Z}, 0\le i<m}$, and m stable families $\{\alpha_i(t) \in \mathcal{L}(\mathcal{F}(t))\}_{t\in\mathbf{Z}}$ for $i = 0, 1, \dots, m-1$. We associate with each $\alpha_i(t)$ a positive integer $r_i \ge 1$ and uniformly bounded families of operators $\mathbf{a}_i(t)$ and $\mathbf{b}_i(t)$ partitioned as follows

$$\mathbf{a}_i(t) = \begin{bmatrix} u_1^{(i)}(t) & u_2^{(i)}(t) & \dots & u_{r_i}^{(i)}(t) \end{bmatrix}, \quad \mathbf{b}_i(t) = \begin{bmatrix} v_1^{(i)}(t) & v_2^{(i)}(t) & \dots & v_{r_i}^{(i)}(t) \end{bmatrix},$$

where $u_j^{(i)}(t) \in \mathcal{L}(\mathcal{U}(t), \mathcal{F}_i)$ and $v_j^{(i)}(t) \in \mathcal{L}(\mathcal{V}(t), \mathcal{F}_i), j = 1, \dots, r_i$, are uniformly bounded families of operators.

PROBLEM 3.3. *Given m stable families $\{\alpha_i(t)\}$ with the associated uniformly bounded data $\mathbf{a}_i(t)$ and $\mathbf{b}_i(t)$, as described above, it is required to find necessary and sufficient conditions for the existence of upper-triangular contractive operators S ($\|S\|_\infty \le 1$) that satisfy*

$$\mathbf{b}_i(t) = \mathbf{a}_i(t) \bullet H_S^{r_i}(\alpha_i(t)) \quad \textit{for} \quad 0 \le i \le m-1 \ \textit{and}\ t \in \mathbf{Z}. \tag{3.2}$$

The first step in the solution consists in constructing three operators $F(t), G(t)$, and $J(t)$ directly from the interpolation data: $F(t)$ contains the information relative to the operators $\{\alpha_i(t)\}$ and the dimensions $\{r_i\}$, $G(t)$ contains the information relative to the direction operators $\{\mathbf{a}_i(t), \mathbf{b}_i(t)\}$, and $J(t) = (I_{\mathcal{U}(t)} \oplus -I_{\mathcal{V}(t)})$. Define, for $i = 0, 1, \dots, m-1$, $\mathcal{R}_i(t) = \mathcal{F}_i \oplus \mathcal{F}_i \oplus \dots \oplus \mathcal{F}_i$ (r_i times), and $\mathcal{R}(t) = \overset{m-1}{\underset{i=0}{\oplus}} \mathcal{R}_i(t)$. The operators $F(t)$ and $G(t)$ are then constructed as follows: we associate with each $\alpha_i(t)$ an operator in Jordan form $\bar{F}_i(t) \in \mathcal{L}(\mathcal{R}_i(t-1), \mathcal{R}_i(t))$ $(= \mathcal{L}(\mathcal{R}_i(t))$, in this case),

$$\bar{F}_i(t) = \begin{bmatrix} \alpha_i(t) & & & \\ 1 & \alpha_i(t) & & \\ & \ddots & \ddots & \\ & & 1 & \alpha_i(t) \end{bmatrix},$$

and two operators $U_i(t)$ and $V_i(t)$, respectively, which are composed of the operators associated with $\alpha_i(t)$, viz.,

$$U_i(t) = \begin{bmatrix} u_1^{(i)}(t) \\ u_2^{(i)}(t) \\ \vdots \\ u_{r_i}^{(i)}(t) \end{bmatrix}, \quad V_i(t) = \begin{bmatrix} v_1^{(i)}(t) \\ v_2^{(i)}(t) \\ \vdots \\ v_{r_i}^{(i)}(t) \end{bmatrix}.$$

Then $F(t) =$ diagonal $\{\bar{F}_0(t), \bar{F}_1(t), \dots, \bar{F}_{m-1}(t)\}$ and

$$G(t) = \begin{bmatrix} U_0(t) & V_0(t) \\ U_1(t) & V_1(t) \\ \vdots & \vdots \\ U_{m-1}(t) & V_{m-1}(t) \end{bmatrix} \equiv \begin{bmatrix} \mathbf{U}(t) & \mathbf{V}(t) \end{bmatrix}.$$

We shall denote the diagonal entries of $F(t)$ by $\{f_i(t)\}_{i=0}^{n-1}$ (for example, $f_0(t) = f_1(t) = \dots = f_{r_0-1}(t) = \alpha_0(t)$). We have thus specified all the elements of a displacement equation as in (2.1).

THEOREM 3.4. *The tangential Hermite-Fejér Problem 3.3 is solvable if, and only if, the displacement equation (2.1), associated with the interpolation data above, has a Pick solution.*

Proof. The result follows by showing that the interpolation conditions (3.2) follow from Theorem 2.2. The assumptions made in the statement of Problem 3.3 guarantee that conditions (2.2a)-(2.2c) are satisfied. If $R(t)$ is a Pick solution then there exists an upper-triangular contraction S that satisfies (2.4). Now, by comparing terms on both sides of (2.4) and by invoking the special constructions of $\{F(t), G(t)\}$ as above, we conclude that expression (2.4) can be rewritten as

$$\mathbf{b}_i(t) = \mathbf{a}_i(t) \bullet H_S^{r_i}(\alpha_i(t)) \ , \quad \text{for} \ \ i = 0, 1, \ldots, m-1 \ ,$$

which is the desired interpolation property (3.2). Conversely, assume there exists an interpolating solution S that satisfies (3.2). Then, by comparing terms on both sides of (3.2), we conclude that the i^{th} entry of $\mathbf{U}(t)P_{\mathcal{U}}(t)S / \underset{j\leq t}{\oplus} \mathcal{V}(j)$ is the i^{th} entry of $\mathbf{V}(t)$. Hence, S satisfies $\mathbf{V}(t) = \mathbf{U}(t)P_{\mathcal{U}}(t)S / \underset{j\leq t}{\oplus} \mathcal{V}(j)$ for every $t \in \mathbb{Z}$. Consequently, $R(t)$ is a Pick solution. □

3.3. A Special Case: The Carathéodory-Fejér Problem. The Hermite-Fejér problem includes as a special case the following so-called Carathéodory-Fejér problem.

PROBLEM 3.5. *Given families of Hilbert spaces $\{\mathcal{U}(t), \mathcal{V}(t)\}_{t\in\mathbb{Z}}$, and n families $\{\beta_i(t)\}$, $i = 0, 1, \ldots, n-1$, of operators $\beta_i(t) \in \mathcal{L}(\mathcal{V}(t), \mathcal{U}(t-n+1))$, it is required to find necessary and sufficient conditions for the existence of an upper-triangular contraction $S \in \mathcal{L}(\underset{t\in\mathbb{Z}}{\oplus} \mathcal{V}(t), \underset{t\in\mathbb{Z}}{\oplus} \mathcal{U}(t))$, $S = [S_{ij}]_{i,j\in\mathbb{Z}}$, such that for all $t \in \mathbb{Z}$ we have $S_{t-i,t} = \beta_i(t)$ for $i = 0, 1, \ldots, n-1$.*

The classical Carathéodory-Fejér-Schur interpolation problem [1] corresponds to the special case $\beta_i(t) = \beta_i$ for all $t \in \mathbb{Z}$ and $i = 0, 1, \ldots, n-1$. Several other contractive completion problems, such as those considered in [4], also follow as special cases of Problem 3.5 by choosing $\beta_i(t) = 0$ for sufficiently large values of t.

To put the above problem into our framework, as described in the previous section, we construct the operators

$$U(t) = \begin{bmatrix} I \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix} \ , \ \ V(t) = \begin{bmatrix} \beta_0(t) \\ \beta_1(t) \\ \vdots \\ \beta_{n-1}(t) \end{bmatrix} \ ,$$

as well as $J(t) = (I_{\mathcal{U}(t)} \oplus -I_{\mathcal{V}(t)})$, $G(t) = \begin{bmatrix} U(t) & V(t) \end{bmatrix}$, and

$$F(t) = \begin{bmatrix} \mathbf{0} & & & \\ I & \mathbf{0} & & \\ & \ddots & \ddots & \\ & & I & \mathbf{0} \end{bmatrix} . \tag{3.3}$$

COROLLARY 3.6. *Problem 3.5 has solutions, if and only if, $\|M(t)\| \leq 1$ for all*

$t \in \mathbb{Z}$, where

$$M(t) = \begin{bmatrix} \beta_0(t) & & & \\ \beta_1(t) & \beta_0(t-1) & & \\ \vdots & & \ddots & \\ \beta_{n-1}(t) & \dots & & \beta_0(t-n+1) \end{bmatrix}.$$

3.4. A Special Case: The Nevanlinna-Pick Problem. The Hermite-Fejér problem also includes as a special case, the following so-called time-variant version of Nevanlinna-Pick introduced in [10] and further studied and extended in [5].

