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# Inverse Scattering Experiments, Structured Matrix Inequalities, and Tensor Algebra<sup>\*</sup>

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## Abstract

This paper deals with a general matrix extension problem with structural constraints and provides a recursive solution in terms of an inverse scattering experiment. Both the stationary and nonstationary cases are considered, in addition to connections to tensor algebra.

*Key words:* scattering, displacement structure, Schur algorithm, matrix extension, matrix inequality, tensor algebra.

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## 1 The Generalized Schur Algorithm

The displacement structure concept provides a convenient framework for the study of matrix problems involving different kinds of structure (see, e.g., [1, 2, 3]). In this paper we use the displacement structure formalism, and the related Schur algorithm, to elaborate on a generalized version of certain classical matrix extension problems. We highlight connections with structured matrix inequalities and also relate the discussion to tensor algebra. Both cases of time-invariant (stationary) and time-variant (nonstationary) structures are considered. We first start with a brief description of the generalized Schur algorithm for a special class of structured matrices.

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Thus let  $F$  be an  $n \times n$  *strictly* lower triangular matrix. Then we say that an  $n \times n$  positive-definite Hermitian matrix  $R$  has *displacement structure* with respect to  $F$  if it satisfies a displacement equation of the form

$$R - FRF^* = GJG^* , \quad J = (I_p \oplus -I_q) , \quad (1)$$

where  $J$  is a signature matrix that specifies the displacement inertia of  $R$ , and  $G$  is an  $n \times r$  so-called generator matrix with  $r \ll n$  and  $r = (p + q)$ . We say that  $R$  has structure when the difference  $R - FRF^*$  is low rank; its rank  $r$  is called the displacement rank of  $R$ . Since  $F$  is strictly lower triangular, the equation (1) has a unique solution  $R$  and, therefore, the triple  $\{F, G, J\}$  fully characterizes  $R$ .

A major result concerning such structured matrices  $R$  is that the successive Schur complements of  $R$ , denoted by  $R_i$ , inherit a similar structure. That is, if  $R_i$  is the Schur complement of the leading  $i \times i$  submatrix of  $R$ , then  $R_i$  also exhibits displacement structure of the form

$$R_i - F_i R_i F_i^* = G_i J G_i^* ,$$

where  $F_i$  is the submatrix obtained after deleting the first  $i$  rows and columns of  $F$ , and the generator  $G_i$  satisfies a recursive construction that we explain below.

**Algorithm 1 (A generalized Schur algorithm)** *Generator matrices  $G_i$  for the successive Schur complements  $R_i$  of a positive-definite structured matrix  $R$ , as in (1), can be recursively constructed as follows. Start with  $G_0 = G$ ,  $F_0 = F$ , and repeat for  $i \geq 0$ :*

- (1) *At step  $i$  we have  $F_i$  and  $G_i$ . Let  $g_i$  denote the top row of  $G_i$ .*
- (2) *Choose any  $J$ -unitary rotation  $\Theta_i$  that reduces  $g_i$  to the form*

$$g_i \Theta_i = \begin{bmatrix} \delta_i & 0 & \dots & 0 \end{bmatrix} \quad (2)$$

*Such a rotation always exists in view of the positive-definiteness of  $R$  and it can be implemented in many different ways, e.g., as a sequence of elementary unitary and hyperbolic rotations. [If such a transformation cannot be performed, then the given matrix  $R$  is not positive-definite; in other words, the generalized Schur algorithm can also be used as a test for positivity.]*

- (3) *Apply  $\Theta_i$  to  $G_i$  leading to the next generator as follows:*

$$\begin{bmatrix} 0 \\ G_{i+1} \end{bmatrix} = G_i \Theta_i \begin{bmatrix} 0 & 0 \\ 0 & I_{r-1} \end{bmatrix} + F_i G_i \Theta_i \begin{bmatrix} 1 & 0 \\ \mathbf{0} & 0 \end{bmatrix} . \quad (3)$$

(4) The columns of the Cholesky factor of  $R$ , viz.,  $R = \bar{L}\bar{L}^*$ , are given by

$$\bar{l}_i = G_i \Theta_i \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \quad (4)$$

◇

Pictorially, we have the following (see Fig. 1):

$$G_i = \begin{bmatrix} \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \\ \vdots & & \vdots \end{bmatrix} \xrightarrow{\Theta_i} \begin{bmatrix} \delta_i & 0 & 0 \\ \times' & \times' & \times' \\ \times' & \times' & \times' \\ \vdots & & \vdots \end{bmatrix} \xrightarrow{F_i} \begin{bmatrix} 0 & 0 & 0 \\ \times'' & \times' & \times' \\ \times'' & \times' & \times' \\ \vdots & & \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ G_{i+1} \end{bmatrix}$$

In words:

- Choose an  $r \times r$   $J$ -unitary rotation  $\Theta_i$  that reduces the top row of  $G_i$  as in (2). We say that  $G_i$  is reduced to proper form.
- Apply  $\Theta_i$  to  $G_i$ .
- Multiply the first column of  $G_i \Theta_i$  by  $F_i$  and keep all other columns unchanged.

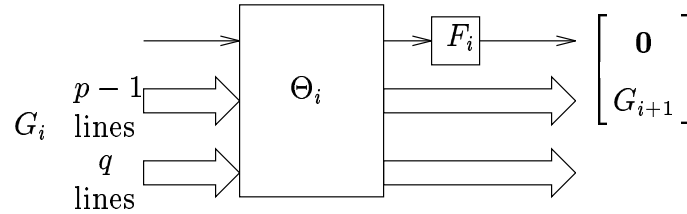


Fig. 1. Pictorial representation of the generalized Schur algorithm.

Another useful conclusion follows by combining the generator recursion (3) with the expression (4) for the  $i$ -th column of the Cholesky factor. Indeed, define

$$l_i \triangleq \sqrt{d_i} \bar{l}_i,$$

where  $1/\sqrt{d_i}$  is the top entry of  $\bar{l}_i$  (and equal to  $|\delta_i|^{-2}$ ). That is, the top entry of  $l_i$  is normalized to unity. Then it can be verified that (3) and (4) lead to the following

expression

$$\begin{bmatrix} l_i & 0 \\ G_{i+1} & \end{bmatrix} = \begin{bmatrix} F_i l_i & G_i \end{bmatrix} \begin{bmatrix} 0 & \frac{\delta_i}{d_i} \begin{bmatrix} 1 & 0 \end{bmatrix} \\ \Theta_i \begin{bmatrix} \delta_i \\ 0 \end{bmatrix} & \Theta_i \begin{bmatrix} 0 & 0 \\ 0 & I_{r-1} \end{bmatrix} \end{bmatrix}. \quad (5)$$

We can therefore regard the transformation that appears on the right-hand side as the system matrix of a first-order linear state-space system; the rows of  $\{G_i\}$  and  $\{G_{i+1}\}$  can be regarded as inputs and outputs of this system, respectively, and the entries of  $\{l_i, F_i l_i\}$  can be regarded as the corresponding current and future states. If we let  $\Theta_i(z)$  denote the transfer function of the linear system (with inputs from the left), viz.,

$$\Theta_i(z) = \Theta_i \begin{bmatrix} 0 & 0 \\ 0 & I_{r-1} \end{bmatrix} + \Theta_i \begin{bmatrix} \delta_i \\ 0 \end{bmatrix} (z-0)^{-1} \frac{\delta_i}{d_i} \begin{bmatrix} 1 & 0 \end{bmatrix},$$

simple algebra will show that the above expression collapses to

$$\Theta_i(z) = \Theta_i \begin{bmatrix} z^{-1} & 0 \\ 0 & I_{r-1} \end{bmatrix}. \quad (6)$$

We therefore see that each step of the generalized Schur recursions can be regarded as giving rise to a first-order section  $\Theta_i(z)$ . A succession of steps of the generalized Schur algorithm would therefore lead to a feedforward cascade of sections, say for  $(n+1)$  steps,

$$\Theta(z) = \Theta_0(z)\Theta_1(z)\dots\Theta_n(z) = \begin{bmatrix} \Theta_{11}(z) & \Theta_{12}(z) \\ \Theta_{21}(z) & \Theta_{22}(z) \end{bmatrix}, \quad (7)$$

which we partition accordingly with  $J = (I_p \oplus -I_q)$ . That is,  $\Theta_{11}(z)$  is  $p \times p$ ,  $\Theta_{12}(z)$  is  $p \times q$ ,  $\Theta_{21}(z)$  is  $q \times p$ , and  $\Theta_{22}(z)$  is  $q \times q$ . The transformation implied by  $\Theta(z)$  is depicted in Fig. 2, with the input terminals denoted by  $\{i_1, i_2\}$  and the output terminals denoted by  $\{o_1, o_2\}$ . It is also a  $J$ -lossless transformation.

