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**RECURSIVE SOLUTIONS OF RATIONAL INTERPOLATION PROBLEMS VIA  
FAST MATRIX FACTORIZATION\***

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We describe a novel approach to analytic rational interpolation problems of the Hermite-Fejér type, based on the fast generalized Schur algorithm for the recursive triangular factorization of structured matrices. We use the interpolation data to construct a convenient so-called generator for the factorization algorithm. The recursive algorithm then leads to a transmission-line cascade of first-order sections that makes evident the interpolation property. We also give state-space descriptions for each section and for the entire cascade.

## 1 INTRODUCTION

Interpolation problems of various types have a long history in mathematics and in circuit theory, control theory, and system theory. In this paper we shall only be concerned with rational interpolating functions that are analytic inside the unit disc. Not surprisingly, the rich subject of interpolation theory can be approached in many ways and in different settings. We cannot pretend to a full appreciation of the many prior contributions to this literature. The books of Foias and Frazho [FF90], Ball, Gohberg, and Rodman [BGR90], and the monographs of Helton [Hel87],

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Kovalishina and Potapov [KP88], and Dym [Dym89a] may be consulted for this. The following remarks are an attempt to give some of the flavor of the different approaches.

The interest in scalar analytic interpolating functions can be traced back to the early work on moment problems by Chebyshev, Stieltjes, Toeplitz, Carathéodory, Pick, Schur, and Nevanlinna. The main tool used during this period was complex function theory. An excellent reference to this early approach to moment problems, and to related questions in analysis, is the book of Akhiezer [Akh65]. Later studies of matrix- and operator-valued interpolating functions were approached via different methods. Sarason, Sz. Nagy, and Foias [Sar67, SNF70] explored the links between operator theory, interpolation theory, and the lifting of commutants. Adamjan, Arov, and Krein [AAK68] discussed the connection of interpolation problems to the approximation of Hankel operators. Potapov [KP88] showed how to reduce various problems in analysis to the so-called Fundamental Matrix Inequality, which was then solved by using the factorization theory of analytic  $J$ -expansive matrix functions. Krein and Nudelman [KN73] employed the methods of convex geometry in their study of moment problems. Ball and Helton [BH83] used the notion of Krein spaces in the study of meromorphic interpolation.

The interest in interpolation problems, and more specifically in rational interpolating functions, was further stimulated by the realization in the last three decades that several problems in system theory, and especially in  $H^\infty$ -control theory, can be reduced to rational interpolation problems. A classical applications paper in circuit theory was that of Youla and Saito [YS67], which was followed up and significantly extended by Helton [Hel87], and then others. Applications in control theory were pioneered by Tannenbaum [Tan82], who applied the Nevanlinna-Pick theory to the robust stabilization problem, and Zames and Francis [Zam82, ZF83], who solved the minimum sensitivity problem via interpolation theory. More recent results appear in the works of Kimura [Kim87, Kim89a] and Limebeer et al. [LA88, LG90].

However, for system and control applications, where linear systems are often described via state-space models (see, *e.g.*, [Kai80]), it was very appealing to look for state-space-based solutions for the interpolation problems. A significant step in this direction was the work of Glover [Glo84] who, motivated by the earlier work of Adamjan, Arov, and Krein [AAK68], showed how to solve the related Hankel-norm model reduction problem in the state-space domain. The corresponding state-space approach to interpolation problems was then followed up by Kimura [Kim88, Kim89b, Kim89a], who was perhaps the first to give a closed form expression for the interpolating solution in state-space form, and which has since then been rederived in different ways (see, *e.g.*, [Dym89b, BGR90]).

Kimura's expression was given in the continuous-time domain, *i.e.*, for rational interpolating functions that are analytic and bounded by one in the right-half plane. The corresponding description in the discrete-time case, where the right-half plane is replaced by the interior of the unit disc, is analogous to expression (7) that will be derived ahead, where  $\Theta(z)$  is a rational  $J$ -lossless matrix function that is used to parametrize all interpolating solutions. The expression for  $\Theta(z)$  involves three quantities  $F, G$  and  $R^{-1}$ , where  $F$  and  $G$  are matrices that are constructed from the interpolation data, and  $R$  is the positive-definite solution (also known as a Pick matrix) of the Lyapunov/Stein (or displacement) equation  $R - FRF^* = GJG^*$ , where  $J$  is a signature matrix (all these constructions and connections will be discussed in detail in the sequel). We shall write  $\Theta(z) = (F, G, R^{-1})$ .

Employing the notion of  $J$ -lossless conjugation, Kimura [Kim89a] then rederived the results of Genin et al. [GDK<sup>+</sup>83], which showed, following the work of Livsic (see, *e.g.*, [LY79]), that the  $J$ -lossless transfer matrix  $\Theta(z)$  could be recursively decomposed into a cascade of first-order  $J$ -lossless sections by computing the successive Schur complements of  $R$ . More specifically, if  $r_{00}$  denotes the  $(0,0)$  entry of  $R$  and  $R_1$  denotes the Schur complement of  $r_{00}$  in  $R$ , then  $\Theta(z)$  can be decomposed into the product of a first-order  $J$ -lossless section  $\Theta_0(z) = (f_0, g_0, r_{00}^{-1})$  and an  $(n-1)$ -order  $J$ -lossless function  $\Theta_1(z) = (F_1, G_1, R_1^{-1})$ , where the quantities  $(f_0, g_0, F_1, G_1)$  are computed from the original  $(F, G)$ . Similar results for the half-plane case were also obtained by Limebeer and Anderson [LA88] in their analysis of the complexity of  $H^\infty$ -controllers. We may remark that the expressions for  $\Theta(z)$ , and the corresponding cascade decomposition, require explicit knowledge of the inverse of  $R$ , the solution of  $R - FRF^* = GJG^*$ , as well as the inverses of the successive Schur complements of  $R$ , which is not attractive from a computational point of view.

In related work, and motivated by the results in  $H^\infty$ -control theory, Ball, Gohberg, and Rodman [BGR90] further developed the state-space framework for rational interpolation problems, making many connections with system theory and transfer function realizations. A detailed theory of the pole-zero structure of rational matrix-valued functions is developed and subsequently employed to devise a state-space characterization of the interpolating solutions. In this framework, priority is given to global descriptions of the solutions rather than to recursive procedures.

Recursive solutions though have a long history, starting with the early works of Schur and Nevanlinna for the scalar case. A more recent formulation appeared in the work of Nudelman [Nud77, Nud81] who showed how to reduce a general interpolation problem to one involving Potapov's fundamental matrix inequality and also devised a recursive construction of the solution.

A related treatment of a recursive procedure for the extraction of elementary  $J$ -lossless sections also appeared in the work of Dewilde and Dym [DD81]. In the setting studied in [DD81], Pick matrices did not play any explicit role as happens in the interpolation problem. A later analysis of the recursive extraction, however, and based on the theory of reproducing kernel Hilbert spaces (RKHS) [AD86], showed that the procedure corresponded to a recursive Schur complementation of a Pick matrix, as explained above. The connections to analytic interpolation problems were further pursued and made more explicit by Dym [Dym89b, Dym89a], who proposed the following procedure for the solution of interpolation problems: use the interpolation data to construct a convenient RKHS that is characterized by a  $J$ -lossless matrix-valued function; this function can then be used to parametrize the solutions and it can also be factored into elementary sections as  $\Theta(z)$  above.

In this paper, we describe a novel recursive solution and derivation for analytic rational interpolation problems. The basis for our approach are some results we have developed over several years on algorithms for the recursive triangular factorization of matrices with displacement structure; we called the algorithm a generalized Schur algorithm, since it grew out of the seminal work of Schur [Sch17]. The recursive algorithm leads to a cascade of  $J$ -lossless first-order sections, each of which has an evident interpolation property. This is due to the fact that linear systems have “transmission zeros”: certain inputs at certain frequencies yield zero outputs. More specifically, each section of the cascade can be characterized by a  $(p + q) \times (p + q)$  rational transfer matrix,  $\Theta_i(z)$  say, that has a left zero-direction vector  $g_i$  at a frequency  $f_i$ , viz.,

$$g_i \Theta_i(f_i) \equiv \begin{bmatrix} u_i & v_i \end{bmatrix} \begin{bmatrix} \Theta_{i,11} & \Theta_{i,12} \\ \Theta_{i,21} & \Theta_{i,22} \end{bmatrix} (f_i) = \mathbf{0} ,$$

which makes evident (with the proper partitioning of the row vector  $g_i$  and the matrix function  $\Theta_i(z)$ ) the following interpolation property:  $u_i \Theta_{i,12} \Theta_{i,22}^{-1}(f_i) = -v_i$ . This suggested to us that one way of solving an interpolation problem is to show how to construct an appropriate cascade so that the local interpolation properties of the elementary sections combine in such a way that the cascade yields a solution to the global interpolation problem. All possible interpolants can be parametrized by attaching various loads to the right-hand side of the cascade system.

The main feature of our derivation is that it approaches the subject from a matrix-factorization point of view, and that it relies almost completely on matrix-based arguments: we use the interpolation data to construct a so-called generator matrix; the generator is then used to start a recursive algorithm for the computation of the Cholesky factor of an associated structured matrix;

each step of the algorithm yields a first-order  $J$ -lossless section with an intrinsic “blocking” or “transmission zero” property; these local blocking relations are then shown to combine to yield the desired interpolation conditions. The paper is quite self-contained, and gives a simple proof of the particular form of the generalized Schur algorithm needed herein.

We emphasize that, unlike many previous approaches, we do not approach the subject by first studying the problem of  $J$ -lossless extraction or cascade factorization. We instead start by factoring a structured matrix and then show that we obtain, as an interesting fallout, a  $J$ -lossless cascade system as well as solutions to analytic interpolation problems. Finally, we note that an earlier version of this paper circulated as a preprint since [KS91], and that the results were also presented at several meetings, see e.g. [SK92, Say92].

## 1.1 Notation

Let  $RH_{p \times q}^\infty$  denote the space of  $p \times q$  rational matrix-valued functions  $K(z)$  that are analytic and bounded inside the open unit disc ( $|z| < 1$ ) with norm  $\|K\|_\infty = \sup_{|z| < 1} \bar{\sigma}[K(z)]$ , where  $\bar{\sigma}[K(z)] = \sup_{\|x\|_2=1} \|K(z)x\|_2$  denotes the spectral norm of  $K$  at  $z$ . A matrix valued function  $S(z) \in RH_{p \times q}^\infty$  that is strictly bounded by unity in  $|z| < 1$  ( $\|S\|_\infty < 1$ ) will be referred to as a function of *Schur type*. We also use the notation  $A_*(z)$  to denote the para-Hermitian conjugate of  $A(z)$ , viz.,  $A_*(z) = [A(1/z^*)]^*$ , where the symbol  $*$  stands for Hermitian conjugation (complex conjugation for scalars). We further write  $\mathcal{H}_A^k(z)$  to refer to the following block-Toeplitz upper-triangular matrix

$$\mathcal{H}_A^k(z) = \begin{bmatrix} A(z) & \frac{1}{1!}A^{(1)}(z) & \frac{1}{2!}A^{(2)}(z) & \dots & \frac{1}{(k-1)!}A^{(k-1)}(z) \\ & A(z) & \frac{1}{1!}A^{(1)}(z) & \dots & \frac{1}{(k-2)!}A^{(k-2)}(z) \\ & & A(z) & \dots & \frac{1}{(k-3)!}A^{(k-3)}(z) \\ & & & \ddots & \vdots \\ & \mathbf{O} & & \ddots & \frac{1}{1!}A^{(1)}(z) \\ & & & & A(z) \end{bmatrix},$$

where  $A(z)$  is a rational matrix function analytic at  $z$ ,  $k \geq 1$  is a positive integer, and  $A^{(i)}(z)$  denotes the  $i^{\text{th}}$  derivative at  $z$ . We denote by  $e_i = \begin{bmatrix} \mathbf{0}_{1 \times i} & 1 & \mathbf{0} \end{bmatrix}$  the  $i^{\text{th}}$  basis vector of the  $n$ -dimensional space of complex numbers  $\mathcal{C}^{1 \times n}$ .