PROBLEM 3.7. *Given families of Hilbert spaces $\{\mathcal{F}_i, \mathcal{U}(t), \mathcal{V}(t)\}_{t\in\mathbb{Z}}$, and n stable families of operators $\{\alpha_i(t)\}$, $i = 0, 1, \dots, n-1$, $\alpha_i(t) \in \mathcal{L}(\mathcal{F}_i)$, with two uniformly bounded families of operators $\{u_i(t), v_i(t)\}_{t\in\mathbb{Z}}$, $i = 0, 1, \dots, n-1$, $u_i(t) \in \mathcal{L}(\mathcal{U}(t), \mathcal{F}_i)$, $v_i(t) \in \mathcal{L}(\mathcal{V}(t), \mathcal{F}_i)$, it is required to find necessary and sufficient conditions for the existence of an upper-triangular contraction $S \in \mathcal{L}(\bigoplus_{t\in\mathbb{Z}} \mathcal{V}(t), \bigoplus_{t\in\mathbb{Z}} \mathcal{U}(t))$ such that for all $t \in \mathbb{Z}$ we have $u_i(t) \bullet S(\alpha_i(t)) = v_i(t), \quad i = 0, 1, \dots, n-1$.*

The classical Nevanlinna-Pick problem [1] corresponds to the special case $\alpha_i(t) = \alpha_i$, $u_i(t) = u_i$, and $v_i(t) = v_i$ for all $t \in \mathbb{Z}$ and $i = 0, 1, \dots, n-1$. Following the construction given in the previous section we get

$$F(t) = \begin{bmatrix} \alpha_0(t) & & & O \\ & \alpha_1(t) & & \\ O & & \ddots & \\ & & & \alpha_{n-1}(t) \end{bmatrix}, \quad G(t) = \begin{bmatrix} u_0(t) & v_0(t) \\ u_1(t) & v_1(t) \\ \vdots & \vdots \\ u_{n-1}(t) & v_{n-1}(t) \end{bmatrix}.$$

COROLLARY 3.8. *The Nevanlinna-Pick Problem 3.7 has solutions if, and only if,*

$$\left[\left\{ u_i(t)u_j^*(t) - v_i(t)v_j^*(t) \right\} \bullet N_{\alpha_j^*}(\alpha_i(t)) \right]_{i,j=0}^{n-1} \geq 0 \quad \textit{for all } t \in \mathbb{Z} ,$$

where, for a stable family $\{\alpha(t)\}_{t\in\mathbb{Z}}$, the upper-triangular operator N_α is defined by $(N_\alpha)_{tt} = I$, and $(N_\alpha)_{t-j,t} = \alpha(t-j+1)\alpha(t-j+2)\dots\alpha(t)$ for $j \geq 1$. (The stability of $\{\alpha(t)\}_{t\in\mathbb{Z}}$ assures that N_α is a well-defined bounded operator).

4. A Recursive Solution. The examples considered in the previous sections reveal the significance of Theorem 2.2 in the solution of moment and interpolation problems. However, the result of Theorem 2.2 only provides an existence statement, *i.e.*, it only assures the existence of an upper-triangular contraction S that satisfies (2.4). It does not show how to construct or find such an S. The several applications mentioned above though, motivate the need for an alternative route that would also lead to a construction of S. In this section we shall, following the arguments in [25, 26, 28, 24], describe a recursive procedure that will lead us to what we shall call a *generalized Schur algorithm*, and which will allow us to provide an implementation for S in terms of a cascade of elementary sections. The derivation that follows, however, is only applicable to the case where the involved Hilbert spaces are finite-dimensional. More specifically, we consider displacement equations as in (2.1), viz.,

$$R(t) - F(t)R(t-1)F^*(t) = G(t)J(t)G^*(t) \quad ,$$

where $R(t) \in \mathcal{L}(\mathcal{R}(t))$, $G(t) \in \mathcal{L}(\mathcal{U}(t) \oplus \mathcal{V}(t), \mathcal{R}(t))$, $F(t) \in \mathcal{L}(\mathcal{R}(t-1), \mathcal{R}(t))$, and $J(t) = (I_{\mathcal{U}(t)} \oplus -I_{\mathcal{V}(t)})$, and assume the following *finite-dimensionality* conditions:

(4.1a) *There exists a finite positive integer* n *such that* $\mathcal{R}(t) = \overset{n-1}{\underset{i=0}{\oplus}} \mathcal{R}_i(t), \ \forall\, t$;

(4.1b) $\dim \mathcal{R}_i(t)$ *are all equal and finite for all* $t \in \mathbb{Z}$ and $i = 0, 1, \ldots, n-1$;

(4.1c) $\dim \mathcal{U}(t) = p(t) < \infty$ *and* $\dim \mathcal{V}(t) = q(t) < \infty$ for all $t \in \mathbb{Z}$.

We further assume that:

(4.1d) $\{F(t)\}$ *is a uniformly bounded family of lower triangular operators with stable families of diagonal entries* $\{f_i(t)\}_{i=0}^{n-1}$;

(4.1e) $\{G(t)\}$ *is a uniformly bounded family.*

By condition (4.1a) we can write $R(t) = [r_{ij}(t)]_{i,j=0}^{n-1}$, with block entries $r_{ij}(t) \in \mathcal{L}(\mathcal{R}_j(t), \mathcal{R}_i(t))$.

4.1. A Time-Variant Embedding Relation. A major tool in our analysis is a so-called embedding result for displacement equations. This result was derived in [14] in the time-invariant case and further explored and discussed in [19]. Its relevance to rational interpolation problems was detailed in [24, 25, 27], and in connection with time-variant interpolation problems in [5, 25, 26]. Here we discuss the general time-variant case following the pattern developed in [25, 26].

For this purpose, we consider again the time-variant displacement equation (2.1) and, in addition, assume that $\{R(t)\}_{t\in\mathbb{Z}}$ is also uniformly bounded from below, viz.,

(4.1f) $\exists\ r_1 > 0$ *such that* $0 < r_1 I \leq R(t)$ *for all* $t \in \mathbb{Z}$.

THEOREM 4.1. *Suppose (4.1a)-(4.1f) hold, then there exist uniformly bounded families of operators* $\{H(t)\}_{t\in\mathbb{Z}}$ *and* $\{K(t)\}_{t\in\mathbb{Z}}$,

$$H(t) \in \mathcal{L}(\mathcal{R}(t-1), \mathcal{U}(t) \oplus \mathcal{V}(t)) \ , \quad K(t) \in \mathcal{L}(\mathcal{U}(t) \oplus \mathcal{V}(t)) \ ,$$

such that the following time-variant embedding relation is satisfied

$$\begin{bmatrix} F(t) & G(t) \\ H(t) & K(t) \end{bmatrix} \begin{bmatrix} R(t-1) & \mathbf{0} \\ \mathbf{0} & J(t) \end{bmatrix} \begin{bmatrix} F(t) & G(t) \\ H(t) & K(t) \end{bmatrix}^* = \begin{bmatrix} R(t) & \mathbf{0} \\ \mathbf{0} & J(t) \end{bmatrix}. \tag{4.2}$$

Proof. It is easy to check (as in [19, 24, 25]) that the following choices for $H(t)$ and $K(t)$ satisfy the embedding relation (we use the notation $\Theta^{-1}(t)$ to mean $(\Theta(t))^{-1}$):

$$H(t) = \Theta^{-1}(t) J(t) G^*(t) \left[R^{\frac{*}{2}}(t) - \tau(t) R^{\frac{*}{2}}(t-1) F^*(t) \right]^{-1} \left[\tau(t) R^{-\frac{1}{2}}(t-1) - R^{-\frac{1}{2}}(t) F(t) \right],$$

$$K(t) = \Theta^{-1}(t) \left\{ I - J(t) G^*(t) \left[R^{\frac{*}{2}}(t) - \tau(t) R^{\frac{*}{2}}(t-1) F^*(t) \right]^{-1} R^{-\frac{1}{2}}(t) G(t) \right\},$$

for an arbitrary $J(t)$–unitary operator $\Theta(t)$ and an arbitrary unitary operator $\tau(t)$, whenever the inverse of $R^{\frac{*}{2}}(t) - \tau(t) R^{\frac{*}{2}}(t-1) F^*(t)$ exists. Here, $R^{\frac{1}{2}}(t)$ denotes the operator defined by $R(t) = R^{\frac{1}{2}}(t) R^{\frac{*}{2}}(t)$. [The finite-dimensionality conditions guarantee that it is always possible to choose a unitary matrix $\tau(t)$ so as to assure the invertibility of $R^{\frac{*}{2}}(t) - \tau(t) R^{\frac{*}{2}}(t-1) F^*(t)$].