The associated scattering or transmission line cascade would then be (see Fig. 3):

$$\Sigma(z) = \begin{bmatrix} \Theta_{11}(z) - \Theta_{12}(z)\Theta_{22}^{-1}(z)\Theta_{21}(z) & -\Theta_{12}(z)\Theta_{22}^{-1}(z) \\ \Theta_{22}^{-1}(z)\Theta_{21}(z) & \Theta_{22}^{-1}(z) \end{bmatrix}. \quad (8)$$

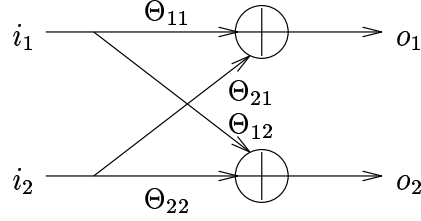


Fig. 2. The feedforward cascade.

Such cascades map any strictly contractive load (or Schur function) that connects  $o_1$  to  $o_2$  to a contractive (transfer) function at the left-hand terminals of the cascade. The key fact is that the flow on the last  $q$  lines is reversed (without affecting the values of the signals inside the cascade).

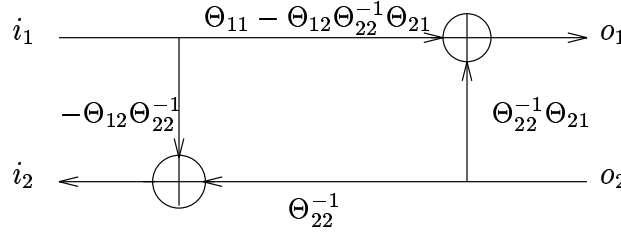


Fig. 3. The scattering cascade.

## 2 Structured Matrix Extension Problems

In this section we investigate the application of the aforementioned generalized Schur algorithm to new matrix extension problems, and to certain structured matrix inequalities.

Thus consider a positive-definite matrix  $R$  with displacement structure (1). We partition  $G$  into  $G = [U \ V]$  accordingly with  $J$ . Here,  $U$  is  $n \times p$  and  $V$  is  $n \times q$ . We also consider a possibly nonlinear function  $\phi(\cdot)$  and a possibly matrix quantity  $\rho$ . We then pose the following extension problem.

**Problem 1 (A General Extension Problem)** *Given  $\{R, U, V, F, J, \phi, \rho\}$  as described above, find a possibly matrix quantity  $x$  (with column dimension  $q$ ) such that the extended displacement equation*

$$\bar{R} - \begin{bmatrix} F & 0 \\ \rho & 0 \end{bmatrix} \bar{R} \begin{bmatrix} F & 0 \\ \rho & 0 \end{bmatrix}^* = \begin{bmatrix} U & V \\ \phi(x) & x \end{bmatrix} J \begin{bmatrix} U & V \\ \phi(x) & x \end{bmatrix}^* \quad (9)$$

*has a positive-semidefinite solution ( $\bar{R} \geq 0$ ). [Note that  $\phi(x)$  has column dimension  $p$ .]*

In other words, we want to extend the dimension of the original matrix  $R$  in such

a way so as to preserve both:

- (1) Its positive semi-definiteness and
- (2) its displacement structure.

A well-known special case of the above formulation is the Toeplitz extension problem, which reads as follows. Given a moment sequence  $\{c_k, c_0 = 1, 0 \leq k \leq n\}$ , we seek a new entry  $c_{n+1}$  such that the corresponding Toeplitz matrix  $T_{n+1}$  remains positive-definite, where the notation  $T_n$  denotes the matrix

$$T_n \triangleq \begin{bmatrix} c_0 & c_{-1} & c_{-2} & \dots & c_{-n} \\ c_1 & c_0 & c_{-1} & \dots & c_{-n+1} \\ c_2 & c_1 & c_0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & c_{-1} \\ c_n & c_{n-1} & \dots & c_1 & c_0 \end{bmatrix}, \quad c_{-k} = c_k^*.$$

Let  $Z$  denote the lower shift triangular matrix (of appropriate dimensions), with ones on the first subdiagonal. Then it is easy to verify that  $T_{n+1}$  satisfies the displacement equation

$$T_{n+1} - ZT_{n+1}Z^* = \underbrace{\begin{bmatrix} 1 & 0 \\ c_1 & c_1 \\ c_2 & c_2 \\ \vdots & \vdots \\ c_n & c_n \\ \hline \boxed{c_{n+1}} & \boxed{c_{n+1}} \end{bmatrix}}_{G'_0} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_J \underbrace{\begin{bmatrix} 1 & 0 \\ c_1 & c_1 \\ c_2 & c_2 \\ \vdots & \vdots \\ c_n & c_n \\ \hline \boxed{c_{n+1}} & \boxed{c_{n+1}} \end{bmatrix}}^*. \quad (10)$$

This shows that the one-step Toeplitz extension problem can be interpreted as one of adding a row of the special form  $[c_{n+1} \ c_{n+1}]$  to  $G_0$ , thus obtaining  $G'_0$ , and requiring that  $G'_0$  still be a generator matrix for a positive-definite Toeplitz matrix. In other words, the above classical problem is a special case of (9) with the identifications  $F = Z$ ,  $\rho = [0 \ 0 \ \dots \ 1]$  (an  $n$ -dimensional vector),  $\phi(x) = x$ , and  $x = c_{n+1}$ , a scalar!

Now the extension problem (9) need not always be solvable (just like a solution to the Toeplitz extension problem need not always exist). It may also have multiple

solutions. Before proceeding to a discussion of its solution, we first note that we can reformulate (9) into an equivalent problem that involves a nonlinear matrix inequality. To see this, we express the matrix  $\bar{R}$  in the form

$$\bar{R} = \begin{bmatrix} R & a \\ a^* & b \end{bmatrix},$$

with the unknown (possibly block) entries denoted by  $a$  and  $b$ . Then the displacement equation (9) requires that the following equality must hold:

$$\begin{bmatrix} R & a \\ a^* & b \end{bmatrix} = \begin{bmatrix} FRF^* & FR\rho^* \\ \rho RF^* & \rho R\rho^* \end{bmatrix} + \begin{bmatrix} GJG^* & U\phi^*(x) - Vx^* \\ \phi(x)U^* - xV^* & \phi(x)\phi^*(x) - xx^* \end{bmatrix}.$$

This equality has three unknowns  $\{a, b, x\}$ . We can solve for  $(a, b)$  in terms of  $x$  to get

$$a = FR\rho^* + U\phi^*(x) - Vx^*, \quad (11)$$

$$b = \rho R\rho^* + \phi(x)\phi^*(x) - xx^*. \quad (12)$$

These constraints that are imposed on the  $(a, b, x)$  are a result of the desired displacement structure of  $\bar{R}$ . There is yet another constraint that follows from the desired positivity of  $\bar{R}$ . More specifically, since  $R > 0$ , the positivity of  $\bar{R}$  is equivalent to determining an  $x$  such that  $b - a^*R^{-1}a \geq 0$ . If we now replace the above expressions for  $a$  and  $b$  in terms of  $x$ , we obtain the following equivalent problem.

**Problem 2 (Nonlinear Matrix Inequality)** *Given  $\{R, U, V, F, J, \phi, \rho\}$ , as described earlier, find, if possible, an  $x$  that satisfies the nonlinear matrix inequality:*

$$\rho R\rho^* + \phi(x)\phi^*(x) - xx^* - \quad (13)$$

$$[FR\rho^* + U\phi^*(x) - Vx^*]^* R^{-1} [FR\rho^* + U\phi^*(x) - Vx^*] \geq 0.$$

Therefore, solving the new extension problem (9) is equivalent to solving and checking the feasibility of a certain nonlinear matrix inequality. This is a general matrix inequality that includes some important special cases.