## 1.2 The Hermite-Fejér Problem

We now state a general Hermite-Fejér interpolation problem that includes many of the classical problems as special cases (see, *e.g.*, [FF90, pp. 294–298] and [BGR90, pp. 407–413]). We consider  $m$  stable points  $\{\alpha_i\}_{i=0}^{m-1}$  (*i.e.*, inside the open unit disc  $|z| < 1$ ) and we associate with each  $\alpha_i$  a positive integer  $r_i \geq 1$  and two row vectors  $\mathbf{a}_i$  and  $\mathbf{b}_i$  partitioned as follows:

$$\mathbf{a}_i = \begin{bmatrix} u_1^{(i)} & u_2^{(i)} & \dots & u_{r_i}^{(i)} \end{bmatrix} \quad \text{and} \quad \mathbf{b}_i = \begin{bmatrix} v_1^{(i)} & v_2^{(i)} & \dots & v_{r_i}^{(i)} \end{bmatrix},$$

where  $u_j^{(i)}$  and  $v_j^{(i)}$  ( $j = 1, \dots, r_i$ ) are  $1 \times p$  and  $1 \times q$  row vectors, respectively. That is,  $\mathbf{a}_i$  and  $\mathbf{b}_i$  are partitioned into  $r_i$  row vectors each. If an interpolation point  $\alpha_i$  is repeated (say,  $\alpha_i = \alpha_{i+1} = \dots = \alpha_{i+j}$ ), we shall then further assume that the following nondegeneracy condition is satisfied:

$$\{u_1^{(i)}, u_1^{(i+1)}, \dots, u_1^{(i+j)}\} \quad \text{are linearly independent.} \quad (1)$$

This assumption rules out degenerate cases and its relevance will become clear in the proof of Theorem 2.1. The tangential Hermite-Fejér problem reads as follows.

**Problem 1.2.1** *Describe all Schur-type functions  $S(z) \in RH_{p \times q}^\infty$  that satisfy*

$$\mathbf{b}_i = \mathbf{a}_i \mathcal{H}_S^{r_i}(\alpha_i) \quad \text{for} \quad 0 \leq i \leq m-1. \quad (2)$$

It is clear that the above statement includes, among others, several important special cases:

- **Scalar Carathéodory-Fejér [Akh65]:**  $m = 1$ ,  $\alpha_0 = 0$ ,  $r_0 = n$ ,  $p = q = 1$ ,  $\mathbf{a}_0 = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}$  and  $\mathbf{b}_0 = \begin{bmatrix} \beta_0 & \beta_1 & \dots & \beta_{n-1} \end{bmatrix}$ . In this case, we are reduced to finding a scalar Schur function  $s(z)$  that satisfies  $\frac{s^{(i)}(0)}{i!} = \beta_i$  for  $i = 0, 1, \dots, n-1$ .
- **Scalar Nevanlinna-Pick [Akh65]:**  $m = n$ ,  $\{\alpha_i\}$  distinct,  $r_i = 1$ ,  $p = q = 1$ ,  $\mathbf{a}_i = 1$  and  $\mathbf{b}_i = \beta_i$ . In this case, we are reduced to finding a scalar Schur function  $s(z)$  that satisfies  $s(\alpha_i) = \beta_i$  for  $i = 0, 1, \dots, n-1$ .
- **Tangential Nevanlinna-Pick [Fed72]:**  $m = n$ ,  $\{\alpha_i\}$  distinct,  $r_i = 1$ ,  $\mathbf{a}_i = u_i$  and  $\mathbf{b}_i = v_i$ , where  $u_i$  and  $v_i$  are  $1 \times p$  and  $1 \times q$  row vectors, respectively. In this case, we are reduced to finding a  $p \times q$  Schur matrix function  $S(z)$  that satisfies the tangential conditions  $u_i S(\alpha_i) = v_i$  for  $i = 0, 1, \dots, n-1$ .

**Remark:** Assume we are required to determine a Schur function  $S(z)$  that satisfies two tangential conditions at the *same* point  $\alpha_0$ , say,  $u_0 S(\alpha_0) = v_0$  and  $u_1 S(\alpha_0) = v_1$ . If the row vectors  $u_0$  and  $u_1$  are equal (or parallel) then  $v_0$  and  $v_1$  must be equal (or parallel) as well, and the two conditions trivially collapse to a single one. To avoid such degenerate cases, we added the additional constraint, stated in (1), that  $u_0$  and  $u_1$  should be linearly independent.

## 2 SOLVABILITY CONDITION

The first step in the solution consists in constructing three matrices  $F, G$ , and  $J$  directly from the interpolation data:  $F$  contains the information relative to the points  $\{\alpha_i\}$  and the dimensions  $\{r_i\}$ ,  $G$  contains the information relative to the direction vectors  $\{\mathbf{a}_i\}$  and  $\{\mathbf{b}_i\}$ , and  $J = (I_p \oplus -I_q)$  is a signature matrix, where  $I_p$  denotes the  $p \times p$  identity matrix. The matrices  $F$  and  $G$  are constructed as follows: we associate with each  $\alpha_i$  a Jordan block  $\bar{F}_i$  of size  $r_i \times r_i$

$$\bar{F}_i = \begin{bmatrix} \alpha_i & & & & \\ & 1 & & & \\ & & \alpha_i & & \\ & & & \ddots & \\ & & & & \ddots & \\ & & & & & 1 & \alpha_i \end{bmatrix},$$

and two  $r_i \times p$  and  $r_i \times q$  matrices  $U_i$  and  $V_i$ , respectively, which are composed of the row vectors associated with  $\alpha_i$ ,

$$U_i = \begin{bmatrix} u_1^{(i)} \\ u_2^{(i)} \\ \vdots \\ u_{r_i}^{(i)} \end{bmatrix} \quad \text{and} \quad V_i = \begin{bmatrix} v_1^{(i)} \\ v_2^{(i)} \\ \vdots \\ v_{r_i}^{(i)} \end{bmatrix}.$$

Then

$$F = \begin{bmatrix} \bar{F}_0 & & & \\ & \bar{F}_1 & & \mathbf{O} \\ & & \ddots & \\ & & & \mathbf{O} \\ & \mathbf{O} & & \bar{F}_{m-1} \end{bmatrix} \quad \text{and} \quad G = \begin{bmatrix} U_0 & V_0 \\ U_1 & V_1 \\ \vdots & \vdots \\ U_{m-1} & V_{m-1} \end{bmatrix} \equiv \begin{bmatrix} \mathbf{U} & \mathbf{V} \end{bmatrix}. \quad (3)$$

Let  $n = \sum_{i=0}^{m-1} r_i$  and  $r = p + q$ , then  $F$  and  $G$  are  $n \times n$  and  $n \times r$  matrices, respectively. We shall denote the diagonal entries of  $F$  by  $\{f_i\}_{i=0}^{n-1}$  (for instance,  $f_0 = f_1 = \dots = f_{r_0-1} = \alpha_0$ ). For the special examples considered in the previous section, the matrices  $F, G$ , and  $J$  are given by:

- **Scalar Carathéodory-Fejér:**  $J = \text{diagonal}\{1, -1\}$ ,

$$F = Z \equiv \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & \ddots & \ddots & & \\ & & & 1 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & \beta_0 \\ 0 & \beta_1 \\ \vdots & \vdots \\ 0 & \beta_{n-1} \end{bmatrix}.$$

- **Scalar Nevanlinna-Pick:**  $J = \text{diagonal}\{1, -1\}$ ,

$$F = \begin{bmatrix} f_0 & & & & \\ & f_1 & & \mathbf{O} & \\ & & \ddots & & \\ & & & \mathbf{O} & \\ & \mathbf{O} & & & f_{n-1} \end{bmatrix}, \quad G = \begin{bmatrix} 1 & \beta_0 \\ 1 & \beta_1 \\ \vdots & \vdots \\ 1 & \beta_{n-1} \end{bmatrix}. \quad (4)$$

- **Tangential Nevanlinna-Pick:**  $J = (I_p \oplus -I_q)$ ,

$$F = \begin{bmatrix} f_0 & & & & \\ & f_1 & & \mathbf{O} & \\ & & \ddots & & \\ & & & \mathbf{O} & \\ & \mathbf{O} & & & f_{n-1} \end{bmatrix}, \quad G = \begin{bmatrix} u_0 & v_0 \\ u_1 & v_1 \\ \vdots & \vdots \\ u_{n-1} & v_{n-1} \end{bmatrix}.$$

We shall show in the next sections that by applying a simple recursive procedure to  $F$  and  $G$  we obtain a cascade structure that satisfies the interpolation conditions (2) and, in fact, parametrizes all solutions. Meanwhile, we shall associate with the interpolation problem the following displacement equation (for the origin of the name see, *e.g.*, [KKM79] and [LAK86]),

$$R - FRF^* = GJG^*, \quad (5)$$

where  $F$  and  $G$  are constructed as in (3). Clearly,  $R$  is unique since  $F$  is a stable matrix ( $|f_i| < 1, \forall i$ ). We say that  $R$  has a Toeplitz-like structure with respect to  $(F, G, J)$  and  $G$  is called its generator matrix [LAK86, LA83]. We also note that equation (5) can be thought of as a special case of discrete-time Lyapunov equations, which are also often called Stein equations. The displacement theory exploits the fact that often  $G$  has many fewer columns than  $R$ . We now verify that the above construction of  $F, G$ , and  $R$ , allows us to prove the necessary and sufficient conditions for the existence of solutions - see also [FF90, pp. 294–298] for a proof based on the lifting of commutants.



**Theorem 2.1** *Under the nondegeneracy condition (1) (or (6)), the tangential Hermite-Fejér problem is solvable if, and only if,  $R$  is positive-definite ( $R > 0$ ).*

[ The proof that follows has some novel and motivating ingredients; however readers interested chiefly in the algorithm may go directly to Section 3].