We now show that we can choose $\Theta(t)$ and $\tau(t)$ adequately so as to guarantee the uniform boundedness of the families $\{H(t), K(t)\}_{t\in\mathbb{Z}}$. By our hypothesis, there exist $r_1 > 0$ and $r_2 > 0$, independent of t, such that $0 < r_1 I \leq R(t) \leq r_2 I$ for all $t \in \mathbb{Z}$. It follows that we can always find $\tau(t)$ such that

$$\left[R^{\frac{*}{2}}(t) - \tau(t)R^{\frac{*}{2}}(t-1)F^*(t)\right]\left[R^{\frac{*}{2}}(t) - \tau(t)R^{\frac{*}{2}}(t-1)F^*(t)\right]^* \geq \epsilon I > 0 \ ,$$

for some $\epsilon > 0$. Indeed, define $A(t) = R^{\frac{1}{2}}(t-1)F^*(t)R^{-\frac{1}{2}}(t)$. If $A(t) = 0$ then the claim is obvious, otherwise write $A(t) = (A_1(t) \oplus \mathbf{0})$ with respect to the decompositions $\mathcal{R}(A^*(t)) \oplus \mathcal{K}er\ A(t)$ and $\mathcal{R}(A(t)) \oplus \mathcal{K}er\ A^*(t)$ of $\mathcal{R}(t)$ and $\mathcal{R}(t-1)$, respectively. We readily conclude that $A_1(t)$ is invertible. If we define $\tau(t) = (-A_1^*(t)\left[A_1(t)A_1^*(t)\right]^{-1/2} \oplus B(t))$, with respect to the above decompositions, and for an arbitrary unitary operator $B(t)$, then it follows that $[\tau^*(t) - A(t)]\,[\tau(t) - A^*(t)] \geq I$. Therefore, $R^{\frac{*}{2}}(t) - \tau(t)R^{\frac{*}{2}}(t-1)F^*(t)$ is invertible and the family

$$\{\left[R^{\frac{*}{2}}(t) - \tau(t)R^{\frac{*}{2}}(t-1)F^*(t)\right]^{-1}\}_{t\in\mathbb{Z}}$$

is uniformly bounded. Taking $\Theta(t) = I$ for all $t \in \mathbb{Z}$ leads to uniformly bounded families $\{H(t), K(t)\}_{t\in\mathbb{Z}}$. □

4.2. Generalized Schur Algorithm. We now use the embedding result of Theorem 4.1 to derive a generalized Schur algorithm for block matrices $R(t) = [r_{ij}(t)]_{i,j=0}^{n-1}$ along the lines presented in [25, 26, 28]. More precisely, we focus on the time-variant displacement equation (2.1) and show that it allows the successive computation of the Schur complements of $R(t)$ to be reduced to a computationally efficient recursive procedure applied to the so-called generator matrix $G(t)$.

Let $R_i(t)$ denote the Schur complement of the leading $i \times i$ submatrix of $R(t)$. If $l_i(t)$ and $d_i(t)$ stand for the the first column and the $(0,0)$ entry of $R_i(t)$, respectively, then (the positive-definiteness of $R(t)$ guarantees $d_i(t) > 0$ for all i)

$$R_i(t) - l_i(t)d_i^{-1}(t)l_i^*(t) = \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & R_{i+1}(t) \end{bmatrix} \equiv \tilde{R}_{i+1}(t). \tag{4.3}$$

Hence, starting with an $n \times n$ matrix $R(t)$ and performing n consecutive Schur complement steps, we obtain the triangular factorization of $R(t)$, viz.,

$$R(t) = l_0(t)d_0^{-1}(t)l_0^*(t) + \begin{bmatrix} 0 \\ l_1(t) \end{bmatrix} d_1^{-1}(t) \begin{bmatrix} 0 \\ l_1(t) \end{bmatrix}^* + \ldots \ = \ L(t)D^{-1}(t)L^*(t) \ ,$$

where $D(t) = \text{diag}\{d_0(t), \ldots, d_{n-1}(t)\}$ ($D^{-1}(t)$ stands for $(D(t))^{-1}$), and the (nonzero parts of the) columns of the lower triangular matrix $L(t)$ are $\{l_0(t), \ldots, l_{n-1}(t)\}$. The point, however, is that this procedure can be speeded up for matrices $R(t)$ that exhibit a time-variant displacement structure as in (2.1). In this case, the above (Gauss/Schur) reduction procedure can be shown to reduce to a recursion on the elements of the generator matrix $G(t)$. The computational advantage then follows from the fact that the column dimension of $G(t)$, viz., $r(t) = p(t) + q(t)$, is usually small when compared to the dimension of $R(t)$. The following theorem shows that the triangular factor at time $(t-1)$, viz., $L(t-1)$, can be time-updated to $L(t)$ via a recursive procedure on $G(t)$.

THEOREM 4.2. *The Schur complements $R_i(t)$ are also structured with generator matrices $G_i(t)$, viz., $R_i(t) - F_i(t)R_i(t-1)F_i^*(t) = G_i(t)J(t)G_i^*(t)$, where $G_i(t)$ is* *a matrix that satisfies, along with* $l_i(t)$*, the following generator recursion:* $G_0(t) = G(t)$, $F_0(t) = F(t)$,

$$\begin{bmatrix} l_i(t) & \mathbf{0} \\ & G_{i+1}(t) \end{bmatrix} = \begin{bmatrix} F_i(t)l_i(t-1) & G_i(t) \end{bmatrix} \begin{bmatrix} f_i^*(t) & h_i^*(t)J(t) \\ J(t)g_i^*(t) & J(t)k_i^*(t)J(t) \end{bmatrix},$$

where $g_i(t)$ *is the top (block) row of* $G_i(t)$, $F_i(t)$ *is the submatrix obtained after deleting the first (block) row and column of* $F_{i-1}(t)$*, and* $h_i(t)$ *and* $k_i(t)$ *are arbitrary matrices chosen so as to satisfy the time-variant embedding relation*

$$\begin{bmatrix} f_i(t) & g_i(t) \\ h_i(t) & k_i(t) \end{bmatrix} \begin{bmatrix} d_i(t-1) & \mathbf{0} \\ \mathbf{0} & J(t) \end{bmatrix} \begin{bmatrix} f_i(t) & g_i(t) \\ h_i(t) & k_i(t) \end{bmatrix}^* = \begin{bmatrix} d_i(t) & \mathbf{0} \\ \mathbf{0} & J(t) \end{bmatrix} , \tag{4.4}$$

where $d_i(t)$ *satisfies the time-update* $d_i(t) = f_i(t)d_i(t-1)f_i^*(t) + g_i(t)J(t)g_i^*(t)$.

Proof. We prove the result for $i = 0$. The same argument is valid for $i \geq 1$. Let $d_0(t), l_0(t)$, and $g_0(t)$, denote the $(0,0)$ (block) entry of $R(t)$, the first (block) column of $R(t)$, and the first (block) row of $G(t)$, respectively. It then follows from the displacement equation (2.1) that $l_0(t) = F(t)l_0(t-1)f_0^*(t) + G(t)J(t)g_0^*(t)$ and $d_0(t) = f_0(t)d_0(t-1)f_0^*(t) + g_0(t)J(t)g_0^*(t)$. Let $F_1(t)$ be the submatrix obtained after deleting the first (block) row and column of $F(t)$. Using the expressions for $l_0(t), d_0(t)$, and (4.3), it is straightforward to check that we can write $\tilde{R}_1(t) - F(t)\tilde{R}_1(t-1)F^*(t) =$

$$\begin{aligned} &= G(t)J(t)\left\{J(t) - g_0^*(t)d_0^{-1}(t)g_0(t)\right\}J(t)G^*(t) \\ &\quad - F(t)l_0(t-1)f_0^*(t)d_0^{-1}(t)g_0(t)J(t)G^*(t) \\ &\quad - G(t)J(t)g_0^*(t)d_0^{-1}(t)f_0(t)l_0^*(t-1)F^*(t) \\ &\quad - F(t)l_0(t-1)\left[d_0^{-1}(t-1) - f_0^*(t)d_0^{-1}(t)f_0(t)\right]l_0^*(t-1)F^*(t) . \end{aligned} \tag{4.5a}$$