**Example 1 (Riccati inequality).** Assume  $\phi(x) = x$  is a matrix and let us denote it by  $X$  (capital letter instead). In this case, the nonlinear inequality reduces to a quadratic expression in the unknown  $X$  of the form

$$XAX^* + BX + X^*B^* + C \geq 0,$$

which is a Riccati inequality. Here, the quantities  $(A, B, C)$  are related through  $(R^{-1}, U, V, \rho)$ . Therefore, solving the corresponding extension problem is equivalent

to checking the feasibility and finding a solution to a Riccati inequality. Note however that this is not an arbitrary Riccati inequality; its parameters  $\{A, B, C\}$  are interrelated. In a sense we have a "structured" Riccati inequality.

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**Example 2 (Linear matrix inequality).** Assume  $\phi(x) = x + c$ , a row vector and partition the entries of  $x$  and  $c$  into

$$x = \begin{bmatrix} x_1 & x_2 & \dots & x_q \end{bmatrix} \quad \text{and} \quad c = \begin{bmatrix} c_1 & c_2 & \dots & c_q \end{bmatrix}.$$

Then determining an  $x$  such that

$$\begin{bmatrix} FRF^* & FR\rho^* \\ \rho RF^* & \rho R\rho^* \end{bmatrix} + \begin{bmatrix} GJG^* & U\phi^*(x) - Vx^* \\ \phi(x)U^* - xV^* & \phi(x)\phi^*(x) - xx^* \end{bmatrix} \geq 0$$

is equivalent to checking the feasibility and solving the linear matrix inequality

$$F_0 + \sum_{k=1}^q x_k F_k \geq 0,$$

where

$$F_0 \triangleq \begin{bmatrix} R & FR\rho^* + Uc^* \\ \rho RF^* + cU^* & \rho R\rho^* + cc^* \end{bmatrix},$$

and

$$F_k \triangleq \begin{bmatrix} 0 & \text{col}_k(U - V) \\ (\text{col}_k(U - V))^* & 2c_k \end{bmatrix}.$$

The notation  $\text{col}_k(\cdot)$  denotes the  $k$ -th column of its argument.

Note again that we do not have an arbitrary linear matrix inequality [4] but rather one in which the  $F_k$  matrices have some structure and are fully determined in terms of the quantities  $\{F, U, V, R, \rho, c\}$ . We may say that we have a structured matrix inequality.

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Returning to the extension problem, it is clear that a solution need not always exist. One condition for the solvability of the problem was given in [5] in terms of



the function  $\phi$  (see the second part of the result below). However, it appeared to be of interest to find a convenient condition that will not make restrictions on  $\phi$ . Such a condition is described by the first part of the following result.

**Theorem 3 (Solvability Conditions)** *Each of the following conditions is by itself sufficient to guarantee solvability of the nonlinear matrix inequality (13) and, consequently, of the new extension problem (9):*

- (1)  $R - UU^* > 0$ , regardless of  $\phi$ .
- (2) A contractive  $\phi$ , viz., any  $\phi$  such that  $\|\phi(x) - \phi(y)\| \leq \|x - y\|$  for any norm and any arguments  $x$  and  $y$ , and  $R > 0$ .

**PROOF.** This is a special case of the general theorem established later in the paper (see Thm. 6 and Thm. 7).

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We now show how to determine a solution  $x$  in each of these two cases.

**Algorithm 2 (The Case  $R - UU^* > 0$ )** *Consider a positive-definite matrix  $R$  with displacement structure (1). We partition  $G$  into  $G = [U \ V]$  accordingly with  $J$ . Here,  $U$  is  $n \times p$  and  $V$  is  $n \times q$ . When  $R - UU^* > 0$ , a solution  $x$  to (9) and (13) exists and it can be found as follows:*

- (1) Apply the generalized Schur algorithm (3) to the generators

$$\left\{ \begin{bmatrix} 0 & 0 \\ 0 & F \end{bmatrix}, \begin{bmatrix} I & 0 \\ & G \end{bmatrix}, J \right\}$$

and construct the feedforward and the scattering cascades  $\Theta(z)$  and  $\Sigma(z)$ .

- (2) Terminate the scattering cascade with any strict Schur function.
- (3) Let  $S(z) = S_1z + S_2z^2 + S_3z^3 + \dots$  denote the strict Schur function that is mapped at the input of the scattering cascade. The condition  $R - UU^* > 0$  guarantees  $S_0 = 0$ .
- (4) Then  $x$  is given by

$$\begin{aligned} x &= \rho U S_1 + \rho F U S_2 + \rho F^2 U S_3 + \rho F^3 U S_4 + \dots & (14) \\ &= \left( \left\{ \dots, \rho F^2 U, \rho F U, \rho U, \boxed{0}, \dots \right\} * \left\{ \dots, 0, \boxed{0}, S_1, S_2, \dots, \right\} \right) (0) \end{aligned}$$

**PROOF.** This construction is again a special case of a general result established later in the paper (see the proof in Thm. 7).

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The construction (14) has a useful physical interpretation. It shows that we should excite the scattering cascade with the following input sequence, which starts in the remote past (say at  $-\infty$ ):

$$\left\{ \dots, \rho F^4 U, \rho F^3 U, \rho F^2 U, \rho F U, \rho U \right\}. \quad (15)$$

The desired  $x$  is then the output at the left-hand side of the cascade at the time instant following the end of the input sequence. Now recall that we are dealing with stable matrices  $F$ . Hence, the powers  $F^k$  tend to zero for increasing  $k$ . In this way, for all practical purposes, the above excitation sequence may not need to be infinitely long.

Consider again the Toeplitz extension problem that we considered earlier. That is, let  $R = T_n$ ,  $F = Z$ , and  $U = \text{col}\{1, c_1, \dots, c_n\}$ . In this case,  $R - UU^* > 0$  since it is equal  $(0 \oplus R_1)$ , where  $R_1$  is the Schur complement of  $R$  with respect to its  $(0, 0)$  entry and  $R$  is assumed positive-definite. Also, using  $\rho = [0 \ 0 \ \dots \ 1]$  and  $F = Z$ , we obtain that the excitation sequence (15) in this case should be

$$\left\{ \dots, 0, 0, \rho Z^{n-1} U, \dots, \rho Z U, \rho U, \boxed{0}, \dots \right\} = \left\{ \dots, 0, 0, 1, \dots, c_{n-1}, c_n, \boxed{0}, \dots \right\}.$$

That is, it has finite duration (since all the entries prior to the value 1 are zero). Hence, we can simply apply the sequence

$$\left\{ \boxed{1}, \dots, c_{n-1}, c_n \right\}$$

to the scattering cascade and measure  $x$  (or  $c_{n+1}$ ) at the left-most output at the time instant following the end of this input sequence, which in this case is at time  $(n+1)$ . We therefore see that that the construction (14) reduces to the well-known perfect reflection experiment (e.g., [6, 7]).

**Algorithm 3 (The Case of a Contractive  $\phi$ )** *Consider a positive-definite matrix  $R$  with displacement structure (1). We partition  $G$  into  $G = [U \ V]$  accordingly with  $J$ . Here,  $U$  is  $n \times p$  and  $V$  is  $n \times q$ . When  $\phi$  is a contraction, a solution  $x$  to (9) and (13) exists and it can be found as follows:*

- (1) *Apply the generalized Schur algorithm (3) to  $\{F, G, J\}$  and construct the feed-forward and the scattering cascades  $\Theta(z)$  and  $\Sigma(z)$ .*
- (2) *Terminate the scattering cascade with any strict Schur function.*
- (3) *Let  $S(z) = S_0 + S_1 z + S_2 z^2 + S_3 z^3 + \dots$  denote the strict Schur function that is mapped at the input of the scattering cascade. It follows that  $S_0$  is strictly contractive.*

(4) Then  $x$  is given by

$$x = \phi(x)S_0 + [\rho US_1 + \rho FUS_2 + \rho F^2US_3 + \dots] \quad (16)$$

$$= \phi(x)S_0 + \left( \left\{ \dots, \rho F^2U, \rho FU, \rho U, \boxed{0}, \dots \right\} * \left\{ \dots, 0, \boxed{0}, S_1, S_2, \dots \right\} \right) (0)$$

(5) By the Contraction Mapping Theorem [15], a solution  $x$  can be determined recursively from

$$x^{(n)} = \phi(x^{(n-1)})S_0 + [\rho US_1 + \rho FUS_2 + \rho F^2US_3 + \dots] ,$$

for any initial condition  $x^{(0)}$ .