**Proof:** If  $R$  is positive-definite then the recursive procedure described in the next sections finds a solution  $S(z)$ . Conversely, assume there exists a solution  $S(z)$  satisfying the interpolation conditions (2) and let  $\{S_i\}_{i=0}^{\infty}$  be the Taylor series coefficients of  $S(z)$  around the origin, viz.,

$$S(z) = S_0 + zS_1 + z^2S_2 + z^3S_3 + \dots$$

Define the (semi-infinite) block lower-triangular Toeplitz matrix

$$\mathcal{S} = \begin{bmatrix} S_0 & & & & \\ S_1 & S_0 & & & \\ S_2 & S_1 & S_0 & & \\ \vdots & & & \ddots & \\ & & & & \mathbf{O} \end{bmatrix},$$

as well as the (semi-infinite) matrices (using (3))

$$\mathcal{U} = \begin{bmatrix} \mathbf{U} & F\mathbf{U} & F^2\mathbf{U} & \dots \end{bmatrix} \quad \text{and} \quad \mathcal{V} = \begin{bmatrix} \mathbf{V} & F\mathbf{V} & F^2\mathbf{V} & \dots \end{bmatrix}.$$

We can easily check, by comparing terms on both sides, that the following relation holds because of (2):  $\mathcal{V} = \mathcal{U}\mathcal{S}$ . But  $R$  in (5) is given by  $R = \mathcal{U}\mathcal{U}^* - \mathcal{V}\mathcal{V}^* = \mathcal{U}(I - \mathcal{S}\mathcal{S}^*)\mathcal{U}^*$ . Moreover,  $\mathcal{S}$  is a strict contraction (since  $S(z)$  is a Schur-type function with  $\|S\|_{\infty} < 1$ ) and it follows from (1) that  $\mathcal{U}$  has full row rank or equivalently that

$$\mathcal{U}\mathcal{U}^* > 0. \tag{6}$$

Hence,  $R > 0$ . Now let us give a proof of (6). This result is a consequence of the non-degeneracy assumption (1). One proof follows by invoking a well-known property of Krylov subspaces (see Theorem 1.2.4 in [BGR90] for an alternative route). For an arbitrary square matrix  $A$ , we let  $K^m(A, \mathbf{x}) \equiv \begin{bmatrix} \mathbf{x} & A\mathbf{x} & \dots & A^{m-1}\mathbf{x} \end{bmatrix}$  denote a Krylov matrix. We also denote by  $\mathcal{K}^m(A, \mathbf{x})$  the span of  $K^m(A, \mathbf{x})$ . It is a well-known fact [Par80] that the subspace  $\mathcal{K}^m(A, \mathbf{x})$  has the following invariance property: for any scalar  $\sigma$ ,  $\mathcal{K}^m(A, \mathbf{x}) = \mathcal{K}^m(A - \sigma I, \mathbf{x})$ . We now use this property to show that  $\mathcal{U} \equiv K^{\infty}(F, \mathbf{U})$  has full rank. For the sake of illustration, we first assume that  $F$  is

composed of only two Jordan blocks with the same eigenvalue  $\alpha_0$ , such as

$$F = \begin{bmatrix} \alpha_0 & & & & \\ & 1 & \alpha_0 & & \\ & & 0 & \alpha_0 & \\ & & & 1 & \alpha_0 \\ & & & & 1 & \alpha_0 \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} u_1^{(0)} \\ u_2^{(0)} \\ u_1^{(1)} \\ u_2^{(1)} \\ u_3^{(1)} \end{bmatrix}.$$

Then  $(F - \alpha_0 I) = Z_2 \oplus Z_3$ , where  $Z_i$  refers to the  $i \times i$  lower triangular shift matrix with ones on the first subdiagonal and zeros elsewhere. Therefore,

$$\mathcal{K}^\infty(F, \mathbf{U}) = \mathcal{K}^\infty(Z_2 \oplus Z_3, \mathbf{U}) = \text{span} \begin{bmatrix} u_1^{(0)} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ u_2^{(0)} & u_1^{(0)} & \mathbf{0} & \mathbf{0} \\ u_1^{(1)} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ u_2^{(1)} & u_1^{(1)} & \mathbf{0} & \mathbf{0} \\ u_3^{(1)} & u_2^{(1)} & u_1^{(1)} & \mathbf{0} \end{bmatrix},$$

and  $\mathcal{K}^\infty(F, \mathbf{U})$  clearly has full row rank since  $u_1^{(0)}$  and  $u_1^{(1)}$  are assumed linearly independent. Assume now that  $F$  still has two Jordan blocks as above, but with distinct eigenvalues  $\alpha_0$  and  $\alpha_1$ , respectively. It is not difficult to see that the same argument goes through, where we now use the invariance property twice: we first shift by  $\alpha_0$  and then by  $\alpha_1 - \alpha_0$ . The same reasoning can be extended to an arbitrary Jordan structure in  $F$  to show that (6) holds because of (1). ■

Notice that the proof of Theorem 2.1 explicitly uses the displacement equation (5) and shows (at least implicitly) that the contraction  $\mathcal{S}$  can be constructed from  $F$  and  $G$  via  $\mathcal{V} = \mathcal{U}\mathcal{S}$ . While an efficient construction will be given below, we remark that the proof establishes an important link between displacement equations and interpolation conditions, where the connection is achieved through the Jordan structure of  $F$ . We further remark that such links between the solution of displacement equations (as in (5)) and interpolation problems have also been pointed out by Ball et al. [BGR90, pp. 324–326] by using calculations based on function residues, and by Dym [Dym89b] through connections with reproducing kernel Hilbert spaces. The significance of the Jordan structure has also been noticed by Helton [Hel78], Rosenblum and Rovnyak [RR85], and Alpay et al. [ABDdS91].

We shall now proceed to show how to recursively construct realizations for contractions  $\mathcal{S}$  that solve the Hermite-Fejér problem. It should be mentioned though, that several existing approaches write down a solution formula for such interpolation problems by considering a global transfer matrix  $\Theta(z)$  of the following form,

$$\Theta(z) = I - (1 - z)JG^*(I - zF^*)^{-1}R^{-1}(I - F)^{-1}G. \quad (7)$$

We shall describe in the next section an alternative *recursive* approach that constructs  $\Theta(z)$  as a cascade of elementary sections. A derivation of the above global expression from the recursive argument developed in this paper will be given in Section 4.2.

### 3 RECURSIVE SCHUR COMPLEMENTATION

The key to our approach is an efficient way of exploiting the displacement structure to find a fast algorithm for the triangular factorization of the (structured) matrix  $R$  defined by the displacement equation (5). Now it is well-known that the triangular factor of any Hermitian matrix with nonzero leading minors can be obtained from the successive leading Schur complements of the matrix. The first step in our approach is to show that the successive computation of the Schur complements of  $R$  in (5) can be carried out in a computationally efficient recursive procedure by exploiting the structure implied by (5). We then prove in the next section that this recursive algorithm leads to a cascade (or transmission-line) structure with the desired interpolation properties.

Let  $R_i$  denote the Schur complement of the leading  $i \times i$  submatrix of  $R$ . If  $l_i$  and  $d_i$  stand for the first column and the  $(0,0)$  entry of  $R_i$ , respectively, then (the positive-definiteness of  $R$  guarantees  $d_i > 0$  for all  $i$ )

$$R_i - l_i d_i^{-1} l_i^* = \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & R_{i+1} \end{bmatrix} \equiv \tilde{R}_{i+1}. \quad (8)$$

Hence, starting with an  $n \times n$  matrix  $R$  and performing  $n$  consecutive Schur complement steps, we obtain the triangular factorization of  $R$ , viz.,

$$R = l_0 d_0^{-1} l_0^* + \begin{bmatrix} 0 \\ l_1 \end{bmatrix} d_1^{-1} \begin{bmatrix} 0 \\ l_1 \end{bmatrix}^* + \begin{bmatrix} 0 \\ 0 \\ l_2 \end{bmatrix} d_2^{-1} \begin{bmatrix} 0 \\ 0 \\ l_2 \end{bmatrix}^* \dots = LD^{-1}L^*,$$

where  $D = \text{diag}\{d_0, \dots, d_{n-1}\}$ , and the (nonzero parts of the) columns of the lower triangular matrix  $L$  are  $\{l_0, \dots, l_{n-1}\}$ . This procedure requires  $O(n^3)$  operations (elementary additions and multiplications).

The point, however, is that the procedure can be speeded up to  $O(rn^2)$  operations in the case of matrices  $R$  that exhibit displacement structure as in (5). In this case, the above (Gauss/Schur) reduction procedure can be shown to reduce to a recursion on the elements of the generator matrix  $G$ . The computational advantage then follows from the fact that  $G$  has  $rn$  elements as compared to  $n^2$  in  $R$ . Hence, great savings in computation are expected when  $r \ll n$ , as happens in important applications. The following results give the array form of the generator recursion (see [LA83, LAK92] for alternative derivations). We first state the results and then follow with a proof.

**Theorem 3.1** *If  $R$  is structured as in (5), then the Schur complements  $R_i$  are also structured with generator matrices  $G_i$ , viz., they satisfy displacement equations of the form  $R_i - F_i R_i F_i^* = G_i J G_i^*$ , where  $F_i$  is the submatrix obtained after deleting the first row and column of  $F_{i-1}$ , and  $G_i$  is an  $(n-i) \times r$  matrix that satisfies a recursive construction as detailed in the next algorithm.*

It is worth pointing out that generator matrices are not unique. If  $G_i$  is a generator for  $R_i$  then  $G_i \Theta_i$  is also a generator for any  $J$ -unitary matrix  $\Theta_i$  ( $\Theta_i J \Theta_i^* = J$ ), since  $G_i \Theta_i J \Theta_i^* G_i^* = G_i J G_i^*$ .

**Algorithm 3.1** *Generator matrices for the successive Schur complements can be recursively computed as follows: start with  $G_0 = G$ ,  $F_0 = F$ , and repeat for  $i = 0, 1, \dots, n-1$ ,*

(i) *At step  $i$  we have  $F_i$  and  $G_i$ .*

(ii) *Let  $g_i$  denote the first row of  $G_i$ . Choose any  $J$ -unitary rotation matrix  $\Theta_i$  that rotates  $g_i$  to  $g_i \Theta_i = \begin{bmatrix} \delta_i & 0 & \dots & 0 \end{bmatrix}$ , where the nonzero scalar entry  $\delta_i$  is in the first column position. We remark that such a  $\Theta_i$  can be implemented in a variety of ways, e.g., by using (hyperbolic) Givens or Householder transformations.*

(iii) *Construct  $G_{i+1}$  as follows:*

$$\begin{bmatrix} \mathbf{0}_{1 \times r} \\ G_{i+1} \end{bmatrix} = \Phi_i G_i \Theta_i \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + G_i \Theta_i \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & I_{r-1} \end{bmatrix}, \quad (9)$$

where  $\Phi_i$  is an  $(n-i) \times (n-i)$  “Blaschke” matrix defined by  $\Phi_i = (F_i - f_i I_{n-i})(I_{n-i} - f_i^* F_i)^{-1}$ . That is, multiply  $G_i$  by  $\Theta_i$  and keep the last  $r-1$  columns; multiply the first column of  $G_i \Theta_i$  by  $\Phi_i$ ; these two steps result in  $G_{i+1}$ .

(iv) *The  $i^{\text{th}}$  column  $l_i$  and the  $i^{\text{th}}$  diagonal entry  $d_i$  are given by*

$$l_i = (I_{n-i} - f_i^* F_i)^{-1} G_i \Theta_i \begin{bmatrix} \delta_i^* \\ \mathbf{0} \end{bmatrix}, \quad d_i = \frac{g_i J g_i^*}{1 - |f_i|^2} = \frac{|\delta_i|^2}{1 - |f_i|^2}. \quad (10)$$

The positive-definiteness of  $R$  is equivalent to the strict positivity of the entries  $\{d_i\}$  and consequently, of the quantities  $\{g_i J g_i^*\}$ ,  $i=0, \dots, n-1$ .

**Proof of Theorem 3.1 and Algorithm 3.1:** We shall give here an immediate verification of the validity of the algorithm. Following this proof however, we shall present an alternative and simple square-root-based argument that reveals the motivation behind the recursion.