We now verify that the right-hand side of the above expression can be put into the form of a *perfect square* by introducing some auxiliary quantities. Consider a (block) column vector $h_0(t)$ and a matrix $k_0(t)$ that are defined to satisfy the following relations (in terms of the quantities that appear on the right-hand side of the above expression - this is always possible as explained ahead)

$$h_0^*(t)J(t)h_0(t) = d_0^{-1}(t-1) - f_0^*(t)d_0^{-1}(t)f_0(t) , \tag{4.5b}$$

$$k_0^*(t)J(t)k_0(t) = J(t) - g_0^*(t)d_0^{-1}(t)g_0(t) , \quad k_0^*(t)J(t)h_0(t) = g_0^*(t)d_0^{-1}(t)f_0(t) .$$

Using $\{h_0(t), k_0(t)\}$, we can factor the right-hand side of (4.5a) as $\tilde{G}_1(t)J(t)\tilde{G}_1^*(t)$, where $\tilde{G}_1(t) = F(t)l_0(t-1)h_0^*(t)J(t) + G(t)J(t)k_0^*(t)J(t)$. Recall that the first (block) row and column of $\tilde{R}_1(t)$ are zero. Hence, the first (block) row of $\tilde{G}_1(t)$ is zero, $\tilde{G}_1(t) = \begin{bmatrix} \mathbf{0} & G_1^{\mathbf{T}}(t) \end{bmatrix}^{\mathbf{T}}$. Moreover, it follows from (4.5b) (and from the expression for $d_0(t)$) that

$$\begin{bmatrix} f_0(t) & g_0(t) \\ h_0(t) & k_0(t) \end{bmatrix}^* \begin{bmatrix} d_0^{-1}(t) & \mathbf{0} \\ \mathbf{0} & J(t) \end{bmatrix} \begin{bmatrix} f_0(t) & g_0(t) \\ h_0(t) & k_0(t) \end{bmatrix} = \begin{bmatrix} d_0^{-1}(t-1) & \mathbf{0} \\ \mathbf{0} & J(t) \end{bmatrix} ,$$

which is equivalent to (4.4) for $i = 0$. □

The existence of uniformly bounded families $\{h_i(t), k_i(t)\}_{t\in\mathbf{Z}}$ that satisfy (4.4) follow as a special case of Theorem 4.1, since $d_i(t)$ satisfies a time-variant displacement equation, viz.,

$$d_i(t) = f_i(t)d_i(t-1)f_i^*(t) + g_i(t)J(t)g_i^*(t) \ ,$$

the finite-dimensionality conditions stated prior to Theorem 4.1 are satisfied, and the families $\{d_i(t), g_i(t)\}_{t\in\mathbf{Z}}$ are uniformly bounded as shown next.

LEMMA 4.3. *The sequences $\{d_i(t)\}_{t\in\mathbf{Z}}$ and $\{g_i(t)\}_{t\in\mathbf{Z}}$ obtained through the recursive Schur reduction procedure are uniformly bounded. More specifically, there exist real numbers b_d, c_d, and c_v (independent of t) such that*

$$0 < b_d I < d_i(t) < c_d I\ ,\quad \|g_i(t)\| < c_v \quad \textit{for all } t.$$

Proof. It is clear $\{d_0(t)\}_{t\in\mathbf{Z}}$ is uniformly bounded from above since $\{f_0(t)\}_{t\in\mathbf{Z}}$ is stable and $\{g_0(t)J(t)g_0^*(t)\}_{t\in\mathbf{Z}}$ is uniformly bounded. A similar argument shows that $\{l_0(t)\}_{t\in\mathbf{Z}}$ is also uniformly bounded. It further follows from $0 < r_1 I \leq R(t)$ that the sequence $\{d_0(t)\}_{t\in\mathbf{Z}}$ is uniformly bounded from below, viz., $d_0(t) \geq r_1 I > 0$ for all t. Hence, by Theorem 4.1, we can always choose uniformly bounded sequences $\{h_0(t)\}_{t\in\mathbf{Z}}$ and $\{k_0(t)\}_{t\in\mathbf{Z}}$ so as to satisfy the embedding relation (4.4). From the generator recursion we get $g_1(t) = e_1 F(t) l_0(t-1) h_0^*(t) J(t) + e_1 G(t) J(t) k_0^*(t) J(t)$. It then follows that $\{g_1(t)\}_{t\in\mathbf{Z}}$ is also uniformly bounded. Repeating this argument we conclude, by induction, that there exist real numbers $c_d > 0$ and $c_v > 0$ such that $d_i(t) < c_d I$ and $\|g_i(t)\| < c_v$ for all $t \in \mathbb{Z}$.

To show that the sequence $\{d_i(t)\}_{t\in\mathbf{Z}}$ is uniformly bounded from *below*, we use the fact that the successive Schur complements $R_i(t)$ also satisfy relations similar to (2.1). For this purpose, we rewrite each step of the Schur reduction procedure (4.3) in the form

$$R_i(t) = \begin{bmatrix} l_i(t)d_i^{-1}(t) & \begin{matrix}\mathbf{0}\\ I_{n-i-1}\end{matrix} \end{bmatrix} \begin{bmatrix} d_i(t) & \mathbf{0} \\ \mathbf{0} & R_{i+1}(t) \end{bmatrix} \begin{bmatrix} l_i(t)d_i^{-1}(t) & \begin{matrix}\mathbf{0}\\ I_{n-i-1}\end{matrix} \end{bmatrix}^*, \tag{4.6}$$

which exhibits a congruence relation. We define, for notational simplicity,

$$A_i(t) \equiv \begin{bmatrix} l_i(t)d_i^{-1}(t) & \begin{matrix}\mathbf{0}\\ I_{n-i-1}\end{matrix} \end{bmatrix},$$

which is an invertible lower triangular matrix. Assume $R_i(t) > \epsilon_i I$ for some $\epsilon_i > 0$ independent of t ($\epsilon_0 = r_1$ since $0 < r_1 I \leq R(t)$). Then clearly $d_i(t) > \epsilon_i I$ and $A_i(t)$ is uniformly bounded. For any nonzero column vector $\mathbf{y}$, we can always write $\mathbf{y} = A_i^*(t)\mathbf{x}$ for some nonzero column vector $\mathbf{x}$, since $A_i(t)$ has full rank. Therefore,

$$\begin{aligned} \mathbf{y}^* \begin{bmatrix} d_i(t) & \mathbf{0} \\ \mathbf{0} & R_{i+1}(t) \end{bmatrix} \mathbf{y} = \mathbf{x}^* A_i(t) \begin{bmatrix} d_i(t) & \mathbf{0} \\ \mathbf{0} & R_{i+1}(t) \end{bmatrix} A_i^*(t)\mathbf{x} \;&=\; \mathbf{x}^* R_i(t)\mathbf{x} \\ > \epsilon_i \|\mathbf{x}\|^2 \;=\; \epsilon_i \|A_i^{-*}(t)\mathbf{y}\|^2 \;&\equiv\; \epsilon_{i+1}\|\mathbf{y}\|^2\ , \end{aligned}$$

where in the last equality we defined ϵ_{i+1} and used the fact that $\{A_i^{-1}(t)\}_{t\in\mathbf{Z}}$ is uniformly bounded. Consequently, $d_{i+1}(t) > \epsilon_{i+1}$ and we can choose $b_d = \min_{0\leq i\leq n-1} \epsilon_i$. We thus conclude that $\{d_i(t)\}_{t\in\mathbf{Z}}$ is uniformly bounded from below. $d_i(t)$

We finally remark that we can also conversely show that if $\{d_i(t)\}_{t\in\mathbf{Z}}$ is uniformly bounded from below, then $\{R(t)\}_{t\in\mathbf{Z}}$ is also uniformly bounded from below. To see this, we apply the same argument and use (4.6) backwards starting with $R_{n-1}(t) = d_{n-1}(t)$ down to $R_0(t) = R(t)$. □

4.3. Recursive Construction of S. The question now is: How does the recursive algorithm in Theorem 4.2 relate to the result of Theorem 2.2? The relevant fact to note here is that each recursive step gives rise to a linear discrete-time system (in state-space form)

$$\begin{bmatrix} f_i^*(t) & h_i^*(t)J(t) \\ J(t)g_i^*(t) & J(t)k_i^*(t)J(t) \end{bmatrix},$$

which appears on the right-hand side of the generator recursion in Theorem 4.2. This can be thought of as the (state-space) transition matrix of a linear system as follows

$$\begin{bmatrix} \mathbf{x}_i(t+1) & \mathbf{y}_i(t) \end{bmatrix} = \begin{bmatrix} \mathbf{x}_i(t) & \mathbf{w}_i(t) \end{bmatrix} \begin{bmatrix} f_i^*(t) & h_i^*(t)J(t) \\ J(t)g_i^*(t) & J(t)k_i^*(t)J(t) \end{bmatrix}, \tag{4.7}$$

where $\mathbf{x}_i(t)$ denotes the state, $\mathbf{w}_i(t)$ denotes an input vector, and $\mathbf{y}_i(t)$ denotes an output vector at time t.