**PROOF.** This construction is also a special case of the general result established in Thm. 6 later in the paper.

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Note that when  $S_0 = 0$  the above solution reduces to the one we had before in the case  $R - UU^* > 0$ . More specifically, the expression for  $x$  becomes independent of  $\phi$ . What we are really doing here is replace the identity feedback in the classical perfect reflection experiment by a nonlinear feedback (at time 0) and study the modeling capabilities of the resulting cascade, as shown in Fig. 4.

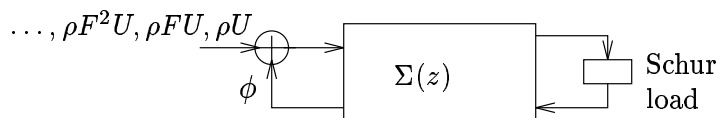


Fig. 4. A generalized reflection experiment.

### 3 Nonstationary Extension Problems

We now extend the results of the earlier sections to time-variant structured matrices and, therefore, to time-variant systems and cascades. In what follows, the symbol  $\mathbf{Z}$  denotes the set of integers.

Let  $\{F(t), t \in \mathbf{Z}\}$  denote a sequence of  $n \times n$  lower triangular block matrices that are assumed to be uniformly bounded, *viz.*, there exists a positive constant  $c_f$ , independent of  $t$ , such that

$$\|F(t)\| < c_f \quad \text{for all } t \in \mathbf{Z}. \quad (17)$$

We also assume that the diagonal entries  $\{f_i(t)\}$  of  $\{F(t)\}$  are uniformly stable, which means that there exists a positive constant  $c$  such that

$$\|f_i(t)\| \leq c < 1, \quad \text{for all } t \text{ and } i = 0, \dots, n-1. \quad (18)$$

We further consider a sequence  $\{G(t) = [U(t) \ V(t)], t \in \mathbf{Z}\}$  of uniformly bounded  $n \times r(t)$  matrices and introduce the signature matrices

$$J(t) = \begin{bmatrix} I_{p(t)} & 0 \\ 0 & -I_{q(t)} \end{bmatrix}, \quad r(t) = p(t) + q(t).$$

**Definition 4 (Time-Variant Displacement)** *A given sequence of  $n \times n$  matrices  $\{R(t)\}_{t \in \mathbf{Z}}$  is said to have a displacement structure with respect to  $\{F(t), G(t)\}_{t \in \mathbf{Z}}$  if the family  $\{R(t)\}_{t \in \mathbf{Z}}$  is uniformly bounded and satisfies the displacement equation*

$$R(t) - F(t)R(t-1)F^*(t) = G(t)J(t)G^*(t). \quad (19)$$

*We say that (19) admits a Pick solution if  $R(t)$  is positive-semidefinite for every  $t \in \mathbf{Z}$ .*

Comparing with the earlier definition (1), we are now dealing with a family of matrices  $\{R(t)\}$ , one for each  $t$ , and not only a single matrix  $R$ . Also, we are not requiring each individual  $R(t)$  to be structured in the sense defined by (1). Instead, we are requiring that the  $R(t)$  change in time in such a way that each  $R(t)$  is obtained via a low rank modification of a term that depends on the previous  $R(t-1)$ , *viz.*,  $F(t)R(t-1)F^*(t)$ . Moreover, the displacement inertia  $J(t)$  itself is also allowed to vary with time. Such time-updates are common in applications, especially in the adaptive filtering literature (*e.g.*, [8, 9]).

The initial time instant in (19) is not specified, and is often assumed to be in the remote past (say  $t \rightarrow -\infty$ ). It may also be at some finite time  $t_0$  for which  $R(t_0)$  is available. However, under the previous assumptions on  $\{F(t), G(t)\}$ , the infinite block matrices

$$\begin{aligned} \mathcal{U}(t) &\triangleq \begin{bmatrix} \dots & F(t)F(t-1)U(t-2) & F(t)U(t-1) & U(t) \end{bmatrix}, \\ \mathcal{V}(t) &\triangleq \begin{bmatrix} \dots & F(t)F(t-1)V(t-2) & F(t)V(t-1) & V(t) \end{bmatrix}, \end{aligned}$$

can be shown to be bounded, and the displacement equation (19) will be guaranteed to have a unique uniformly bounded solution that is given by

$$R(t) = \mathcal{U}(t)\mathcal{U}^*(t) - \mathcal{V}(t)\mathcal{V}^*(t). \quad (20)$$

Note that in view of the stability condition on the  $\{f_i(t)\}$ , this expression for  $R(t)$  is independent of the boundary condition  $R(t \rightarrow -\infty)$ . Note further that the same conclusion is reached in the special, though frequent, case when  $F(t) = 0$  and  $G(t) = 0$  for  $|t|$  sufficiently large.

### 3.1 The Generalized Schur Algorithm

Given  $R(t-1)$ , we need to know  $\{F(t), G(t), J(t)\}$  in order to determine  $R(t)$  using (19). In a manner similar to the time-invariant case (1), we shall not seek to determine  $R(t)$  by explicitly applying (19). Instead, we shall use the fact that  $R(t)$  is a “low rank” modification of  $R(t-1)$  and exploit it to determine  $R(t)$  more efficiently. This will be achieved by extending the generalized Schur algorithm to this case. This extension will operate as follows. It will use as input data  $\{F(t), G(t), J(t)\}$  and the Cholesky factor of  $R(t-1)$ , say  $\bar{L}(t-1)$ , and then compute the Cholesky factor of  $R(t)$  without determining  $R(t)$ ,

$$R(t) = \bar{L}(t)\bar{L}^*(t). \quad (21)$$

The following algorithm of [10, 11] tells us how to compute the columns of  $\bar{L}(t)$  from the columns of  $\bar{L}(t-1)$  and knowledge of  $\{F(t), G(t), J(t)\}$ . Here we describe the algorithm for positive-definite matrices  $R(t)$ .

Let  $\bar{l}_i(t)$  denote the nonzero part of the  $i$ -th column of  $\bar{L}(t)$ . Let also  $1/\sqrt{d_i(t)}$  denote the top entry of  $\bar{l}_i(t)$  and define

$$l_i(t) = \sqrt{d_i(t)} \bar{l}_i(t).$$

That is, the top entry of  $l_i(t)$  is normalized to 1, and we also obtain the equivalent triangular factorization for  $R(t)$ ,

$$R(t) = L(t)D^{-1}(t)L^*(t), \quad (22)$$

where the diagonal entries of  $D(t)$  are the  $\{d_i(t)\}$  and the columns of  $L(t)$  are the  $\{l_i(t)\}$ . Here,  $L(t)$  is a unit diagonal lower triangular matrix.

Let also  $R_i(t)$  denote the Schur complement of  $R(t)$  with respect to its leading  $i \times i$  block. All these definitions are straightforward extensions of what we did in the time-invariant case - just drop the letter  $t$ . In fact, an easy way to appreciate

the following statement is to simply freeze the time variable  $t$  and focus on the corresponding matrix  $R(t)$ .

**Algorithm 4 (The Generalized Schur Algorithm)** *Assume we know*

$$\{L(t-1), D(t-1), G(t), F(t), J(t)\},$$

and that  $R(t)$  satisfies the displacement equation (19). Then the Schur complements of  $R(t)$  satisfy a similar displacement equation,

$$R_i(t) - F_i(t)R_i(t-1)F_i^*(t) = G_i(t)J(t)G_i^*(t), \quad (23)$$

where the  $G_i(t)$ , and the triangular factorization (22) of  $R(t)$ , can be obtained from the following recursive construction:

$$\begin{bmatrix} 0 \\ l_i(t) \\ G_{i+1}(t) \end{bmatrix} = \begin{bmatrix} F_i(t)l_i(t-1) & G_i(t) \end{bmatrix} \begin{bmatrix} f_i^*(t) & h_i^*(t)\Theta_i(t) \\ J(t)g_i^*(t) & k_i^*(t)\Theta_i(t) \end{bmatrix}, \quad (24)$$

where  $g_i(t)$  is the top row of  $G_i(t)$ ,  $h_i(t)$  and  $k_i(t)$  are given by

$$h_i^*(t) = \frac{1 - \tau_i(t)f_i^*(t)}{\tau_i(t)d_i(t) - d_i(t-1)f_i(t)} g_i(t) \quad (25)$$

$$k_i^*(t) = \left( I_{r(t)} - \frac{\tau_i(t)J(t)g_i^*(t)g_i(t)}{\tau_i(t)d_i(t) - d_i(t-1)f_i(t)} \right), \quad (26)$$

$\Theta_i(t)$  is an arbitrary  $J(t)$ -unitary matrix, ( $\Theta_i(t)J(t)\Theta_i^*(t) = J(t)$ ), and  $\tau_i(t)$  is an arbitrary complex number satisfying

$$|\tau_i(t)|^2 = d_i(t-1)/d_i(t).$$

Moreover, it can be shown that by choosing  $\Theta_i(t) = I_{r(t)}$  and  $\tau_i(t)$  in the opposite direction of  $f_i(t)$ , then  $\{h_i(t)\Theta_i^*(t), k_i(t)\Theta_i^*(t)\}_{t \in \mathbf{Z}}$  are guaranteed to be uniformly bounded sequences.