We prove the result for  $R_1$ . The same argument is valid for  $i > 1$ . It follows from the displacement equation (5) that the  $(0, 0)$  entry of  $R$  is given by:  $d_0 = g_0 J g_0^* / (1 - |f_0|^2)$ , where  $g_0$  denotes the first row of  $G$ . But  $d_0 > 0$  and  $|f_0| < 1$ . Consequently,  $g_0 J g_0^* > 0$ . That is,  $g_0$  has positive  $J$ -norm. Hence, we can always find a  $J$ -unitary rotation  $\Theta_0$  ( $\Theta_0 J \Theta_0^* = J$ ) that reduces  $g_0$  to the form  $g_0 \Theta_0 = \begin{bmatrix} \delta_0 & 0 & \dots & 0 \end{bmatrix}$  (see Lemma 3.1.1 below). By comparing the  $J$ -norm on both sides of this equality we conclude that  $g_0 \Theta_0 J \Theta_0^* g_0^* = g_0 J g_0^* = |\delta_0|^2$ . That is,  $|\delta_0|^2 = d_0(1 - |f_0|^2) > 0$ . It further follows from the displacement equation (5) that the first column of  $R$  satisfies the equation:  $l_0 - F l_0 f_0^* = G J g_0^* = G \Theta_0 J \Theta_0^* g_0^*$ . Consequently,  $l_0 = (I_n - f_0^* F)^{-1} G \Theta_0 \begin{bmatrix} \delta_0^* & \mathbf{0} \end{bmatrix}^T$ . We still need to verify that the matrix  $G_1$  constructed via (9) is indeed a generator for the first Schur complement  $R_1$ . This follows by direct verification. Using (8) we have  $\tilde{R}_1 = R - l_0 d_0^{-1} l_0^*$ . Substituting for  $l_0$  and  $d_0$  we readily verify that  $\tilde{R}_1 - F \tilde{R}_1 F^* =$

$$\begin{aligned} & G J G^* - (1 - |f_0|^2) (I - f_0^* F)^{-1} G \Theta_0 \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \Theta_0^* G^* (I - f_0 F^*)^{-1} \\ & + (1 - |f_0|^2) F (I - f_0^* F)^{-1} G \Theta_0 \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \Theta_0^* G^* (I - f_0 F^*)^{-1} F^* , \end{aligned}$$

which, upon simplification, and using the fact that  $F(I - f_0^* F)^{-1} = (I - f_0^* F)^{-1} F$ , yields

$$\tilde{R}_1 - F \tilde{R}_1 F^* = G \Theta_0 \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_{p-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -I_q \end{bmatrix} \Theta_0^* G^* + \Phi_0 G \Theta_0 \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \Theta_0^* G^* \Phi_0^* .$$

The right-hand side of the above expression is precisely  $\begin{bmatrix} \mathbf{0} \\ G_1 \end{bmatrix} J \begin{bmatrix} \mathbf{0} & G_1^* \end{bmatrix}$ , where  $G_1$  is given by (9). Hence,  $R_1 - F_1 R_1 F_1^* = G_1 J G_1^*$ . ■

We remark here that the recursive algorithm (9) can be depicted graphically as a “cascade network” of elementary sections, one of which is shown in Figure 1;  $\Theta_i$  is any  $J$ -unitary matrix that rotates the first row of the  $i^{\text{th}}$  generator to  $\begin{bmatrix} \delta_i & \mathbf{0} \end{bmatrix}$ . The rows of  $G_i$  enter the section one

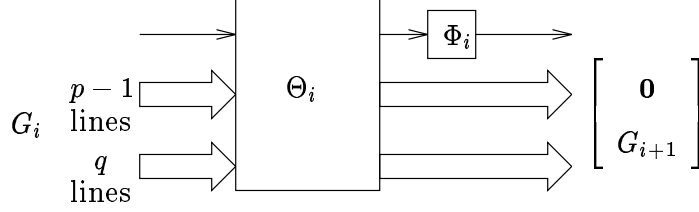


Figure 1: Feedforward signal-flow diagram of one step of the array recursion.

row at a time. The left-most entry of each row is applied through the top line, while the remaining entries are applied through the bottom lines. The “Blasckhe” matrix  $\Phi_i$  then acts on the entries of the top line. When  $F_i = Z$ , the lower shift matrix,  $\Phi_i = Z$ , a delay unit. In general, note that the first row of each  $\Phi_i$  is zero, and in this sense  $\Phi_i$  acts as a generalized delay element. To clarify this, observe that when the entries of the first row of  $G_i$  are processed by  $\Theta_i$  and  $\Phi_i$ , the values of the outputs of the section will all be zero. The rows of  $G_{i+1}$  will start appearing at these outputs only when the second and higher rows of  $G_i$  are processed by the section.

We further remark that the above algorithm is simply one version, among many other possible versions, of a general recursive procedure that is given in Section 3.3 ahead. It corresponds to a special choice of the rotation matrices  $\Theta_i$ , viz., those that rotate the successive rows  $g_i$  to the form

$$g_i \Theta_i = \begin{bmatrix} \delta_i & 0 & \dots & 0 \end{bmatrix}.$$

But other choices for  $\Theta_i$  are also possible, even  $\Theta_i = I$ . Each choice would then lead to new expressions that replace (9) and (10). Different choices for  $\Theta_i$  may result in different numerical implications.

### 3.1 A Square-Root Argument

The proof given in the previous section establishes the validity of the recursive procedure of Algorithm 3.1. But one may still wonder how did we guess expression (9)? In order to justify this result further, and in order to introduce a state-space description that will be useful later in (16), we shall now provide a simple motivation for (9). For this purpose, we first state a simple result in matrix theory that plays an important role in the derivation of all so-called square-root algorithms. We include a simple proof for completeness.

**Lemma 3.1.1** *Consider two  $n \times m$  ( $n \leq m$ ) matrices  $A$  and  $B$ . If  $AJA^* = BJB^*$  is of full rank, for some  $m \times m$  signature matrix  $J = (I_p \oplus -I_q)$ ,  $p + q = m$ , then there exists an  $m \times m$   $J$ -unitary*

matrix  $\Theta$  ( $\Theta J \Theta^* = J$ ) such that  $A = B\Theta$ .

**Proof:** One proof follows by invoking the hyperbolic singular value decompositions of  $A$  and  $B$  (see, e.g., [OSB91] and [Ack91, pp. 44-45]), viz.,

$$A = U_A \begin{bmatrix} \Sigma_{A,+} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \Sigma_{A,-} & \mathbf{0} \end{bmatrix} V_A^* , \quad B = U_B \begin{bmatrix} \Sigma_{B,+} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \Sigma_{B,-} & \mathbf{0} \end{bmatrix} V_B^* ,$$

where  $U_A$  and  $U_B$  are  $n \times n$  unitary matrices,  $V_A$  and  $V_B$  are  $m \times m$   $J$ -unitary matrices, and  $\Sigma_{A,+}, \Sigma_{A,-}, \Sigma_{B,+}$ , and  $\Sigma_{B,-}$ , are  $p' \times p', q' \times q', p' \times p'$ , and  $q' \times q'$  diagonal matrices, respectively, with  $p' + q' = n$ . It further follows from the full rank condition and the equality  $AJA^* = BJB^*$ , that  $\Sigma_{A,+} = \Sigma_{B,+}$ ,  $\Sigma_{A,-} = \Sigma_{B,-}$ , and that we can choose  $U_A = U_B$ . Let  $\Theta = JV_BJV_A^*$  then  $\Theta J \Theta^* = J$  and  $B\Theta = A$ . ■

Returning to the displacement equation (5), we first remark that the positive-definiteness of  $R$  guarantees the existence of a unique (lower triangular) Cholesky factor,  $\bar{L} = LD^{-1/2}$ , such that  $R = \bar{L}\bar{L}^*$ . We shall denote the *nonzero* parts of the columns of  $\bar{L}$  by  $\{\bar{l}_i\}_{i=0}^{n-1}$  ( $\bar{l}_i = l_i d_i^{-1/2}$ ). It then follows from (5) that we can write

$$\begin{bmatrix} \bar{L} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \bar{L}^* \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} F\bar{L} & G \end{bmatrix} \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & J \end{bmatrix} \begin{bmatrix} \bar{L}^* F^* \\ G^* \end{bmatrix} .$$

This last expression fits into the statement of Lemma 3.1.1. Hence, there exists an  $(I \oplus J)$ -unitary matrix  $\mathbf{\Gamma}$  such that

$$\begin{bmatrix} \bar{L} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} F\bar{L} & G \end{bmatrix} \mathbf{\Gamma} . \tag{11}$$

By examining this identity more closely we shall see that (9) will readily follow. We first note that the  $(I \oplus J)$ -unitary transformation  $\mathbf{\Gamma}$  can be achieved through a sequence of elementary transformations, say  $\mathbf{\Gamma}_0, \mathbf{\Gamma}_1, \mathbf{\Gamma}_2, \dots$ , that produces the block zero in the postarray in (11) by introducing one zero row at a time. We first implement  $\mathbf{\Gamma}_0$  as a sequence of two rotations  $\Theta_0$  and  $\Gamma_0$ . The first rotation  $\Theta_0$  reduces the first row of  $G$  to the form  $\begin{bmatrix} \delta_0 & \mathbf{0} \end{bmatrix}$ , and the second rotation  $\Gamma_0$  annihilates the remaining nonzero entry  $\delta_0$ . The overall effect is to annihilate the first row of the  $G$  matrix. We then proceed to implement  $\mathbf{\Gamma}_1, \mathbf{\Gamma}_2, \dots$  in a similar fashion. So let  $\Theta_0$  be the desired  $J$ -unitary matrix, viz.,

$$\begin{bmatrix} F\bar{L} & G \end{bmatrix} \begin{bmatrix} I \\ \Theta_0 \end{bmatrix} = \begin{bmatrix} & \delta_0 & 0 & 0 \\ F\bar{L} & x & x & x \\ & x & x & x \end{bmatrix} . \tag{12}$$

In order to annihilate the remaining nonzero entry  $\delta_0$  in the first row of the postarray, we still need an elementary unitary (Givens) rotation, say  $\Gamma_0$ . This can be done by “pivoting” against the first column of  $F\bar{L}$ , while keeping all other columns unchanged. This operation produces the first column of  $\bar{L}$  (because of (11)), and a matrix  $G_1$  whose significance we shall verify very soon:

$$\begin{bmatrix} \boxed{f_0 d_0^{1/2}} & \mathbf{0} & \boxed{\delta_0} & 0 & 0 \\ x & F_1 \bar{L}_1 & x & x & x \\ x & & x & x & x \end{bmatrix} \begin{bmatrix} c & & s & & \\ & I & & & \\ s^* & & -c & & \\ & & & & I \end{bmatrix} = \begin{bmatrix} \bar{l}_0 & \mathbf{0} & \boxed{0} & 0 & 0 \\ & F_1 \bar{L}_1 & & & \\ & & G_1 & & \end{bmatrix}, \quad (13)$$

where  $F_1$  and  $\bar{L}_1$  are the submatrices obtained after deleting the first row and column of  $F$  and  $\bar{L}$ , respectively. The letters  $c$  and  $s$  denote the (cosine and sine) parameters of the rotation matrix. Let  $\bar{x}_0$  and  $x_1$  denote  $\begin{bmatrix} \delta_0 & x & x \end{bmatrix}^T$  and the first column of  $G_1$ , respectively. From expression (13) we see that, ignoring the columns that remain unchanged and are thus common to the pre- and post-arrays,

$$\begin{bmatrix} F\bar{l}_0 & \bar{x}_0 \end{bmatrix} \begin{bmatrix} c & s \\ s^* & -c \end{bmatrix} = \begin{bmatrix} \bar{l}_0 & \mathbf{0} \\ & x_1 \end{bmatrix}. \quad (14)$$

The rotation parameters are clearly given by

$$c = \frac{1}{\sqrt{1 + |\rho_0|^2}}, \quad s = \frac{\rho_0}{\sqrt{1 + |\rho_0|^2}}, \quad \text{where} \quad \rho_0 = \frac{\delta_0}{f_0 d_0^{1/2}}.$$

That is, we can rewrite (14) more explicitly as follows:

$$\begin{bmatrix} F\bar{l}_0 & \bar{x}_0 \end{bmatrix} \begin{bmatrix} f_0^* & \frac{\delta_0}{d_0^{1/2}} \\ \frac{\delta_0}{d_0^{1/2}} & -f_0 \end{bmatrix} = \begin{bmatrix} \bar{l}_0 & \mathbf{0} \\ & x_1 \end{bmatrix}, \quad (15)$$

which leads to

$$\begin{bmatrix} 0 \\ x_1 \end{bmatrix} = \Phi_0 \bar{x}_0 = \Phi_0 G \Theta_0 \begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix}.$$

Hence,  $G_1$  is obtained as follows: choose a  $J$ -unitary rotation  $\Theta_0$  that converts the first row of  $G$  to the form  $\begin{bmatrix} \delta_0 & \mathbf{0} \end{bmatrix}$  and apply  $\Theta_0$  to  $G$  as in (12); keep the last  $(r-1)$  columns of  $G\Theta_0$  unchanged and multiply the first column by  $\Phi_0$ ; this results in  $G_1$ . This is precisely the same description as in (9).