The second important observation, which we shall verify very soon, is that each such section exhibits an intrinsic blocking property. The cascade of n sections would then exhibit certain global blocking properties, which will be shown to be equivalent to the desired result (2.4). Interesting enough, these blocking properties simply follow from the fact that each step of the Schur reduction procedure yields a matrix with a new zero row and column (as in (4.3)), which translates to a generator matrix with a new zero row as in Theorem 4.2.

4.3.1. Properties of the First-Order Sections. Let $T_i = \begin{bmatrix} T_{lj}^{(i)} \end{bmatrix}_{l,j=-\infty}^{\infty}$ denote the upper-triangular transfer operator associated with (4.7), where the $\{T_{lj}^{(i)}\}$ denote the time-variant Markov parameters of T_i and are given by

$$T_{ll}^{(i)} = J(l)k_i^*(l)J(l) \quad , \quad T_{l,l+1}^{(i)} = J(l)g_i^*(l)h_i^*(l+1)J(l+1) \ ,$$
$$T_{lj}^{(i)} = J(l)g_i^*(l)f_i^*(l+1)f_i^*(l+2)\dots f_i^*(j-1)h_i^*(j)J(j) \ , \quad \text{for } j > l+1.$$

The output and input sequences of T_i are clearly related by

$$\begin{bmatrix} \dots & \mathbf{y}_i(-1) & \boxed{\mathbf{y}_i(0)} & \mathbf{y}_i(1) & \dots \end{bmatrix} = \begin{bmatrix} \dots & \mathbf{w}_i(-1) & \boxed{\mathbf{w}_i(0)} & \mathbf{w}_i(1) & \dots \end{bmatrix} T_i.$$

After n recursive steps (recall that $G(t)$ has n rows) we obtain a cascade of sections $\mathbf{T} = T_0 T_1 \dots T_{n-1}$, which may be regarded as a generalized transmission line.

LEMMA 4.4. *Each first-order section T_i is a bounded upper-triangular linear operator.*

Proof. We already know that $\{f_i(t)\}_{t\in\mathbf{Z}}$ and $g_i(t)\}_{t\in\mathbf{Z}}$ are stable and uniformly bounded sequences, respectively, and that $\{h_i(t), k_i(t)\}_{t\in\mathbf{Z}}$ can always be chosen to be uniformly bounded sequences as well. It is then a standard result that the corresponding transfer operator T_i is bounded (see, *e.g.*, [15]). □

Moreover, if we define the direct sum $\mathbf{J} = \bigoplus_{t\in\mathbf{Z}} J(t)$, it then follows that each T_i also satisfies the following $\mathbf{J}$-losslessness property.

LEMMA 4.5. *Each first-order section T_i satisfies $T_i\mathbf{J}T_i^* = \mathbf{J}$ and $T_i^*\mathbf{J}T_i = \mathbf{J}$.*

Proof. The proof is a direct consequence of the embedding construction (4.4), which leads to the relations

$$f_i^*(t)d_i^{-1}(t)f_i(t) + h_i^*(t)J(t)h_i(t) = d_i^{-1}(t-1).$$
$$f_i^*(t)d_i^{-1}(t)g_i(t) + h_i^*(t)J(t)k_i(t) = \mathbf{0}.$$
$$g_i^*(t)d_i^{-1}(t)g_i(t) + k_i^*(t)J(t)k_i(t) = J(t).$$

Therefore, we can expand $d_i^{-1}(t)$ and write

$$\begin{aligned} d_i^{-1}(t) = {} & h_i^*(t+1)J(t+1)h_i(t+1) + \\ & f_i^*(t+1)h_i^*(t+2)J(t+2)h_i(t+2)f_i(t+1) + \\ & f_i^*(t+1)f_i^*(t+2)h_i^*(t+3)J(t+3)h_i(t+3)f_i(t+2)f_i(t+1) + \ldots \end{aligned}$$

Now the t^{th} element on the main diagonal of $T_i\mathbf{J}T_i^*$ (denoted by λ_{tt}) is given by

$$\begin{aligned} \lambda_{tt} = {} & J(t)\left[k_i^*(t)J(t)k_i(t) + g_i^*(t)h_i^*(t+1)J(t+1)h_i(t+1)g_i(t) + \right. \\ & \left. g_i^*(t)f_i^*(t+1)h_i^*(t+2)J(t+2)h_i(t+2)f_i(t+1)g_i(t) + \ldots\right] J(t). \end{aligned}$$

Using the expression for $d_i^{-1}(t)$ we obtain

$$\lambda_{tt} = J(t) - J(t)g_i^*(t)\left[d_i^{-1}(t) - d_i^{-1}(t)\right] g_i(t)J(t) = J(t)$$

The same argument can be used to show that the off-diagonal elements of $T_i\mathbf{J}T_i^*$ are zero. For proving that $T_i^*\mathbf{J}T_i = \mathbf{J}$ we use a similar procedure. □

Furthermore, each section T_i satisfies an important blocking property in the following sense.

THEOREM 4.6. *Each first-order section T_i satisfies*

$$\left[\ \ldots \quad f_i(t)f_i(t-1)g_i(t-2) \quad f_i(t)g_i(t-1) \quad g_i(t) \quad ?\ \right] T_i = \left[\ \mathbf{0} \quad ?\ \right] ,$$

where $g_i(t)$ is at the t^{th} position of the row vector. Consequently, $g_i(t) \bullet T_i(f_i(t)) = \mathbf{0}$.

Proof. This follows directly from the embedding result (4.4) (as well as from the fact that each step of the generator recursion in Theorem 2.2 produces a new zero row). The output of T_i at time t is given by

$$\begin{aligned} \mathbf{y}_i(t) &= \ldots + f_i(t)f_i(t-1)g_i(t-2)T_{t-2,t} + f_i(t)g_i(t-1)T_{t-1,t} + g_i(t)T_{tt} \\ &= \left[-d_i(t-1) + d_i(t-1)\right] f_i(t)h_i^*(t)J(t) = \mathbf{0} \ , \end{aligned}$$

where we substituted the expressions for the Markov parameters $\{T_{jt}\}_{j\leq t}$ and used

$$\begin{aligned} d_i(t) = {} & g_i(t)J(t)g_i^*(t) + f_i(t)g_i(t-1)J(t-1)g_i^*(t-1)f_i^*(t) + \\ & f_i(t)f_i(t-1)g_i(t-2)J(t-2)g_i^*(t-2)f_i^*(t-1)f_i^*(t) + \ldots \end{aligned}$$

The same argument holds for the previous outputs. □

In general terms, the blocking property means that when $g_i(t)$ (which is the first row of $G_i(t)$) is applied to T_i we obtain a zero output at $f_i(t)$ at time t. We say that $f_i(t)$ is a time-variant *transmission-zero* of T_i and $g_i(t)$ is the associated time-variant *left-zero direction.* We remark that the concepts of transmission zeros and blocking directions are central to many problems in network theory and linear systems [17].

We can now put together the two main pieces proved so far: the Schur reduction procedure and the blocking properties of the elementary sections. This leads to the following constructive proof of Theorem 2.2, assuming finite-dimensional spaces $\{\mathcal{R}(t), \mathcal{U}(t), \mathcal{V}(t)\}_{t\in\mathbf{Z}}$ and the supplementary nondegeneracy condition $\mathbf{U}(t)\mathbf{U}^*(t) \geq \mu > 0$ for all $t \in \mathbf{Z}$, where μ is a fixed constant.