The above description includes two degrees of freedom; the rotations  $\Theta_i(t)$  and the complex numbers  $\tau_i(t)$ . Choices of  $\{\Theta_i(t), \tau_i(t)\}$  that result in array form descriptions are also possible and are described in detail in [10]. We shall choose  $\tau_i = (1 + f_i)/(1 + f_i^*)$ .

We may also remark that in view of their definitions, the quantities

$$\{f_i(t), g_i(t), h_i(t), k_i(t)\}$$

satisfy the so-called embedding relation

$$\begin{bmatrix} f_i(t) & g_i(t) \\ h_i(t) & k_i(t) \end{bmatrix} \begin{bmatrix} d_i(t-1) & 0 \\ 0 & J(t) \end{bmatrix} \begin{bmatrix} f_i(t) & g_i(t) \\ h_i(t) & k_i(t) \end{bmatrix}^* = \begin{bmatrix} d_i(t) & 0 \\ 0 & J(t) \end{bmatrix}. \quad (27)$$

Such relations are useful in the characterization of lossless systems.

### 3.2 The Scattering Cascade

The transformation matrix

$$\begin{bmatrix} f_i^*(t) & h_i^*(t)\Theta_i(t) \\ J(t)g_i^*(t) & k_i^*(t)\Theta_i(t) \end{bmatrix}$$

that appears in (24) can be regarded as the state-space description of an elementary section (just like (5) for  $\Theta_i(z)$ ). Here we shall write  $\mathcal{T}_i$  to denote the upper triangular operator of Markov parameters that is associated with this elementary section; it is now a time-variant system (rather than time-invariant and described by a transfer function  $\Theta_i(z)$  or a Toeplitz upper-triangular operator).

Let  $[T_{lj}^{(i)}]$  denote the Markov parameters of  $\mathcal{T}_i$  (they can be determined explicitly in terms of the state-space representation of  $\mathcal{T}_i$  [10].) Then it can be shown that each  $\mathcal{T}_i$  is a bounded upper-triangular linear operator that satisfies

$$\mathcal{T}_i \mathcal{J} \mathcal{T}_i^* = \mathcal{J} = \mathcal{T}_i^* \mathcal{J} \mathcal{T}_i = \mathcal{J}, \quad (28)$$

where we defined

$$\mathcal{J} \triangleq \bigoplus_{t \in \mathbf{Z}} J(t).$$

This so-called  $\mathcal{J}$ -losslessness property is the extension to the time-variant scenario of the  $J$ -losslessness of each section  $\Theta_i(z)$  in the time-invariant case. It is also a direct consequence of the embedding relation (27). In this way, and assuming  $R(t)$  is  $n \times n$ , we obtain a cascade of  $n$  time-variant elementary sections that we describe as (see Fig. 5)

$$\mathcal{T} = \mathcal{T}_0 \mathcal{T}_1 \mathcal{T}_2 \dots \mathcal{T}_{n-1}.$$

The  $\mathcal{J}$ -losslessness property of each section  $\mathcal{T}_i$  reflects on the entire cascade  $\mathcal{T}$  since it can be shown that  $\mathcal{T}$  is also a bounded upper-triangular linear operator that

$$\mathcal{T} \mathcal{J} \mathcal{T}^* = \mathcal{T}^* \mathcal{J} \mathcal{T} = \mathcal{J}.$$

Related discussion on the properties of such time-variant systems can also be found in the approaches developed in [12, 13, 14].

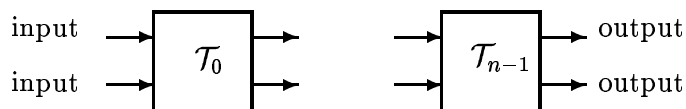


Fig. 5. *Cascade of first-order time-variant sections.*

In order to obtain the scattering cascade that corresponds to  $\mathcal{T}$ , we note that the Markov parameters  $T_{lj}^{(i)}$  of each first-order transfer operator  $\mathcal{T}_i$  are  $r(l) \times r(j)$  matrix entries. We partition  $T_{lj}^{(i)}$  accordingly with  $J(l)$  and  $J(j)$ , *viz.*,

$$T_{lj}^{(i)} \triangleq \begin{bmatrix} T_{11,i}^{lj} & T_{12,i}^{lj} \\ T_{21,i}^{lj} & T_{22,i}^{lj} \end{bmatrix},$$

where  $T_{11,i}^{lj}$ ,  $T_{12,i}^{lj}$ ,  $T_{21,i}^{lj}$ , and  $T_{22,i}^{lj}$  are  $p(l) \times p(j)$ ,  $p(l) \times q(j)$ ,  $q(l) \times p(j)$ , and  $q(l) \times q(j)$  matrices, respectively. We further define the upper-triangular operators

$$\begin{aligned} \mathcal{T}_{11}^{(i)} &\triangleq \left[ T_{11,i}^{lj} \right]_{l,j=-\infty}^{\infty}, & \mathcal{T}_{12}^{(i)} &\triangleq \left[ T_{12,i}^{lj} \right]_{l,j=-\infty}^{\infty}, \\ \mathcal{T}_{21}^{(i)} &\triangleq \left[ T_{21,i}^{lj} \right]_{l,j=-\infty}^{\infty}, & \mathcal{T}_{22}^{(i)} &\triangleq \left[ T_{22,i}^{lj} \right]_{l,j=-\infty}^{\infty} \end{aligned}$$

[This step is equivalent to the partitioning of  $\Theta(z)$  in the time-invariant case into its block entries  $\Theta_{ij}(z)$  – see, *e.g.*, (7).] It can be shown [10] that when  $R(t)$  is uniformly positive-definite, *i.e.*,  $R(t) > \epsilon I > 0$ , for all  $t$  and for some  $\epsilon$  independent of  $t$ , then the operator

$$\mathcal{S} \triangleq \mathcal{T}_{12}^{(i)} \mathcal{T}_{22}^{- (i)} \tag{29}$$

is upper-triangular and strictly contractive. Moreover, the scattering cascade that corresponds to  $\mathcal{T}$  is described in operator form by

$$\Sigma = \begin{bmatrix} \mathcal{T}_{11} - \mathcal{T}_{12} \mathcal{T}_{22}^{-1} \mathcal{T}_{21} & -\mathcal{T}_{12} \mathcal{T}_{22}^{-1} \\ \mathcal{T}_{22}^{-1} \mathcal{T}_{21} & \mathcal{T}_{22}^{-1} \end{bmatrix}. \tag{30}$$

The operator  $\mathcal{S}$  is the transfer operator at the input of the scattering cascade (from its left-most input to output ports). This cascade also maps strictly contractive loads to strictly contractive operators at its inputs (see Fig. 6).



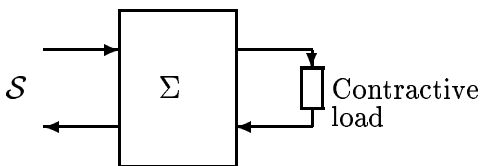


Fig. 6. *Scattering interpretation.*

### 3.3 A Nonstationary Extension Problem

We now develop the time-variant version of the extension problem we studied earlier (*viz.*, (9)).