We still need to verify that the matrix  $G_1$  so obtained is a generator of  $R_1$ . For this purpose, we compare the  $(I \oplus J)$ -norm on both sides of (13) and write

$$F\bar{L}\bar{L}^*F^* + \bar{G}J\bar{G}^* = \bar{l}_0\bar{l}_0^* + \begin{bmatrix} \mathbf{0} \\ F_1\bar{L}_1 \end{bmatrix} \begin{bmatrix} \mathbf{0} & \bar{L}_1^*F_1^* \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ G_1 \end{bmatrix} J \begin{bmatrix} \mathbf{0} & G_1^* \end{bmatrix}.$$



But the Cholesky factor of the first Schur complement  $R_1$  is  $\bar{L}_1$  itself. Hence, using (5), we get

$$R - \bar{l}_0 \bar{l}_0^* = \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & F_1 R_1 F_1^* \end{bmatrix} + \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & G_1 J G_1^* \end{bmatrix}.$$

Consequently (recall (8)),  $R_1 - F_1 R_1 F_1^* = G_1 J G_1^*$ , which shows that  $G_1$  is a generator matrix of the Schur complement  $R_1$ . Therefore, the effect of the transformation  $\mathbf{\Gamma}_0$ , which we implemented as a sequence of two rotations  $\Theta_0$  and  $\Gamma_0$ , is to annihilate the first row of  $G$ ,

$$\begin{bmatrix} F\bar{L} & G \end{bmatrix} \mathbf{\Gamma}_0 = \begin{bmatrix} \bar{l}_0 & \mathbf{0} & \mathbf{0} \\ & F_1 \bar{L}_1 & G_1 \end{bmatrix}.$$

We can now proceed by annihilating the first row of  $G_1$ ,

$$\begin{bmatrix} F_1 \bar{L}_1 & G_1 \end{bmatrix} \mathbf{\Gamma}_1 = \begin{bmatrix} \bar{l}_1 & \mathbf{0} & \mathbf{0} \\ & F_2 \bar{L}_2 & G_2 \end{bmatrix},$$

where  $F_2$  and  $\bar{L}_2$  are the submatrices obtained after deleting the first row and column of  $F_1$  and  $\bar{L}_1$  respectively, and so on. In summary, we are led to the same recursive procedure as in Algorithm 3.1.

### 3.2 First-Order $J$ -Lossless Sections.

It follows from the square-root argument (using (11)-(15)), or alternatively from Algorithm 3.1, that the expressions for  $l_i$  and  $G_i$  can be grouped together into the following revealing expression:

$$\begin{bmatrix} l_i & \mathbf{0} \\ & G_{i+1} \end{bmatrix} = \begin{bmatrix} F_i l_i & G_i \end{bmatrix} \begin{bmatrix} f_i^* & \frac{\delta_i}{d_i} \begin{bmatrix} 1 & \mathbf{0} \end{bmatrix} \\ \Theta_i \begin{bmatrix} \delta_i \\ \mathbf{0} \end{bmatrix} & \Theta_i \begin{bmatrix} -f_i & \mathbf{0} \\ \mathbf{0} & I_{r-1} \end{bmatrix} \end{bmatrix}. \quad (16)$$

We can regard the  $r \times r$  matrix in (16) as the so-called system matrix of a first-order state-space linear system. Let  $\Theta_i(z)$  denote its  $r \times r$  transfer matrix (with inputs from the left), viz.,

$$\Theta_i(z) = \Theta_i \begin{bmatrix} -f_i & \mathbf{0} \\ \mathbf{0} & I_{r-1} \end{bmatrix} + \Theta_i \begin{bmatrix} \delta_i \\ \mathbf{0} \end{bmatrix} (z^{-1} - f_i^*)^{-1} \frac{\delta_i}{d_i} \begin{bmatrix} 1 & \mathbf{0} \end{bmatrix}.$$

It then readily follows, upon simplification, that

$$\Theta_i(z) = \Theta_i \begin{bmatrix} \frac{z - f_i}{1 - z f_i^*} & \mathbf{0} \\ \mathbf{0} & I_{r-1} \end{bmatrix}. \quad (17)$$

Before proceeding we remark that a transfer matrix that is analytic in the unit disc, say  $\Theta_i(z)$ , is said to be  $J$ -lossless if it is unitary on the unit circle, i.e.,  $\Theta_i(z)J\Theta_i^*(z) = J$  on  $|z| = 1$ , and  $J$ -contractive inside the open unit disc, i.e.,  $\Theta_i(z)J\Theta_i^*(z) < J$  in  $|z| < 1$ . It turns out that the  $\Theta_i(z)$  in (17) is indeed  $J$ -lossless.

**Lemma 3.2.1** *Each step of the generator recursion gives rise to a first-order  $J$ -lossless transfer matrix  $\Theta_i(z)$  as in (17).*

**Proof:** The discussion preceding the statement of the Lemma shows that we can associate with each recursive step a first-order transfer matrix as in (17). Moreover,  $\Theta_i(z)$  is clearly analytic in  $|z| < 1$  since  $|f_i| < 1$ . It also satisfies  $\Theta_i(z)J\Theta_i^*(z) = J$  on  $|z| = 1$  and  $\Theta_i(z)J\Theta_i^*(z) < J$  in  $|z| < 1$  since  $(z - f_i)/(1 - zf_i^*)$  is a Blaschke factor and  $\Theta_i$  is  $J$ -unitary. ■

Furthermore, each such section has an important and evident blocking property.

**Lemma 3.2.2** *Each first-order section  $\Theta_i(z)$  has a transmission zero at  $f_i$  and along the direction defined by  $g_i$ , viz.,  $g_i\Theta_i(f_i) = \mathbf{0}$ .*

**Proof:** This is clear from the relation

$$g_i\Theta_i(f_i) = g_i\Theta_i \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & I_{r-1} \end{bmatrix} = \begin{bmatrix} \delta_i & \mathbf{0} \end{bmatrix} \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & I_{r-1} \end{bmatrix} = \mathbf{0}.$$
■

The blocking property means that when  $g_i$  (which is the first row of  $G_i$ ) is applied to  $\Theta_i(z)$  we obtain a zero output at the “frequency”  $z = f_i$ . We remark that the concepts of transmission zeros and blocking directions are central to many problems in network theory and linear systems, see, *e.g.*, [Kai80, BGR90]. In the time-domain, the local blocking property is equivalent to the fact that each step of the generator recursion introduces one more zero row into the generator. This time-domain interpretation is helpful for extensions of the approach of this paper to the time-variant setting as detailed in [SCK94, CSK94, Say92].

### 3.3 The General Form of the Generator Recursion.

We further remark that the array recursion (9), as stated in Algorithm 3.1 and later rederived in Section 3.1, is in fact a special case of a more general recursion. This is already suggested by the

square-root argument of Section 3.1, where we employed particular choices for the rotation matrices  $\Gamma_i$  rather than writing down a general form for each rotation. If we follow this line of reasoning we then obtain the following general forms of the generator recursion and the first-order sections - the reader is referred to [LA83, LAK92] for earlier derivations and to [Say92, Chapter 2] for a derivation along the lines of this paper:

$$\begin{bmatrix} \mathbf{0} \\ G_{i+1} \end{bmatrix} = \left\{ G_i + (\Phi_i - I_{n-i}) G_i \frac{J g_i^* g_i}{g_i J g_i^*} \right\} \Theta_i , \quad (18)$$

$$\Theta_i(z) = \left\{ I + [B_i(z) - 1] \frac{J g_i^* g_i}{g_i J g_i^*} \right\} \Theta_i , \quad (19)$$

where  $B_i(z)$  is a Blaschke factor of the form

$$B_i(z) = \frac{z - f_i}{1 - z f_i^*} \frac{1 - \tau_i f_i^*}{\tau_i - f_i} ,$$

$\Phi_i$  is a ‘‘Blaschke’’ matrix of the form,

$$\Phi_i = \frac{1 - \tau_i f_i^*}{\tau_i - f_i} (F_i - f_i I_{n-i}) (I_{n-i} - f_i^* F_i)^{-1} ,$$

$\Theta_i$  is an arbitrary  $J$ -unitary matrix and  $\tau_i$  is an arbitrary unit-modulus scalar. Observe that it is also evident here that  $g_i \Theta_i(f_i) = \mathbf{0}$  since

$$g_i \Theta_i(f_i) = \left\{ g_i + [B_i(f_i) - 1] \frac{g_i J g_i^*}{g_i J g_i^*} g_i \right\} \Theta_i = \{g_i - g_i\} \Theta_i = \mathbf{0} .$$

We also note that the generator recursion (18) has two free parameters:  $\Theta_i$  and  $\tau_i$ . If we choose  $\tau_i$  so as to satisfy

$$\frac{1 - \tau_i f_i^*}{\tau_i - f_i} = 1 ,$$

or equivalently,  $\tau_i = (1 + f_i)/(1 + f_i^*)$ , and choose  $\Theta_i$  such that  $g_i$  is reduced to the form  $g_i \Theta_i = \begin{bmatrix} \delta_i & \mathbf{0} \end{bmatrix}$ , then it is straightforward to check that (18) collapses to (9). Other choices for  $\Theta_i$  and  $\tau_i$  would lead to alternative implementations.

## 4 RELATION TO THE HERMITE-FEJÉR PROBLEM

The question now is: How does the recursive procedure of Algorithm 3.1 relate to the Hermite-Fejér interpolation problem? The relevant fact to note here is that each recursive step gives rise to a first-order  $J$ -lossless section  $\Theta_i(z)$ , which has an intrinsic blocking property as detailed above. After  $n$  recursive steps (recall that  $G$  has  $n$  rows) we obtain a cascade  $\Theta(z)$  defined by

$$\Theta(z) = \Theta_0(z) \Theta_1(z) \dots \Theta_{n-1}(z) . \quad (20)$$

We shall now show how, by the proper constructions of  $F$  and  $G$  in (3), we can use the local blocking properties to induce the desired interpolation properties into the global (cascade) system  $\Theta(z)$ . To begin with, we note that the  $J$ -losslessness of each section clearly reflects on the entire cascade  $\Theta(z)$ , viz.,  $\Theta(z)$  is also a  $J$ -lossless  $r \times r$  rational transfer matrix. This follows immediately from the definition of  $\Theta(z)$  in (20) and from the fact that each first-order section  $\Theta_i(z)$  is  $J$ -lossless as proved in Lemma 3.2.1.