THEOREM 4.7. *Assuming finite-dimensional spaces $\{\mathcal{R}(t), \mathcal{U}(t), \mathcal{V}(t)\}_{t\in\mathbf{Z}}$ and the nondegeneracy condition $\mathbf{U}(t)\mathbf{U}^*(t) \geq \mu > 0,\ \forall\ t$, the time-variant displacement equation (2.1) has a Pick solution $R(t)$ such that $R(t) > \epsilon I > 0$ for a constant ϵ*

and for all t, if, and only if, there exists an upper-triangular strict contraction S ($\|S\| < 1$), $S \in \mathcal{L}(\underset{t\in\mathbf{Z}}{\oplus}\mathcal{V}(t), \underset{t\in\mathbf{Z}}{\oplus}\mathcal{U}(t))$, such that

$$\mathbf{V}(t) = \mathbf{U}(t)P_{\mathcal{U}}(t)S/\underset{j\leq t}{\oplus}\mathcal{V}(j) \quad \textit{for every } t \in \mathbb{Z}. \tag{4.8}$$

Proof. One implication is immediate. We now give a constructive proof of the converse statement. So assume the displacement equation (2.1) has a Pick solution $R(t)$ such that $R(t) > \epsilon I > 0$ for a constant ϵ and for all t. Then applying the Schur reduction procedure (or the generalized Schur algorithm) to $\{F(t), G(t)\}$ leads to a cascade of elementary sections, viz., $\mathbf{T} = T_0 T_1 \ldots T_{n-1}$. Following an argument similar to that presented in [19] for the time-invariant case and in [24, 26] for the time-variant case, we readily conclude that the entire cascade admits the following state-space description:

$$\begin{bmatrix} \mathbf{x}(t+1) & \mathbf{y}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{x}(t) & \mathbf{w}(t) \end{bmatrix} \begin{bmatrix} F^*(t) & H^*(t)J(t) \\ J(t)G^*(t) & J(t)K^*(t)J(t) \end{bmatrix} ,$$

where $\{H(t), K(t)\}_{t\in\mathbf{Z}}$ are, due to our assumptions, uniformly bounded operators that satisfy the embedding relation

$$\begin{bmatrix} F(t) & G(t) \\ H(t) & K(t) \end{bmatrix} \begin{bmatrix} R(t-1) & \mathbf{0} \\ \mathbf{0} & J(t) \end{bmatrix} \begin{bmatrix} F(t) & G(t) \\ H(t) & K(t) \end{bmatrix}^* = \begin{bmatrix} R(t) & \mathbf{0} \\ \mathbf{0} & J(t) \end{bmatrix}. \tag{4.9}$$

Moreover, it follows from the blocking properties of the sections T_i that the entire cascade $\mathbf{T}$ satisfies the global blocking relation

$$\begin{bmatrix} \ldots & F(t)F(t-1)G(t-2) & F(t)G(t-1) & G(t) & \mathbf{0} & \ldots \end{bmatrix} \mathbf{T} = \begin{bmatrix} \mathbf{0} & ? \end{bmatrix} , \tag{4.10}$$

where $G(t)$ appears in the t^{th} position.

We further partition the matrix entries T_{lj} of the cascade $\mathbf{T}$ accordingly with $J(l)$ and $J(j)$,

$$T_{lj} = \begin{bmatrix} T_{11}^{lj} & T_{12}^{lj} \\ T_{21}^{lj} & T_{22}^{lj} \end{bmatrix} ,$$

and consider the triangular operators

$$\mathbf{T}_{11} = \left[T_{11}^{lj}\right], \ \mathbf{T}_{21} = \left[T_{21}^{lj}\right], \ \mathbf{T}_{12} = \left[T_{12}^{lj}\right], \ \mathbf{T}_{22} = \left[T_{22}^{lj}\right], \ \text{for} \quad -\infty < l, j < \infty.$$

We now verify that $\mathbf{T}_{22}^{-1}$ is an upper-triangular and bounded operator and that $\mathbf{T}_{12}\mathbf{T}_{22}^{-1}$ is a strictly contractive upper-triangular operator, such that

$$\mathbf{V}(t) = -\mathbf{U}(t)P_{\mathcal{U}}(t)\mathbf{T}_{12}\mathbf{T}_{22}^{-1}/\underset{j\leq t}{\oplus}\mathcal{V}(j) \quad \text{for all} \ \ t \in \mathbb{Z}.$$

For this purpose, note that it follows from the $\mathbf{J}$−losslessness property of $\mathbf{T}$ that

$$\mathbf{T}_{22}\mathbf{T}_{22}^* \geq I \ , \ \ \mathbf{T}_{22}^*\mathbf{T}_{22} \geq I. \tag{4.11}$$

Hence, $\mathbf{T}_{22}$ is invertible and $\|\mathbf{T}_{22}^{-1}\| \leq 1$. Define $X(t) = P_{\mathcal{V}}(t)\mathbf{T}_{22}/\underset{j\leq t}{\oplus}\mathcal{V}(j)$, then it follows from (4.11) that $X^*(t)X(t) \geq I$. Define $\mathbf{T}(t) = P_{\mathcal{U}\oplus\mathcal{V}}(t)\mathbf{T}/\underset{j\leq t}{\oplus}\mathcal{U}(j)\oplus\mathcal{V}(j)$ and $J_t = \underset{j\leq t}{\oplus} J(j)$. It follows from the embedding relation (4.9) that

$$J_t - \mathbf{T}(t)J_t\mathbf{T}^*(t) = \begin{bmatrix} \dots & F(t)G(t-1) & G(t) \end{bmatrix}^* R^{-1}(t) \begin{bmatrix} \dots & F(t)G(t-1) & G(t) \end{bmatrix} \geq 0.$$

Hence, $X(t)X^*(t) \geq I$ and we conclude that $X(t)$ is invertible for every $t \in \mathbb{Z}$ and that the family $\{X^{-1}(t)\}_{t\in\mathbb{Z}}$ is uniformly bounded by 1. Define the following operators (acting on the same space as $\mathbf{T}_{22}$),

$$\tilde{X}(t) = \begin{bmatrix} X(t) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Then $\tilde{X}(t+1)$ and $\tilde{X}(t)$ satisfy the following nested property (they differ by just one block column)

$$\tilde{X}(t+1) = \begin{bmatrix} \tilde{X}(t) & ? \\ \mathbf{0} & ? \end{bmatrix} , \tag{4.12}$$

where ? denotes irrelevant entries. Hence, $\{\tilde{X}(t)\}_{t\in\mathbb{Z}}$ strongly converges to a bounded operator $\tilde{X}$ as $t \to \infty$. It is easily checked that $\tilde{X}$ is upper-triangular and that it actually coincides with $\mathbf{T}_{22}^{-1}$.

The fact that $\mathbf{T}_{12}\mathbf{T}_{22}^{-1} \in \mathcal{L}(\underset{t\in\mathbb{Z}}{\oplus}\mathcal{V}(t), \underset{t\in\mathbb{Z}}{\oplus}\mathcal{U}(t))$, is an upper-triangular strictly contractive operator is a consequence of the $\mathbf{J}$-losslessness of $\mathbf{T}$. We thus conclude that $S = -\mathbf{T}_{12}\mathbf{T}_{22}^{-1} \in \mathcal{L}(\underset{t\in\mathbb{Z}}{\oplus}\mathcal{V}(t), \underset{t\in\mathbb{Z}}{\oplus}\mathcal{U}(t))$ is a strictly contractive upper-triangular operator that satisfies (4.8). □

Remark: The above argument is based on a recursive construction of $\mathbf{T}$. We can also give a direct (nonrecursive) proof of the same result as follows: first prove the embedding relation (4.9) as in Theorem 4.1 and the blocking property (4.10) as in Theorem 4.6. We then conclude the argument as above.

4.3.2. Parametrization of all Solutions. We now show how to parametrize all solutions S that satisfy (4.8).

THEOREM 4.8. *Assuming finite-dimensional spaces $\{\mathcal{R}(t),\mathcal{U}(t),\mathcal{V}(t)\}_{t\in\mathbb{Z}}$ and the nondegeneracy condition $\mathbf{U}(t)\mathbf{U}^*(t) \geq \mu > 0$, $\forall\, t$, and that the displacement equation (2.1) has a Pick solution $R(t)$ such that $R(t) > \epsilon I > 0$ for a constant ϵ and for all t. Then all strictly contractive upper-triangular solutions $S \in \mathcal{L}(\underset{t\in\mathbb{Z}}{\oplus}\mathcal{V}(t), \underset{t\in\mathbb{Z}}{\oplus}\mathcal{U}(t))$ are given by*

$$S = -\left[\mathbf{T}_{11}K + \mathbf{T}_{12}\right]\left[\mathbf{T}_{21}K + \mathbf{T}_{22}\right]^{-1} ,$$

for arbitrary upper-triangular contractive operators

$$K \in \mathcal{L}(\underset{t\in\mathbb{Z}}{\oplus}\mathcal{V}(t), \underset{t\in\mathbb{Z}}{\oplus}\mathcal{U}(t)) \quad \textit{with} \quad \|\mathcal{K}\| < 1.$$

Proof. One implication is immediate. Consider a K as above. Since $\mathbf{T}_{22}^{-1}$ is a bounded upper-triangular operator, it follows that

$$S = -\left[\mathbf{T}_{11}K + \mathbf{T}_{12}\right]\left[\mathbf{T}_{21}K + \mathbf{T}_{22}\right]^{-1}$$

is bounded upper-triangular and, using the $\mathbf{J}(t)-$losslessness of $\mathbf{T}$, we conclude that $\|S\| < 1$. Let $S_1 = \mathbf{T}_{11}K + \mathbf{T}_{12}$ and $S_2 = \mathbf{T}_{21}K + \mathbf{T}_{22}$. Then,

$$\begin{bmatrix} S_1 \\ S_2 \end{bmatrix} = \mathbf{T} \begin{bmatrix} K \\ I \end{bmatrix} ,$$

and, because of the blocking property of $\mathbf{T}$, we obtain that S is a solution of (4.8).