**Problem 5 (Extension Problem)** *Given a family  $\{R(t)\}_{t \in \mathbf{Z}}$  of matrices that satisfy the displacement equation (19), a family  $\{\rho(t)\}_{t \in \mathbf{Z}}$  of appropriate dimensions and a function  $\phi$ , it is required to find conditions for the existence of a family  $\{x(t)\}_{t \in \mathbf{Z}}$  such that the displacement equation*

$$\bar{R}(t) - \bar{F}(t)\bar{R}(t-1)\bar{F}^*(t) = \bar{G}(t)J(t)\bar{G}^*(t),$$

*admits a Pick solution that is an extension of  $\{R(t)\}_{t \in \mathbf{Z}}$ , i.e.,*

$$\bar{R}(t) = \begin{bmatrix} R(t) & * \\ * & * \end{bmatrix} \tag{31}$$

*for all  $t \in \mathbf{Z}$ , and*

$$\bar{F}(t) = \begin{bmatrix} F(t) & 0 \\ \rho(t) & 0 \end{bmatrix}, \quad \bar{G}(t) = \begin{bmatrix} G(t) & \\ \phi(x(t)) & x(t) \end{bmatrix}.$$

It is clear that a necessary condition for the solvability of the extension problem is that  $\{R(t)\}_{t \in \mathbf{Z}}$  must be a Pick family itself. In general, however, this condition is not sufficient. Consider, for instance, a complex number  $z$  such that  $0 < |z| < 1$ ,  $F(t) = 0$ ,  $G(t) = [1 \ z]$ ,  $\rho(t) = 0$  for  $t \in \mathbf{Z}$  and

$$\phi(\lambda) = \begin{cases} 0 & \lambda \neq 0 \\ 1 & \lambda = 0. \end{cases}$$

It is then easy to check that the displacement equation (19) associated with these choices has a Pick solution, but that the extension problem with this data has no solution. The following result was obtained in [5], but we repeat the proof here for

the sake of completeness.

**Theorem 6 (Solvability Condition)** *Assume the nondegeneracy condition*

$$\mathcal{U}(t)\mathcal{U}^*(t) \geq \mu > 0 ,$$

for all  $t \in \mathbf{Z}$ , and assume that  $\phi$  is a contractive function,

$$\|\phi(x) - \phi(y)\| \leq \|x - y\|.$$

If the displacement equation (19) has a Pick solution  $\{R(t)\}_{t \in \mathbf{Z}}$  such that

$$R(t) > \epsilon I > 0 , \quad (32)$$

for a constant  $\epsilon$  and for all  $t$ , then the extension problem is solvable.

**PROOF.** The proof is constructive. Due to our assumptions on the data we deduce from Thm. 3.1 in [10] (see also Eq. (29) above) that there exists an upper triangular strict contraction  $\mathcal{S} = [S_{ij}]_{i,j \in \mathbf{Z}}$  ( $\|\mathcal{S}\| < 1$ ), such that

$$\mathcal{V}(t) = \mathcal{U}(t) \begin{bmatrix} \ddots & & & \vdots \\ & S_{t-2,t-2} & S_{t-2,t-1} & S_{t-2,t} \\ & & S_{t-1,t-1} & S_{t-1,t} \\ & & & S_{tt} \end{bmatrix} \quad (33)$$

for all  $t \in \mathbf{Z}$ . Define

$$\tilde{S}_t \triangleq \begin{bmatrix} \vdots \\ S_{t-2,t} \\ S_{t-1,t} \end{bmatrix}.$$

Since  $\mathcal{S}$  is a strict contraction, it follows that  $\|S_{tt}\|_{2,ind} < 1$  for all  $t \in \mathbf{Z}$  and an application of the Contraction Mapping Theorem [15] shows that the equation

$$x(t) = \phi(x(t))S_{tt} + \rho(t)\mathcal{U}(t-1)\tilde{S}_t, \quad (34)$$

has a unique solution  $x_0(t)$ . Now, if we define

$$\bar{F}(t) \triangleq \begin{bmatrix} F(t) & 0 \\ \rho(t) & 0 \end{bmatrix}, \quad \bar{G}(t) \triangleq \begin{bmatrix} G(t) \\ \phi(x_0(t)) \quad x_0(t) \end{bmatrix},$$

then we see that

$$\bar{\mathcal{V}}(t) \triangleq \left[ \cdots \begin{bmatrix} \bar{F}(t) & 0 \\ \rho(t) & 0 \end{bmatrix} \begin{bmatrix} V(t-1) \\ x_0(t-1) \end{bmatrix} \begin{bmatrix} V(t) \\ x_0(t) \end{bmatrix} \right] = \begin{bmatrix} \mathcal{V}(t) \\ \rho(t)\mathcal{V}(t-1) \quad x_0(t) \end{bmatrix}.$$

Similarly, we obtain that

$$\bar{\mathcal{U}}(t) \triangleq \begin{bmatrix} \mathcal{U}(t) \\ \rho(t)\mathcal{U}(t-1) \quad \phi(x_0(t)) \end{bmatrix}.$$

Since  $x_0(t)$  satisfies (34), it follows that

$$\bar{\mathcal{V}}(t) = \bar{\mathcal{U}}(t) \begin{bmatrix} \cdots & & \cdots \\ & S_{t-2,t-2} & S_{t-2,t-1} & S_{t-2,t} \\ & & S_{t-1,t-1} & S_{t-1,t} \\ & & & S_{tt} \end{bmatrix} \triangleq \bar{\mathcal{U}}(t)[S_{ij}]_{i,j \leq t}, \quad (35)$$

and this relation shows that the displacement equation

$$\bar{R}(t) - \bar{F}(t)\bar{R}(t-1)\bar{F}^*(t) = \bar{G}(t)J(t)\bar{G}^*(t), \quad (36)$$

admits the solution

$$\begin{aligned} \bar{R}(t) &= \bar{\mathcal{U}}(t)\bar{\mathcal{U}}^*(t) - \bar{\mathcal{V}}(t)\bar{\mathcal{V}}^*(t) \\ &= \bar{\mathcal{U}}(t) \left( I - [S_{ij}]_{i,j \leq t} [S_{ij}]_{i,j \leq t}^* \right) \bar{\mathcal{U}}^*(t). \end{aligned}$$

Since  $\mathcal{S}$  is a contraction, it follows that  $\{\bar{R}(t)\}_{t \in \mathbf{Z}}$  is a Pick solution of (36). Finally, we note that each  $\bar{R}(t)$ ,  $t \in \mathbf{Z}$ , is an extension of  $R(t)$ , in the sense that the  $(1,1)$  block entry of  $\bar{R}(t)$  is exactly  $R(t)$ . Therefore,  $\{x_0(t)\}_{t \in \mathbf{Z}}$  is a solution of Prob. 5.

◇

Expression (34) shows how to construct an  $x(t)$ , *viz.*,

$$x(t) = \phi(x(t))S_{tt} + \rho(t)U(t-1)S_{t-1,t} + \rho(t)F(t-1)U(t-2)S_{t-2,t} + \dots$$

The  $\{S_{ij}\}$  are the Markov parameters of the time-variant transfer operator at the input of the scattering cascade of Fig. 6. Hence, we have here the time-variant analogue of the experiment shown in Fig. 4.

Now note that if it happens that there exists an  $\mathcal{S}$  satisfying (33) and such that  $S_{tt} = 0$ , then the solution of the extension problem can be described simply by the formula

$$x(t) = \rho(t)U(t-1)S_{t-1,t} + \rho(t)F(t-1)U(t-2)S_{t-2,t} + \dots ,$$

which is independent of  $\phi$ . Therefore, if we can ensure that the additional constraint  $S_{tt} = 0$  is satisfied by a solution  $\mathcal{S}$ , then a result on the solvability of Prob. 5 can be obtained without any assumption on  $\phi$ . In fact, this is possible as it is showed by the main result of this section.