Next, we shall show that the local blocking properties and the Jordan structure of  $F$  translate to global blocking properties on the entire cascade  $\Theta(z)$ . More specifically, the next theorem states that the rows of the generator matrix  $G$  are zero directions of the transfer matrix  $\Theta(z)$  at the ‘frequencies’  $\{\alpha_i\}$ . This is a consequence of the following intuitive reasoning: the generator recursion guarantees that the relevant zero directions will appear at the input of the appropriate first-order sections, thus annihilating the output of the entire cascade. In other words, the first row of  $G$  annihilates  $\Theta(z)$  at  $z = \alpha_0 (= f_0)$ , due to the local blocking property of the first section  $\Theta_0(z)$ . When the second row of  $G$  goes through  $\Theta_0(z)$ , we get the first row of  $G_1$  at the input of  $\Theta_1(z)$  (due to the generator recursion), which then annihilates  $\Theta(z)$  at  $z = \alpha_0 (= f_1)$ , and so on. This reasoning is made precise in the proof of the next result. [ We may mention that the global blocking property stated below is referred to as a homogeneous interpolation problem in the terminology of [BGR90] ].

We first clarify our notation. Recall that  $F$  is block-diagonal with Jordan structure, viz,  $F = \bar{F}_0 \oplus \bar{F}_1 \oplus \dots \oplus \bar{F}_{m-1}$ , where  $\bar{F}_i$  is an  $r_i \times r_i$  Jordan matrix with eigenvalue at  $\alpha_i$ . The first  $r_0$  rows of  $G$  (i.e., rows 0 through  $r_0 - 1$ ) are associated with  $\alpha_0$  and hence, with the first Jordan block  $\bar{F}_0$ . The next  $r_1$  rows of  $G$  (i.e., rows  $r_0$  through  $r_0 + r_1 - 1$ ) are associated with  $\alpha_1$  and hence, with the second Jordan block  $\bar{F}_1$ . More generally, let  $s_i = r_0 + r_1 + \dots + r_{i-1}$ ,  $s_0 = 0$ , denote the total number of rows in  $G$  that are associated with the first  $i$  Jordan blocks  $\bar{F}_0, \dots, \bar{F}_{i-1}$ , e.g.,  $s_0 = 0, s_1 = r_0, s_2 = r_0 + r_1$ . Then the rows in  $G$  that are associated with  $\bar{F}_0$  are  $\left[ e_{s_0}G \quad e_{s_0+1}G \dots e_{s_0+r_0-1}G \right]$ , and so on.

**Theorem 4.1** *The transfer matrix  $\Theta(z)$  satisfies the global blocking property*

$$\left[ e_{s_i}G \quad e_{s_i+1}G \quad \dots \quad e_{s_i+r_i-1}G \right] \mathcal{H}_{\Theta}^{r_i}(\alpha_i) = \mathbf{0}. \quad (21)$$

**Proof:** We prove the result for the first Jordan block  $\bar{F}_0$ , i.e.,  $\left[ e_0G \quad \dots \quad e_{r_0-1}G \right] \mathcal{H}_{\Theta}^{r_0}(\alpha_0) = \mathbf{0}$ . The same argument is valid for the other Jordan blocks.

It follows from the local blocking property of  $\Theta_0(z)$  that  $g_0\Theta_0(\alpha_0) = \mathbf{0}$ . Consequently,  $g_0\Theta(\alpha_0) = \mathbf{0}$ , which shows that the first row of  $G$  annihilates the entire cascade at  $\alpha_0$ . In fact, the Jordan

structure of  $\bar{F}_0$  (with eigenvalue  $\alpha_0 = f_0 = f_1 = \dots = f_{r_0-1}$ ) imposes a stronger condition on  $\Theta(z)$ , as we readily verify. For example, by comparing the second row on both sides of (9) for  $i = 0$ , and using the Jordan structure of  $\bar{F}_0$ , we conclude that

$$\begin{aligned} g_1 &= e_1 G \Theta_0 \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} + e_1 \Phi_0 G \Theta_0 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = e_1 G \Theta_0(f_0) + \frac{1}{1 - |f_0|^2} e_0 G \Theta_0 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ &= e_1 G \Theta_0(f_0) + g_0 \Theta_0^{(1)}(f_0) = \begin{bmatrix} e_0 G & e_1 G \end{bmatrix} \begin{bmatrix} \Theta_0^{(1)}(f_0) \\ \Theta_0(f_0) \end{bmatrix}. \end{aligned}$$

That is, the first row of  $G_1$  (viz.,  $g_1$ ) is obtained as a linear combination of the first two rows of  $G$ . This result can be readily extended (by comparing the third and later rows on both sides of (9) for  $i = 0$ ) to show that the  $k^{\text{th}}$  row of  $G_1$  ( $k < r_0$ ) is obtained as a linear combination of the first  $(k + 1)$  rows of  $G$ , and so on. More specifically,

$$\begin{bmatrix} e_0 G & e_1 G & \dots & e_{r_0-1} G \end{bmatrix} \mathcal{H}_{\Theta_0}^{r_0}(\alpha_0) = \begin{bmatrix} \mathbf{0} & e_0 G_1 & e_1 G_1 & \dots & e_{r_0-2} G_1 \end{bmatrix}.$$

In other words, when the first  $r_0$  rows of  $G$  are applied to  $\Theta_0(z)$  we obtain the first  $r_0 - 1$  rows of  $G_1$  at  $z = \alpha_0$ . Similarly, when the first  $r_0 - 1$  rows of  $G_1$  are applied to  $\Theta_1(z)$  we obtain the first  $r_0 - 2$  rows of  $G_2$  at  $z = \alpha_0$  (by comparing the second and later rows on both sides of (9) for  $i = 1$ ),

$$\begin{bmatrix} \mathbf{0} & e_0 G_1 & e_1 G_1 & e_{r_0-2} G_1 \end{bmatrix} \mathcal{H}_{\Theta_1}^{r_0}(\alpha_0) = \begin{bmatrix} \mathbf{0} & \mathbf{0} & e_0 G_2 & e_1 G_2 & \dots & e_{r_0-3} G_2 \end{bmatrix},$$

and so on. Repeating this argument for the first  $r_0$  sections we obtain

$$\begin{bmatrix} e_0 G & e_1 G & \dots & e_{r_0-1} G \end{bmatrix} \mathcal{H}_{\Theta_0}^{r_0}(\alpha_0) \mathcal{H}_{\Theta_1}^{r_0}(\alpha_0) \dots \mathcal{H}_{\Theta_{r_0-1}}^{r_0}(\alpha_0) = \mathbf{0}.$$

That is, when the first  $r_0$  rows of  $G$  are processed by the first  $r_0$  sections we obtain a zero output at  $z = \alpha_0$ . But  $\mathcal{H}_{\Theta}^{r_0}(\alpha_0) = \mathcal{H}_{\Theta_0}^{r_0}(\alpha_0) \dots \mathcal{H}_{\Theta_{r_0-1}}^{r_0}(\alpha_0) \mathcal{H}_{\Theta_{r_0}}^{r_0}(\alpha_0) \dots \mathcal{H}_{\Theta_{n-1}}^{r_0}(\alpha_0)$ . Consequently,  $\begin{bmatrix} e_0 G & \dots & e_{r_0-1} G \end{bmatrix} \mathcal{H}_{\Theta}^{r_0}(\alpha_0) = \mathbf{0}$ . Similar arguments can be used to show that if the next  $r_1$  rows of  $G$  are applied through the first  $r_0$  sections, whose product is denoted by  $\bar{\Theta}_0(z)$ , we then obtain the first  $r_1$  rows of  $G_{r_0}$  at  $z = \alpha_1$ , that is,

$$\begin{bmatrix} e_{r_0} G & e_{r_0+1} G & \dots & e_{r_0+r_1-1} G \end{bmatrix} \mathcal{H}_{\bar{\Theta}_0}^{r_1}(\alpha_1) = \begin{bmatrix} e_0 G_{r_0} & e_1 G_{r_0} & \dots & e_{r_1-1} G_{r_0} \end{bmatrix},$$

which in turn annihilate the output of the next  $r_1$  sections denoted by,  $\bar{\Theta}_1(z) = \Theta_{r_0}(z) \dots \Theta_{r_0+r_1-1}(z)$ , at  $z = \alpha_1$ , and so on. ■

## 4.1 From Blocking Properties to Interpolation Properties

We now verify that the global blocking property (21) is equivalent to the desired interpolation properties. For this purpose, we first note that the row vector that appears on the left hand-side of (21) is composed of the  $r_i$  row vectors in  $\begin{bmatrix} U_i & V_i \end{bmatrix}$  that are associated with  $\alpha_i$ , viz.,

$$\begin{bmatrix} u_1^{(i)} & v_1^{(i)} & u_2^{(i)} & v_2^{(i)} & \dots & u_{r_i}^{(i)} & v_{r_i}^{(i)} \end{bmatrix}.$$

If we partition  $\Theta(z)$  accordingly with  $J = (I_p \oplus -I_q)$ ,

$$\Theta(z) = \begin{bmatrix} \Theta_{11}(z) & \Theta_{12}(z) \\ \Theta_{21}(z) & \Theta_{22}(z) \end{bmatrix},$$

it is then a standard result that the rational transfer matrix  $S(z) = -\Theta_{12}(z)\Theta_{22}^{-1}(z)$  is a Schur-type function due to the  $J$ -losslessness of  $\Theta(z)$  (see, *e.g.*, [Dym89a, pp. 14-16]). It follows from (21) that

$$\begin{bmatrix} u_1^{(i)} & v_1^{(i)} & u_2^{(i)} & v_2^{(i)} & \dots & u_{r_i}^{(i)} & v_{r_i}^{(i)} \end{bmatrix} \mathcal{H}^{r_i} \begin{bmatrix} \Theta_{12} \\ \Theta_{22} \end{bmatrix} (\alpha_i) = \mathbf{0},$$

or equivalently,

$$\begin{bmatrix} u_1^{(i)} & u_2^{(i)} & \dots & u_{r_i}^{(i)} \end{bmatrix} \mathcal{H}_S^{r_i}(\alpha_i) = \begin{bmatrix} v_1^{(i)} & v_2^{(i)} & \dots & v_{r_i}^{(i)} \end{bmatrix}.$$

This shows that  $S(z)$  is one possible solution of the Hermite-Fejér problem. In fact, all solutions can be parametrized as stated in the following theorem.