For the converse implication we follow the pattern developed in [6] and adapted for the time-variant Nevanlinna-Pick problem in [5]. Because our framework is more general, we shall indicate the necessary changes. For an upper-triangular operator $X \in \mathcal{L}(\mathcal{E}, \mathcal{K})$, where $\mathcal{E} = \bigoplus_{t\in\mathbf{Z}} \mathcal{E}(t)$ and $\mathcal{K} = \bigoplus_{t\in\mathbf{Z}} \mathcal{K}(t)$ are two families of Hilbert spaces, we define $X(t) = P_{\mathcal{E}}(t)X / \bigoplus_{j\leq t} \mathcal{K}(j)$. We also denote by U_p the set of upper-triangular operators in $\mathcal{L}(\bigoplus_{t\in\mathbf{Z}} \mathcal{V}(t), \bigoplus_{t\in\mathbf{Z}} (\mathcal{U}(t) \oplus \mathcal{V}(t)))$. We claim that $\mathbf{T}U_p = \{X \in U_p \,/ \mathbf{G}(t)X(t) = \mathbf{0},\quad t \in \mathbb{Z}\}$, where $\mathbf{G}(t) = \begin{bmatrix} \dots & F(t)F(t-1)G(t-2) & F(t)G(t-1) & G(t) \end{bmatrix}$. Indeed, take $Y \in U_p$ then $\mathbf{G}(t)(\mathbf{T}Y)(t) = \mathbf{G}(t)\mathbf{T}(t)Y(t) = \mathbf{0}$, by the blocking property of $\mathbf{T}$. Conversely, take $X \in U_p$, $\mathbf{G}(t)X(t) = \mathbf{0}$ for all $t \in \mathbb{Z}$ and define $Y = \mathbf{T}^{-1}X = \mathbf{J}\mathbf{T}^*\mathbf{J}X$. Due to the structure of the Markov parameters of $\mathbf{T}$, it is readily checked that all the entries of Y under the main diagonal are zero. That is, $Y \in U_p$ and the claim is proved. From now on the arguments in Theorem 3.1 of [5] for getting the required representation of the solution of (4.8) apply directly. □

5. Schur Parameters. There are special choices of the parameters $\{h_i(t), k_i(t)\}$ in (4.4) that would greatly simplify the generator recursion of Theorem 4.2 and lead to the so-called Schur parameters and the corresponding lattice sections. These parameters, which first appeared in the classical paper of Schur [30], have encountered applications in several areas including the study of orthogonal polynomials, inverse scattering, digital filtering, etc.. [18]. They were also studied in a general operatorial framework in [3, 7]. However, we want to emphasize that in our paper the Schur parameters are not the parameters associated with the load (*i.e.*, the upper-triangular contractive operator K in Theorem 4.8), but rather the parameters associated with the recursive construction of the strictly contractive solution $S = -\mathbf{T}_{12}\mathbf{T}_{22}^{-1}$.

We shall not go into the details of the lattice structures here. The reader is referred to [24, 26] for a detailed derivation. We shall instead show how certain so-called time-variant Schur (or reflection) parameters appear in two important special cases. In both cases we assume $\dim \mathcal{R}_i(t) = 1$ but in the second case we further assume that $F(t)$ is strictly lower triangular.

We continue to require the finite dimensionality assumptions and the nondegeneracy condition $\mathbf{U}(t)\mathbf{U}^*(t) > \mu > 0$, $\forall\, t$, of the previous section. But we now further assume that $\dim \mathcal{R}_i(t) = 1$ and that there exists $b > 0$ such that

$$b \leq |g_i(t)J(t)g_i^*(t)| \quad \textit{for all } t \in \mathbb{Z}. \tag{5.1}$$

We remark that conditions (5.1) and $\{d_i(t)\}$ *bounded from below* are independent, as can be shown by simple examples. We distinguish between two special cases: $g_i(t)J(t)g_i^*(t) > 0$ or $g_i(t)J(t)g_i^*(t) < 0$. That is, $g_i(t)$ has either positive or negative $J(t)-$norm. We partition $g_i(t)$ accordingly with $J(t)$, viz., $g_i(t) = \begin{bmatrix} u_i(t) & v_i(t) \end{bmatrix}$.

5.1. The Positive Case. In the positive case, we have

$$g_i(t)J(t)g_i^*(t) = u_i(t)u_i^*(t) - v_i(t)v_i^*(t) > 0,$$

and, by a well known factorization result (see [21]), it follows that there exists a contraction $\tilde{\gamma}_i(t) : \mathcal{R}(v_i^*(t)) \to \mathcal{R}(u_i^*(t))$ such that $v_i(t) = u_i(t)\tilde{\gamma}_i(t)$, and $\|\tilde{\gamma}_i(t)\| < 1$. We can extend this contraction by zero to another contraction $\gamma_i(t) \in \mathcal{L}(\mathcal{U}(t),\ \mathcal{V}(t))$ that still satisfies $\|\gamma_i(t)\| < 1$ and $v_i(t) = u_i(t)\gamma_i(t)$. If we now construct the $J(t)-$unitary operator

$$\Theta_i(t) = \begin{bmatrix} I_{\mathcal{U}(t)} & -\gamma_i(t) \\ -\gamma_i^*(t) & I_{\mathcal{V}(t)} \end{bmatrix} \begin{bmatrix} (I - \gamma_i(t)\gamma_i^*(t))^{-1/2} & \mathbf{0} \\ \mathbf{0} & (I - \gamma_i^*(t)\gamma_i(t))^{-1/2} \end{bmatrix} ,$$

we readily conclude that $\Theta_i(t)$ reduces $g_i(t)$ to the form $g_i(t)\Theta_i(t) = \begin{bmatrix} \delta_i(t) & \mathbf{0} \end{bmatrix}$, for a certain $\delta_i(t)$. Now note that

$$\delta_i(t)\delta_i^*(t) = g_i(t)\Theta_i(t)J(t)\Theta_i^*(t)g_i^*(t) = g_i(t)J(t)g_i^*(t) > b \ ,$$

and consequently, using the boundedness of $\{g_i(t)J(t)g_i^*(t)\}$ from below, there exists a constant $k > 0$ such that $\|\Theta_i^{-1}(t)\| \leq k$ for all t.

5.2. The Negative Case. In the negative case, we have

$$g_i(t)J(t)g_i^*(t) = u_i(t)u_i^*(t) - v_i(t)v_i^*(t) < 0,$$

and, by the same argument as above, we conclude that there exists a contraction $\gamma_i(t) \in \mathcal{L}(\mathcal{U}(t),\ \mathcal{V}(t))$ ($\|\gamma_i(t)\| < 1$) such that $u_i(t) = v_i(t)\gamma_i(t)$. If we now define the $J(t)-$unitary operator

$$\Theta_i(t) = \begin{bmatrix} I_{\mathcal{U}(t)} & -\gamma_i^*(t) \\ -\gamma_i(t) & I_{\mathcal{V}(t)} \end{bmatrix} \begin{bmatrix} (I - \gamma_i(t)^*\gamma_i(t))^{-1/2} & \mathbf{0} \\ \mathbf{0} & (I - \gamma_i(t)\gamma_i^*(t))^{-1/2} \end{bmatrix} ,$$

we readily conclude that $\Theta_i(t)$ reduces $g_i(t)$ to the form $g_i(t)\Theta_i(t) = \begin{bmatrix} \mathbf{0} & \delta_i(t) \end{bmatrix}$, for a certain $\delta_i(t)$. It also follows from

$$-\delta_i(t)\delta_i^*(t) = g_i(t)\Theta_i(t)J(t)\Theta_i^*(t)g_i^*(t) = g_i(t)J(t)g_i^*(t) < -b \ ,$$

that $\|\Theta_i^{-1}(t)\| \leq k$ for some $k > 0$.