**Theorem 7 (Alternative Condition)** *Let  $\{R(t)\}_{t \in \mathbf{Z}}$  satisfy the displacement equation (19). If the matrices  $R(t) - U(t)U^*(t)$  are positive semidefinite for  $t \in \mathbf{Z}$ , then the extension problem is solvable.*

**PROOF.** The condition  $R(t) - U(t)U^*(t) \geq 0$  means that the displacement equation

$$R'(t) - \begin{bmatrix} 0 & 0 \\ 0 & F(t) \end{bmatrix} R'(t-1) \begin{bmatrix} 0 & 0 \\ 0 & F(t) \end{bmatrix}^* = \begin{bmatrix} I & 0 \\ U(t) & V(t) \end{bmatrix} J(t) \begin{bmatrix} I & 0 \\ U(t) & V(t) \end{bmatrix}^*$$

admits a Pick solution. By Thm. 2.2 in [11], we deduce that there exists an upper triangular contraction  $\mathcal{S} = [S_{ij}]_{i,j \in \mathbf{Z}}$  ( $\|\mathcal{S}\| \leq 1$ ), such that

$$\begin{bmatrix} 0 \\ \mathcal{V}(t) \end{bmatrix} = \begin{bmatrix} \circ I \\ \mathcal{U}(t) \end{bmatrix} [S_{ij}]_{i,j \leq t}$$

for all  $t \in \mathbf{Z}$ . It follows that  $S_{tt} = 0$  for all  $t \in \mathbf{Z}$  and

$$\mathcal{V}(t) = \mathcal{U}(t) [S_{ij}]_{i,j \leq t}.$$

The same construction as in the proof of the previous theorem concludes the argument.

◇

As done before, we can reformulate the extension problem as a matrix inequality problem. Indeed, taking into account the formulae for  $\bar{R}(t)$ ,  $\bar{F}(t)$ , and  $\bar{G}(t)$ , we obtain that the extension problem (31) is equivalent to the following inequality for the unknown  $x(t)$ :

$$\begin{bmatrix} R(t) & F(t)R(t-1)\rho^*(t) \\ \rho(t)R(t-1)F^*(t) & \rho(t)R(t-1)\rho^*(t) \end{bmatrix} + \begin{bmatrix} 0 & U(t)\phi^*(x(t)) - V(t)x^*(t) \\ \phi(x(t))U^*(t) - x(t)V^*(t) & \phi(x(t))\phi^*(x(t)) - x^*(t)x(t) \end{bmatrix} \geq 0. \quad (37)$$

**Example 3. (Nonstationary linear matrix inequality)** Suppose the entries of all involved matrices are real numbers. Suppose  $U(t)$ ,  $V(t)$  are  $p \times n$  matrices and  $\rho(t)$  is a  $p \times 1$  row vector. So,  $x(t)$  is a  $1 \times n$  row vector,  $x(t) = [x_1(t) \quad x_2(t) \quad \dots \quad x_n(t)]$ .

Set  $\phi(x) = x + c$ ,  $c = [c_1 \ c_2 \ \dots \ c_n]$ , so that (37) becomes (with  $T$  denoting matrix transposition):

$$\begin{aligned} & \begin{bmatrix} R(t) & F(t)R(t-1)\rho_1^T(t) + U(t)c^T \\ \rho(t)R(t-1)F^T(t) + cU^T(t) & \rho(t)R(t-1)\rho^T(t) + cc^T \end{bmatrix} \\ & + \begin{bmatrix} 0 & (U(t) - V(t))x^T(t) \\ \phi(x(t))(U(t) - V(t))^T & x(t)c^T + cx^T(t) \end{bmatrix} \geq 0, \end{aligned}$$

or, equivalently,

$$F(t) + \sum_{k=1}^n x_k(t)F_k(t) \geq 0,$$

where

$$F(t) = \begin{bmatrix} R(t) & F(t)R(t-1)\rho^T(t) + U(t)c^T \\ \rho(t)R(t-1)F^T(t) + cU^T(t) & \rho(t)R(t-1)\rho^T(t) + cc^T \end{bmatrix},$$

and

$$F_k(t) = \begin{bmatrix} 0 & \text{col}_k(U(t) - V(t)) \\ \text{col}_k(U(t) - V(t))^T & 2c_k \end{bmatrix}.$$

This is again an example of a (now time-variant) linear matrix inequality (LMI) that can be solved using Thm. 6.

◇

**Example 4. (KYP Lemma)** Consider the same situation as above. It follows from the proof of Thm. 6 that there exists a matrix  $P$  such that  $P = P^T$  and

$$F(t) + \begin{bmatrix} 0 & (U(t) - V(t))P \\ P(U(t) - V(t))^T & Pc^T + cP \end{bmatrix} \geq 0, \quad (38)$$

if, and only if, the displacement equation

$$R(t) - F(t)R(t-1)F^T(t) = G(t)J(t)G^T(t),$$

admits a Pick solution and the corresponding  $\mathcal{S}$  (from the proof of Thm. 6) is such that

$$X \triangleq (cS_{tt} + \rho(t)\mathcal{U}(t-1)\tilde{S}_t)(I - S_{tt})^{-1} = X^T. \quad (39)$$

The inequality (38) is a Riccati inequality that is usually attached to the Kalman–Yakubovich–Popov Lemma (see, for instance, [16]). Checking the condition (39) appears as an additional constraint. This seems to be a question of broader interest and will be addressed elsewhere.

◇

### 3.4 Application to Tensor Algebras

In this section we describe a rather unexpected connection of the displacement structure theory with tensor algebras.

The tensor algebra  $E_{\otimes}$  over  $E = \mathbf{C}^N$  is defined by  $E_{\otimes} = \mathbf{C} \oplus E_1 \oplus E_2 \oplus \dots$ , where  $E_n = E \otimes E \otimes \dots \otimes E$  ( $n$  copies of  $E$ ) is the  $n$ -fold algebraic tensor product. The elements of  $E_{\otimes}$  are terminating sequences  $f = (f_0, f_1, \dots, f_k, 0, \dots)$ , where  $f_p \in E_p$  is called the  $p$ th homogeneous component of  $f$  (note that  $E_0 = \mathbf{C}$ ,  $E_1 = E$ ). The addition and multiplication of elements in  $E_{\otimes}$  are defined componentwise:

$$(f + g)_n = f_n + g_n$$

and

$$(fg)_n = \sum_{k+l=n} f_k \otimes g_l, \quad (f_0 \otimes g_n = g_n \otimes f_0 = f_0 g_n).$$

We use the fact that  $E_{\otimes}$  can be represented by upper triangular matrices of a special type. Thus, let  $F(E)$  be the full Fock space associated to  $E$ , that is, the Hilbert space

$$F(E) = \oplus_{n \geq 0} E_n$$

obtained by taking the direct sum of the corresponding spaces on which we consider the tensor Hilbert space structure that is induced by the Euclidean norm on  $E = \mathbf{C}^N$ . If  $\{e_1, e_2, \dots, e_N\}$  is the standard basis of  $\mathbf{C}^N$ , then

$$\{e_{i_1} \otimes \dots \otimes e_{i_k} \mid i_1, \dots, i_k \in \{1, 2, \dots, N\}\}$$

is an orthonormal basis for  $E_k$ . We use to write the elements of these bases in lexicographic order. Thus, for  $N = 2$ ,

$$1 < e_1 < e_2 < e_1 \otimes e_1 < e_1 \otimes e_2 < e_2 \otimes e_1 < e_2 \otimes e_2 < \dots$$

In this way,  $E_k$  can be identified with the direct sum of  $N$  copies of  $E_{k-1}$  (see [17] for more details).

We now consider upper triangular matrices with respect to the decomposition  $\oplus_{n \geq 0} E_n$  of  $F(E)$ . We define  $\mathcal{T}_N^0$  to be the set of upper triangular matrices  $T = (T_{ij})_{i,j=0}^{\infty}$  with the property that for  $i \leq j$ ,

$$T_{ij} = T_{i-1,j-1} \oplus T_{i-1,j-1} \oplus \dots \oplus T_{i-1,j-1} \quad (N \text{ copies}) \quad (40)$$

and  $T_{ij} = 0$  for  $j$  sufficiently large. We notice that the entries  $T_{0j}$ ,  $j \geq 0$ , determine the matrix  $T$ .

We define an algebra isomorphism

$$\Phi : E_{\otimes} \rightarrow \mathcal{T}_N^0$$

as follows: for  $f \in E_{\otimes}$ ,  $f = (f_0, f_1, \dots)$ , we write  $f_0 = c_0$  and if  $j > 0$ ,

$$f_j = \sum_{i_1, \dots, i_j \in \{1, \dots, N\}} c_{i_1, \dots, i_j} e_{i_1} \otimes \dots \otimes e_{i_j}.$$

Then we define the matrices  $T_{0j} = [c_{1, \dots, 1}, \dots, c_{i_1, \dots, i_j}, \dots, c_{N, \dots, N}]$ ,  $j \geq 0$ . As noticed before, these matrices will determine a unique element  $T = \Phi(f)$  in  $\mathcal{T}_N^0$ . It is then easily checked that the mapping  $\Phi$  is an algebra isomorphism.