**Theorem 4.1.1** *All solutions  $S(z)$  of the tangential Hermite-Fejér problem are given by a linear fractional transformation of a Schur matrix function  $K(z)$  ( $\|K\|_\infty < 1$ )*

$$S(z) = -[\Theta_{11}(z)K(z) + \Theta_{12}(z)][\Theta_{21}(z)K(z) + \Theta_{22}(z)]^{-1}. \quad (22)$$

**Proof:** We first show that an  $S(z)$  as above is indeed a solution. Define  $S_1(z)$  and  $S_2(z)$  as follows:

$$\begin{bmatrix} S_1(z) \\ S_2(z) \end{bmatrix} = \Theta(z) \begin{bmatrix} K(z) \\ I \end{bmatrix}.$$

Then

$$\mathcal{H}^{r_i} \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} (\alpha_i) = \mathcal{H}_\Theta^{r_i}(\alpha_i) \mathcal{H}^{r_i} \begin{bmatrix} K \\ I \end{bmatrix} (\alpha_i),$$

and consequently, using the global blocking property of  $\Theta(z)$ , we conclude that

$$\begin{bmatrix} u_1^{(i)} & v_1^{(i)} & u_2^{(i)} & v_2^{(i)} & \dots & u_{r_i}^{(i)} & v_{r_i}^{(i)} \end{bmatrix} \mathcal{H}_S^{r_i} \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} (\alpha_i) = \mathbf{0}.$$

Therefore,  $\begin{bmatrix} u_1^{(i)} & \dots & u_{r_i}^{(i)} \end{bmatrix} \mathcal{H}_S^{r_i} (\alpha_i) = \begin{bmatrix} v_1^{(i)} & \dots & v_{r_i}^{(i)} \end{bmatrix}$ , where  $S(z) = -S_1(z)S_2^{-1}(z)$ . But the  $J$ -losslessness of  $\Theta(z)$  implies that  $S(z)$  is a Schur matrix function (see, *e.g.*, [BGR90, p. 404] and [Dym89a, p. 63]). Hence,  $S(z)$  is a Schur type solution of the Hermite-Fejér problem.

The converse is more elaborate. Assume that  $S(z)$  is a Schur-type solution of the Hermite-Fejér problem and define the matrix function

$$K(z) = -[S(z)\Theta_{21}(z) + \Theta_{11}(z)]^{-1} [\Theta_{12}(z) + S(z)\Theta_{22}(z)].$$

It can then be shown, using the  $J$ -losslessness of  $\Theta(z)$  and the fact that  $S(z)$  is an interpolating solution, that  $K(z)$  is a Schur matrix function. The details are spelled out in [BGR90, pp. 404-406, pp. 418-422] and [Dym89a, pp. 63-64, p. 73].

■

The solutions  $S(z)$  have a scattering interpretation as shown in Figure 2, where  $\Sigma(z)$  is the scattering matrix defined by

$$\Sigma(z) = \begin{bmatrix} \Theta_{11} - \Theta_{12}\Theta_{22}^{-1}\Theta_{21} & -\Theta_{12}\Theta_{22}^{-1} \\ \Theta_{22}^{-1}\Theta_{21} & \Theta_{22}^{-1} \end{bmatrix} (z).$$

That is,  $S(z)$  is the transfer matrix from the top left  $(1 \times p)$  input to the bottom left  $(1 \times q)$  output, with a Schur-type load  $(-K(z))$  at the right end. Here we only remark that the scattering matrix  $\Sigma(z)$  is a so-called inner dilation of  $-\Theta_{12}(z)\Theta_{22}^{-1}(z)$  and satisfies  $\Sigma(z)\Sigma^*(z) = I$  on  $|z| = 1$ . The  $\Sigma(z)$  can also be obtained as a cascade of elementary sections  $\Sigma_i(z)$ , which are defined in terms of the  $\Theta_i(z)$ . In the  $\Sigma_i(z)$ -cascade, signals flow in both directions; this yields a so-called (generalized) transmission line, which gives a nice interpretation of the result of Theorem 4.1.1. The blocking properties of the line mean that when the input row vectors  $\{u_1^{(i)}, \dots, u_{r_i}^{(i)}\}$  are applied to the top  $p$  lines of the cascade then the corresponding output row vectors  $\{v_1^{(i)}, \dots, v_{r_i}^{(i)}\}$  in the bottom  $q$  lines, at the “frequency”  $\alpha_i$ , will be independent of the “load”  $-K(z)$  at the right-end.

In our framework, the interpolating solution  $S(z)$  is constructed recursively by adding a new first-order section to the scattering cascade for each interpolation constraint. The additional section does not influence the interpolation properties of the previous sections; at the same time it

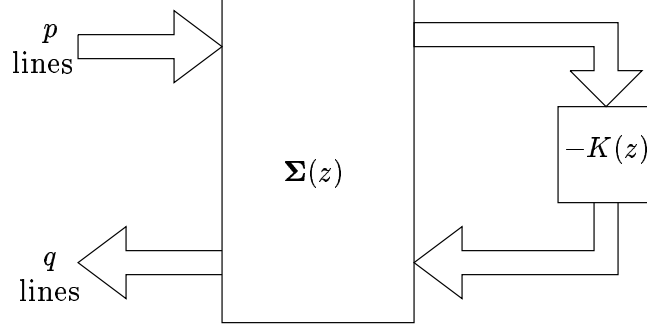


Figure 2: Scattering interpretation.

is determined so that the transfer function of the composite system will satisfy the additional interpolation constraint.

## 4.2 A Global State-Space Description

Each first-order section  $\Theta_i(z)$  has a state-space description as shown on the right-hand side of (16). In order to connect with many previous results in the literature, we can further explore how our results also yield a global state-space description for the entire cascade  $\Theta(z)$  in terms of the original matrices  $F$  and  $G$ . This can be achieved by recursively combining the state-space descriptions of the first-order sections: we first use the last two sections and compute the state-space representation of their cascade, say  $W_{n-2}(z) = \Theta_{n-2}(z)\Theta_{n-1}(z)$ . We then incorporate one more section and compute the state-space representation of  $W_{n-3}(z) = \Theta_{n-3}(z)W_{n-2}(z)$ , and so on. After incorporating the last section  $\Theta_0(z)$  we clearly get a state-space description for  $\Theta(z)$ .

We shall, for brevity, omit the details here. But the interested reader is referred to [LAK92, GDK<sup>+</sup>83] and [Say92, Chapter 2] for a detailed derivation. Following the above argument, and using the generator recursion (9), we obtain the following result.

**Theorem 4.2.1** *The cascade  $\Theta(z)$  admits an  $n$ -dimensional state-space description,*

$$\begin{bmatrix} \mathbf{x}_{j+1} & \mathbf{y}_j \end{bmatrix} = \begin{bmatrix} \mathbf{x}_j & \mathbf{w}_j \end{bmatrix} \begin{bmatrix} F^* & H^*J \\ JG^* & JK^*J \end{bmatrix}, \quad (23)$$



where  $\mathbf{w}_j$  and  $\mathbf{y}_j$  are  $1 \times r$  row input and output vectors, respectively, at time  $j$ ,  $\mathbf{x}_j$  is the  $n$ -dimensional state, and  $H$  and  $K$  are  $r \times n$  and  $r \times r$  matrices, respectively, that satisfy the embedding relation

$$\begin{bmatrix} F & G \\ H & K \end{bmatrix} \begin{bmatrix} R & \mathbf{0} \\ \mathbf{0} & J \end{bmatrix} \begin{bmatrix} F & G \\ H & K \end{bmatrix}^* = \begin{bmatrix} R & \mathbf{0} \\ \mathbf{0} & J \end{bmatrix}. \quad (24)$$

That is,  $\Theta(z) = JK^*J + JG^*(z^{-1}I - F^*)^{-1}H^*J$ . Furthermore,  $H$  and  $K$  can be expressed in terms of  $F, G$ , and  $R^{-1}$  as follows:

$$H = \Theta^{-1}JG^*[I_n - \tau F^*]^{-1}R^{-1}(\tau I_n - F), \quad K = \Theta^{-1}[I_r - JG^*[I_n - \tau F^*]^{-1}R^{-1}G],$$

where  $\Theta$  is a  $J$ -unitary matrix and  $\tau$  is a scalar on the unit circle (the values of  $\Theta$  and  $\tau$  depend on the values of  $\Theta_i$ ). Using the expressions for  $H$  and  $K$  we can rewrite  $\Theta(z)$  in the form

$$\Theta(z) = \left\{ I - (1 - z\tau)JG^*(I - zF^*)^{-1}R^{-1}(I - \tau F)^{-1}G \right\} \Theta. \quad (25)$$

We may remark that the system matrix that appears on the right-hand side of (23) can also be regarded as a state-space realization of the inverse system

$$\begin{bmatrix} F & G \\ H & K \end{bmatrix}^{-1},$$

since it follows from the embedding relation (24) that

$$\begin{bmatrix} F & G \\ H & K \end{bmatrix}^{-1} = \begin{bmatrix} RF^*R^{-1} & RH^*J \\ JG^*R^{-1} & JK^*J \end{bmatrix}.$$

Expressions similar to (25) (but with  $\Theta = I$  and  $\tau = 1$ ) have been derived by many authors [Kim88, Dym89b, BGR90] and have been used to write down a solution formula for rational interpolation problems; see the recent monograph [BGR90] for an extensive discussion and references. We may remark that the freedom in choosing the parameters  $\tau$  and  $\Theta$  can be helpful in simplifying the calculations in specific problems. Also note that the above global expression involves the inverse of  $R$  explicitly, whereas the recursive solution described here avoids the computation of  $R^{-1}$ , or even  $R$ , and only uses the matrices  $F$  and  $G$  that are constructed directly from the interpolation data.

### 4.3 The Recursive Algorithm

We have thus developed a recursive solution of a general Hermite-Fejér interpolation problem, which includes as special cases several other problems that were studied in the literature such as

the scalar and tangential versions of the Carathéodory-Fejér and Nevanlinna-Pick formulations. We summarize here, for clarity and ease of reference, the main steps of the recursive solution.

**Algorithm 4.3.1** *The tangential Hermite-Fejér Problem 1.2.1 (and any related special case) can be recursively solved in  $O(rn^2)$  operations as follows:*

- *Collect the interpolation points  $\{\alpha_i\}$  and the tangential directions  $\{u_j^{(i)}, v_j^{(i)}\}$  into the matrices  $F, G$ , and  $J$ , as described in Section 2 (expression (3)).*
- *Apply the recursive procedure of Algorithm 3.1, steps (i-iii), or any other suitable variant of (18), for  $i = 0, 1, \dots, n - 1$ .*
- *This leads to a recursive construction of a cascade  $\Theta(z)$  that is composed of  $n$  first-order sections as in (17) (or (19)).*
- *The cascade  $\Theta(z)$  then parametrizes all solutions  $S(z)$  of the interpolation problem as stated in (22).*

We again stress the fact that we only have a *single* algorithm (namely recursion (9)), which solves the Hermite-Fejér problem *as well as* its several special cases. That is, while problems such as the Carathéodory-Fejér and Nevanlinna-Pick are usually treated separately in the literature (for example, the scalar Carathéodory-Fejér problem is solved via Schur’s recursion [Sch17], while the scalar Nevanlinna-Pick problem is solved via Nevanlinna’s recursion [Akh65]), our approach develops a single algorithm that simultaneously solves all these and related problems. All we have to do is to construct the appropriate  $(F, G, J)$  (as we did for three special cases in Section 2) and then apply recursion (9). Also note that the recursive algorithm, as described above, only involves rotation operations and multiplication by a “Blaschke” matrix, which are applied to an array of numbers  $G_i$ ; no function recursions or expressions are directly involved. That is, we start with an array  $G_i$ , apply rotation and shift operations to its entries leading to  $G_{i+1}$ , and then repeat. This is a computationally attractive and convenient description.