The contractions $\{\gamma(t)\}_{t\in\mathbb{Z}}$ are called the *Schur parameters* of the underlying displacement structure (2.1).

5.3. Strictly Lower-Triangular $\mathbf{F(t)}$. An important special case that often arises is the case of *strictly* lower triangular $F(t)$. That is, $f_i(t) = 0$ for all $t \in \mathbb{Z}$ and $i = 0, 1, \ldots, n-1$, and consequently, $d_i(t) = g_i(t)J(t)g_i^*(t)$. But $\{d_i(t)\}$ is uniformly bounded from below, viz., $d_i(t) > \epsilon > 0$ for all t. Hence, we now always have

$$u_i(t)u_i^*(t) - v_i(t)v_i^*(t) > \epsilon > 0 \quad \text{for all} \ \ t \in \mathbb{Z} \ ,$$

and there always exist Schur parameters $\gamma_i(t)$ such that $v_i(t) = u_i(t)\gamma_i(t)$, with the corresponding $J(t)-$unitary operator defined by

$$\Theta_i(t) = \begin{bmatrix} I_{\mathcal{U}(t)} & -\gamma_i(t) \\ -\gamma_i^*(t) & I_{\mathcal{V}(t)} \end{bmatrix} \begin{bmatrix} (I - \gamma_i(t)\gamma_i^*(t))^{-1/2} & \mathbf{0} \\ \mathbf{0} & (I - \gamma_i^*(t)\gamma_i(t))^{-1/2} \end{bmatrix}.$$

The generator recursion in Theorem 4.2 can then be shown to reduce to the compact form (see also [29])

$$\begin{bmatrix} \mathbf{0} \\ G_{i+1}(t) \end{bmatrix} = F_i(t)G_i(t-1)\Theta_i(t-1) \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + G_i(t)\Theta_i(t) \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & I_{(r(t)-1)} \end{bmatrix} ,$$

which has the the following interpretation: multiply $G_i(t)$ by $\Theta_i(t)$ and keep the last columns; multiply the first column of $G_i(t-1)\Theta_i(t-1)$ by $F_i(t)$; these two steps result in $G_{i+1}(t)$.

6. Concluding Remarks. We proved a general result (Theorem 2.2) concerning time-variant displacement equations of the form (2.1) with Pick operator solutions $R(t)$. We considered several moment, completion, and interpolation problems whose solutions followed as special cases of Theorem 2.2. These problems were stated in a general operator setting, including a time-variant version of the tangential Hermite-Fejér interpolation problem. Under supplementary finite-dimensionality and nondegeneracy conditions, a recursive procedure was derived that led to a recursive construction and parametrization of all solutions of the general result of Theorem 2.2. We also considered special cases where further simplifications were possible.

REFERENCES

[1] N. I. Akhiezer, *The Classical Moment Problem and Some Related Questions in Analysis*, Hafner Publishing Company, NY, 1965.

[2] D. Alpay, P. Dewilde, and H. Dym, *On the existence and construction of solutions to the partial lossless inverse scattering problem with applications to estimation theory*, IEEE Transactions on Information Theory, 35 (1989), pp. 1184–1205.

[3] G. Arsene, Z. Ceausescu, and C. Foias, *On interwining dilations VIII*, J. Operator Theory, 4 (1980), pp. 55–91.

[4] J. A. Ball and I. Gohberg, *A commutant lifting theorem for triangular matrices with diverse applications*, Integral Equations and Operator Theory, 8 (1985), pp. 205–267.

[5] J. A. Ball, I. Gohberg, and M. A. Kaashoek, *Nevanlinna-Pick interpolation for time-varying input-output maps: The discrete case*, Operator Theory: Advances and Applications, 56 (1992), pp. 1–51.

[6] J. A. Ball, I. Gohberg, and L. Rodman, *Interpolation of Rational Matrix Functions*, vol. 45 of Operator Theory: Advances and Applications, Birkhäuser, Basel, 1990.

[7] Z. Ceausescu and C. Foias, *On interwining dilations V*, Acta Sci. Math., 40 (1978), pp. 9–32.

[8] T. Constantinescu, *Some aspects of nonstationarity – I*, Acta Sci. Math., 54 (1990), pp. 379–389.

[9] T. Constantinescu, A. H. Sayed, and T. Kailath, *Structured matrices and moment problems*, in Proc. of the Workshop on Challenges of a Generalized System Theory, Amsterdam, Netherlands, June 1992.

[10] P. Dewilde, *A course on the algebraic Schur and Nevanlinna-Pick interpolation problems*, in Algorithms and Parallel VLSI Architectures, E. F. Deprettere and A. J. van der Veen, eds., Elsevier Science Publications, (1991), pp. 13–69.

[11] P. Dewilde and H. Dym, *Interpolation for upper triangular operators*, Operator Theory: Advances and Applications, 56 (1992), pp. 153–260.

[12] H. Dym and I. Gohberg, *Extensions of band matrices with band inverses*, Linear Algebra and Its Applications, 36 (1981), pp. 1–24.

[13] C. Foias and A. E. Frazho, *The Commutant Lifting Approach to Interpolation Problems*, vol. 44 of Operator Theory: Advances and Applications, 1990.

[14] Y. Genin, P. V. Dooren, T. Kailath, J. Delosme, and M. Morf, *On $\Sigma-$lossless transfer functions and related questions*, Linear Algebra and Its Applications, 50 (1983), pp. 251–275.

[15] I. Gohberg and M. A. Kaashoek, *Time–varying linear systems with boundary conditions and integral operators, I. The transfer operator and its applications*, Integral Equations and Operator Theory, 7 (1984), pp. 325–391.

[16] J. W. Helton, *Operator Theory, Analytic Functions, Matrices and Electrical Engineering*, CBMS, AMS, Providence, RI, 1987.

[17] T. Kailath, *Linear Systems*, Prentice Hall, Englewood Cliffs, NJ, 1980.

[18] ———, *Signal processing applications of some moment problems*, in Moments in Mathematics, H. Landau, ed., American Mathematical Society, 37 (1987), pp. 71–109.

[19] H. Lev-Ari and T. Kailath, *State-space approach to factorization of lossless transfer functions and structured matrices*, Linear Algebra and Its Applications, 162–164 (1992), pp. 273–295.

[20] A. A. Nudelman, *On a generalization of classical interpolation problems*, Dokl. Akad. Nauk SSR, 256 (1981), pp. 790–793.

[21] M. Rosenblum and J. Rovnyak, *Hardy Classes and Operator Theory*, Oxford Univ. Press, 1985.

[22] L. A. Sakhnovich, *Interpolation problems, inverse spectral problems and nonlinear equations*, Operator Theory: Advances and Applications, 59 (1992), pp. 292–304.

[23] D. Sarason, *Generalized interpolation in* H^∞ , American Mathematical Society Transactions, 127 (1967), pp. 179–203.

[24] A. H. Sayed, *Displacement Structure in Signal Processing and Mathematics*, PhD thesis, Stanford University, Stanford, CA, August 1992.

[25] A. H. Sayed, T. Constantinescu, and T. Kailath, *Square-root algorithms for structured matrices, interpolation, and completion problems*, IMA volumes in Mathematics and Its Applications, to appear.

[26] ———, *Time-variant displacement structure and interpolation problems*, IEEE Transactions on Automatic Control, to appear.

[27] A. H. Sayed and T. Kailath, *Recursive solutions to rational interpolation problems*, in Proc. IEEE International Symposium on Circuits and Systems, San Diego, CA, (1992), pp. 2376–2379.

[28] ———, *Triangular factorization of Toeplitz-like matrices with singular leading minors*, SIAM J. Matrix Anal. Appl., to appear.

[29] A. H. Sayed, H. Lev-Ari, and T. Kailath, *Time-variant displacement structure and triangular arrays*, IEEE Transactions on Signal Processing, to appear.

[30] I. Schur, *Über potenzreihen die im Inneren des Einheitskreises beschränkt sind*, Journal für die Reine und Angewandte Mathematik, 147 (1917), pp. 205–232. (English translation in *Operator Theory: Advances and Applications*, vol. 18, pp. 31–88, edited by I. Gohberg, Birkhäuser, Boston, 1986).

[31] B. Sz. Nagy and C. Foias, *Harmonic Analysis of Operators on Hilbert Space*, North Holland Publishing Co., Amsterdam-Budapest, 1970.