This representation of the tensor algebra suggests that we can extend  $E_{\otimes}$  by considering the algebra  $\mathcal{T}_N$  of all upper triangular bounded operators satisfying the condition (40). Recently, this algebra was related to interpolation and factorization in several noncommuting variables, see for instance [18], [19], and the references therein. The associated Schur class  $\mathcal{S}_N$  consists of all contractions in  $\mathcal{T}_N$ . Relevant to the tensor algebra are the so-called left creation operators  $S_k$  defined by  $S_k f = e_k \otimes f$  for  $f \in F(E)$ . The next result relates the Schur class  $\mathcal{S}_N$  to displacement structure.

**Theorem 8** *Let  $T = (T_{ij})_{i,j=0}^{\infty} \in \mathcal{S}_N$  and  $A = I - T^*T$ . Then*

$$A - \sum_{k=1}^N S_k A S_k^* = G J G^*,$$

where

$$G = \begin{bmatrix} 1 & T_{00}^* \\ 0 & T_{01}^* \\ \vdots & \vdots \end{bmatrix} \quad \text{and} \quad J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

**PROOF.** Since  $S_k T^* = T^* S_k$  for all  $k = 1, \dots, N$  we deduce that

$$I - T^*T - \sum_{k=1}^N S_k (I - T^*T) S_k^* = I - \sum_{k=1}^N S_k S_k^* - T^* (I - \sum_{k=1}^N S_k S_k^*) T.$$

Notice that  $I - \sum_{k=1}^N S_k S_k^* = P_0$ , the projection onto  $E_0$ , so we can deduce that

$$A - \sum_{k=1}^N S_k A S_k^* = G J G^*.$$

◇

The displacement equation  $A - \sum_{k=1}^N S_k A S_k^* = G J G^*$  can be rewritten in the form of a time dependent displacement equation of the type analysed in [10]. Therefore, the theory developed so far for displacement structures can be applied, in particular

the Schur algorithm. It is interesting to note that the associated Schur parameters (as introduced in Section 5.3, [11]) satisfy a relation similar to (40).

**Theorem 9** *Let  $T = (T_{ij})_{i,j=0}^{\infty} \in \mathcal{S}_N$  and let  $\{\Gamma_{ij}\}$  be the set of its Schur parameters obtained from the Schur algorithm. Then*

$$\Gamma_{ij} = \Gamma_{i-1,j-1} \oplus \Gamma_{i-1,j-1} \oplus \dots \Gamma_{i-1,j-1} \quad (N \text{ copies}).$$

**PROOF.** It follows from Theorem 2.1 in [20].

◇

These results motivate the study of the class of matrices  $R$  satisfying the displacement equation:

$$R - \sum_{k=1}^N F_k R F_k^* = G J G^*, \quad (41)$$

where  $F_1, F_2, \dots, F_N$  and  $G$  are the generators of the equation. The main result about this class of matrices refers to an associated scattering experiment. Let  $G = [U, V]$  with respect to the decomposition  $J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ . We also introduce the (wave) operators

$$\begin{aligned} \mathcal{U} &\triangleq [\dots F_1 F_2 U \quad F_1^2 U \quad F_N U \dots F_1 U \quad U]^t, \\ \mathcal{V} &\triangleq [\dots F_1 F_2 V \quad F_1^2 V \quad F_N V \dots F_1 V \quad V]^t, \end{aligned}$$

where  $t$  denotes the matrix transpose. We now obtain the main result of this section.

**Theorem 10** *The displacement equation  $R - \sum_{k=1}^N F_k R F_k^* = G J G^*$  admits a positive-semidefinite solution  $R$  if, and only if, there is  $S \in \mathcal{S}_N$  such that  $\mathcal{V} = S^* \mathcal{U}$ .*

**PROOF.** This is a direct application of Theorem 2.2 in [11] and we can omit the details.

◇

We can use this result in order to solve interpolation problems in several noncommuting variables and factorization problems in the tensor algebra just as particular cases of similar problems in [10, 11].



**Example 5. (Nevanlinna-Pick problem for several noncommuting variables)** Suppose for simplicity  $N = 2$ . A time-variant evaluation of an upper triangular bounded operator can be defined as follows ([10]): for a stable sequence of scalar points  $\{f(t)\}_{t \in \mathbf{Z}}$  (viz. there is  $c > 0$  such that  $|f(t)| < c < 1$  for all  $t \geq t_0$ ) and  $T = [T_{ij}]$  an upper triangular bounded operator, the evaluation of  $T$  at  $\{f(t)\}_{t \in \mathbf{Z}}$  is

$$T(f(t)) = T_{tt} + f(t)T_{t-1,t} + f(t)f(t-1)T_{t-2,t} + \dots$$

Let  $\lambda = (\lambda^{(1)}, \lambda^{(2)})$  be a point in  $\mathcal{B}^2$ , the open unit ball in  $\mathbf{C}^2$ . We notice that the components of the column vector

$$L = \begin{bmatrix} 1 \\ \lambda^{(1)} \\ \lambda^{(2)} \\ (\lambda^{(1)})^2 \\ \lambda^{(1)}\lambda^{(2)} \\ \lambda^{(2)}\lambda^{(1)} \\ (\lambda^{(2)})^2 \\ \vdots \end{bmatrix}$$

form a stable family, also denoted by  $L$ . If  $T$  is an element of the Schur class  $\mathcal{S}_2$  then the evaluation of  $T$  at  $L$  shows that the column vector  $L$  is an eigenvector of  $T$  corresponding to an eigenvalue  $T(\lambda^{(1)}, \lambda^{(2)})$  (whose expression in terms of  $\lambda_1, \lambda_2$  and the entries of  $T$  is easy to be obtained). The number  $T(\lambda) = T(\lambda^{(1)}, \lambda^{(2)})$  is introduced in [18], [19] and references therein as the evaluation of  $T$  at the point  $\lambda = (\lambda^{(1)}, \lambda^{(2)}) \in \mathcal{B}^2$ .

Now, a Nevanlinna-Pick problem can be formulated in this framework: given distinct points  $\lambda_1, \lambda_2, \dots, \lambda_k$  in  $\mathcal{B}^2$  and complex numbers  $b_1, b_2, \dots, b_k$ , describe all elements  $T \in \mathcal{S}_2$  such that  $T(\lambda_j) = b_j$  for all  $j = 1, \dots, k$ .

A solution to this problem was obtained in [19] and references therein; here we show that this follows from Theorem 10, hence from the results in [10] and [11]. Thus, we define

$$F_1 = \begin{bmatrix} \lambda_1^{(1)} & 0 & \dots & 0 \\ 0 & \lambda_2^{(1)} & & \\ \vdots & & \ddots & \\ 0 & & & \lambda_k^{(1)} \end{bmatrix}, \quad F_2 = \begin{bmatrix} \lambda_1^{(2)} & 0 & \dots & 0 \\ 0 & \lambda_2^{(2)} & & \\ \vdots & & \ddots & \\ 0 & & & \lambda_k^{(2)} \end{bmatrix},$$

and

$$[U, V] = \begin{bmatrix} 1 & b_1 \\ 1 & b_2 \\ \vdots & \vdots \\ 1 & b_k \end{bmatrix}.$$

By Theorem 10, the Nevanlinna-Pick problem has a solution if and only if the displacement equation with the generators defined above admits a positive-semidefinite solution  $R$ . By direct computation, we deduce that  $R = [\frac{1-b_i\overline{b_j}}{1-\langle\lambda_i,\lambda_j\rangle}]$  ( $\langle\cdot,\cdot\rangle$  denotes the Euclidean inner product), so we obtain the criterion in [19]. The parametrization of all solutions can be obtained now as a particular case of the results in [10] and [11].

## 4 Concluding Remarks

The purpose of this work has been to formulate a general extension problem and to solve it via inverse scattering experiments, in both cases of stationary and non-stationary structures. In addition, connections with tensor algebra were also highlighted.

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