#### 4.4 A General Result

We have limited our discussion so far to the case of a lower triangular matrix  $F$  in the displacement equation (5). The result of Theorem 2.1, however, can be extended to more general structures. If we instead start with a displacement equation of the form  $\bar{R} - \bar{F}\bar{R}\bar{F}^* = \bar{G}J\bar{G}^*$ , where  $\bar{F}$  is an

arbitrary stable matrix (not necessarily in Jordan or lower triangular form),  $\bar{G} = \begin{bmatrix} \bar{U} & \bar{V} \end{bmatrix}$  and  $J = (I_p \oplus -I_q)$ , then the proof of Theorem 2.1 can be modified to establish the following result (this formulation establishes a connection between displacement structure theory and the interpolation problems considered by Nudelman [Nud81], Rosenblum and Rovynak [RR85], and Foias and Frazho [FF90]).

**Theorem 4.4.1** *The displacement equation  $\bar{R} - \bar{F}\bar{R}\bar{F}^* = \bar{G}J\bar{G}^*$ , has a positive semi-definite solution  $\bar{R}$  if, and only if, there exists a block lower-triangular Toeplitz contraction  $\bar{T}$  such that  $\bar{V} = \bar{T}\bar{U}$ , where  $\bar{V} = \begin{bmatrix} \bar{V} & \bar{F}\bar{V} & \bar{F}^2\bar{V} & \dots \end{bmatrix}$  and  $\bar{U} = \begin{bmatrix} \bar{U} & \bar{F}\bar{U} & \bar{F}^2\bar{U} & \dots \end{bmatrix}$ .*

**Proof:** See [CSK94] for a detailed proof in a more general operatorial setting and along the lines of this paper. ■

In the strictly positive-definite case ( $\bar{R} > 0$ ) we can associate a  $J$ -lossless transfer matrix  $\bar{\Theta}(z)$  with  $\bar{R}$  as in (25), viz.,

$$\bar{\Theta}(z) = \left\{ I - (1 - z\tau)J\bar{G}^*(I - z\bar{F}^*)^{-1}\bar{R}^{-1}(I - \tau\bar{F})^{-1}\bar{G} \right\} \Theta. \quad (26)$$

This transfer matrix can be recursively constructed as follows: we first convert the displacement equation for  $\bar{R}$  to a displacement equation with a lower-triangular  $F$ . For this purpose we invoke a similarity transformation  $T$  that transforms  $\bar{F}$  to Jordan form, viz.,  $F = T\bar{F}T^{-1}$ , where  $F$  is the Jordan canonical form of  $\bar{F}$ . If we further introduce the matrices  $R = T\bar{R}T^*$  and  $G = T\bar{G}$ , then the displacement equation for  $\bar{R}$  reduces to the equivalent equation  $R - FRF^* = GJG^*$  with a lower-triangular  $F$ . The recursive algorithm applied to  $(F, G)$  now constructs a cascade  $\Theta(z)$  as in (25), which can be easily verified to be equal to  $\bar{\Theta}(z)$  because of the similarity relations between  $(R, F, G)$  and  $(\bar{R}, \bar{F}, \bar{G})$ , viz.,  $\bar{\Theta}(z) = \Theta(z)$ . Furthermore, we already know from our previous arguments (Theorem 4.1) that the transfer matrix  $\Theta(z)$  satisfies the following global blocking property

$$\begin{bmatrix} e_{s_i}G & e_{s_i+1}G & \dots & e_{s_i+r_i-1}G \end{bmatrix} \mathcal{H}_{\Theta}^{r_i}(\alpha_i) = \mathbf{0},$$

where the zero directions are the appropriate rows of  $G$  that correspond to the eigenvalue at  $\alpha_i$ . It then follows that  $\bar{\Theta}(z)$  in (26) satisfies the following (generalized) blocking property

$$\begin{bmatrix} e_{s_i}TG & e_{s_i+1}TG & \dots & e_{s_i+r_i-1}TG \end{bmatrix} \mathcal{H}_{\Theta}^{r_i}(\alpha_i) = \mathbf{0},$$

That is, the blocking directions are also determined by the rows of the similarity transformation  $T$ .

## 5 AN EXAMPLE: THE NEVANLINNA-PICK CASE

One might wonder whether recursion (9) in the Carathéodory-Fejér and Nevanlinna-Pick cases differs from the conventional Schur and Nevanlinna-Pick recursions. In the former case ( $F = Z$ ), it can be easily seen that recursion (9) is precisely the array form associated with the Schur algorithm, as discussed in [Kai86]. In the Nevanlinna-Pick case, however, recursion (9) has a slightly different (but simpler) array interpretation than that associated with the classical Nevanlinna-Pick algorithm and based on Newton-series expansions, as we further elaborate for the sake of illustration.

In the Nevanlinna-Pick problem we are interested in finding a scalar Schur function  $s(z)$  that satisfies

$$s(\alpha_i) = \beta_i \quad \text{for } i = 0, 1, \dots, n-1, \quad |\alpha_i| < 1, \quad \{\alpha_i\} \text{ distinct.} \quad (27)$$

Following the construction of Section 2 we define matrices  $F, G$ , and  $J$  as in (4). The corresponding “Blaschke” matrix  $\Phi_i$  in (9) will then be diagonal, since  $F$  is diagonal. We can also write down an explicit expression for the hyperbolic rotation  $\Theta_i$ , viz.,

$$\Theta_i = \frac{1}{\sqrt{1 - |\gamma_i|^2}} \begin{bmatrix} 1 & -\gamma_i \\ -\gamma_i^* & 1 \end{bmatrix}, \quad \gamma_i = \frac{v_{ii}}{u_{ii}},$$

where we partitioned  $g_i$  as follows:  $g_i = \begin{bmatrix} u_{ii} & v_{ii} \end{bmatrix}$ . The generator recursion (9) then leads to a cascade  $\Theta(z)$  that satisfies the global blocking property (which is straightforward to verify in this case):  $\begin{bmatrix} 1 & \beta_i \end{bmatrix} \Theta(\alpha_i) = \mathbf{0}$ . Moreover, the matrix  $R$  that solves the displacement equation  $R - FRF^* = GJG^*$ , is precisely the so-called Pick matrix associated with the Nevanlinna-Pick problem, viz.,

$$R = \left[ \frac{1 - \beta_i \beta_j^*}{1 - \alpha_i \alpha_j^*} \right]_{i,j=0}^{n-1}, \quad (28)$$

and hence, the solvability condition of Theorem 2.1 is equivalent to the positivity of the above Pick matrix.

### 5.1 The Classical Nevanlinna Algorithm

We now consider the classical Nevanlinna recursion [Akh65], which maps Schur functions  $s_i(z)$  to Schur functions  $s_{i+1}(z)$  as follows:

$$s_{i+1}(z) = \frac{1 - \alpha_i^* z}{z - \alpha_i} \frac{s_i(z) - \gamma_i}{1 - \gamma_i^* s_i(z)}, \quad \gamma_i = s_i(\alpha_i), \quad s_0(z) = s(z), \quad i \geq 0. \quad (29)$$

The first  $n$  steps can be used to solve the interpolation problem (27). Now expression (29) is a nonlinear recursion in  $s_i(z)$ , which can be linearized by expressing  $s_i(z)$  as the ratio of two power

series,  $s_i(z) = v_i(z)/u_i(z)$ . This would then allow us to get an alternative representation of (29) that is closer to the array language used in (9). Using (29) we conclude that we can rewrite it in the form

$$(z - \alpha_i) \begin{bmatrix} u_{i+1}(z) & v_{i+1}(z) \end{bmatrix} = \begin{bmatrix} u_i(z) & v_i(z) \end{bmatrix} \Theta_i \begin{bmatrix} \frac{z - \alpha_i}{1 - \alpha_i^* z} & 0 \\ 0 & 1 \end{bmatrix}, \quad (30)$$

where  $\Theta_i$  is the elementary hyperbolic rotation,

$$\Theta_i = \frac{1}{\sqrt{1 - |\gamma_i|^2}} \begin{bmatrix} 1 & -\gamma_i \\ -\gamma_i^* & 1 \end{bmatrix}, \quad \gamma_i = \lim_{z \rightarrow \alpha_i} \frac{v_i(z)}{u_i(z)}.$$

We are now a step closer to a form similar to (9). The array interpretation can be obtained by invoking the *Newton power-series* expansions of  $u_i(z)$  and  $v_i(z)$  and comparing terms on both sides of (30): let  $P_i(z)$  denote the Newton-series basis associated with the points  $\{\alpha_i, \alpha_{i+1}, \dots\}$ , viz.,

$$P_i(z) = \begin{bmatrix} 1 & (z - \alpha_i) & (z - \alpha_i)(z - \alpha_{i+1}) & (z - \alpha_i)(z - \alpha_{i+1})(z - \alpha_{i+2}) & \dots \end{bmatrix},$$

and assume we expand  $u_i(z)$  and  $v_i(z)$  with respect to this Newton-series basis,

$$\begin{aligned} u_i(z) &= u_{ii} + u_{i+1,i}(z - \alpha_i) + u_{i+2,i}(z - \alpha_i)(z - \alpha_{i+1}) + \dots \\ v_i(z) &= v_{ii} + v_{i+1,i}(z - \alpha_i) + v_{i+2,i}(z - \alpha_i)(z - \alpha_{i+1}) + \dots \end{aligned} \quad (31)$$

[ We remark that if we are given a function  $h(z)$  then the coefficients of its Newton series expansion with respect to given points  $\{f_0, f_1, f_2, \dots\}$ ,

$$h(z) = h_0 + h_1(z - f_0) + h_2(z - f_0)(z - f_1) + \dots,$$

can be computed recursively via the so-called *divided difference recursion* as follows: start with  $h_0(z) = h(z)$  and then use

$$h_i(z) = \frac{h_{i-1}(z) - h_{i-1}}{z - f_{i-1}}, \quad h_i = h_i(f_i) ] .$$

Returning to (31), if we now introduce the two-column (semi-infinite) matrix  $G_i$  composed of the power series coefficients of  $u_i$  and  $v_i$ ,

$$G_i = \begin{bmatrix} u_{ii} & v_{ii} \\ u_{i+1,i} & v_{i+1,i} \\ u_{i+2,i} & v_{i+2,i} \\ \vdots & \vdots \end{bmatrix} \equiv \begin{bmatrix} \mathbf{u}_i & \mathbf{v}_i \end{bmatrix}.$$

Then  $\begin{bmatrix} u_i(z) & v_i(z) \end{bmatrix} = P_i(z)G_i$  and we can rewrite (30) in the equivalent array form

$$\begin{bmatrix} 0 & 0 \\ G_{i+1} \end{bmatrix} = \Phi_i G_i \Theta_i \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + G_i \Theta_i \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad (32)$$

where  $\Phi_i$  is the (semi-infinite) ‘‘Blaschke’’ matrix given by  $\Phi_i = (F_i - \alpha_i I)(I - \alpha_i^* F_i)^{-1}$ , and  $F_i$  is the submatrix obtained after deleting the first  $i$  columns and rows of the following *bidiagonal* matrix

$$F = \begin{bmatrix} \alpha_0 & & & & \\ 1 & \alpha_1 & & & \\ & 1 & \alpha_2 & & \\ & & & \ddots & \ddots \\ & & & & \ddots \end{bmatrix}. \quad (33)$$

The forms (32) and (33) are justified because if we multiply (32) by  $P_i(z)$  from the left then (30) follows since  $P_i(z)\Phi_i = \frac{z-\alpha_i}{1-\alpha_i^*z}P_i(z)$ . The point to note here is that (32) has the same form as (9) except that it assumes a *bidiagonal* matrix  $F$  in contrast to (4), which uses a diagonal  $F$ . In (4) we also show how to choose an appropriate  $G$  for the diagonal  $F$ . We still need to show how this choice of  $G$  is handled in the bidiagonal case in order to solve the interpolation problem.

So assume we start with

$$G = G_0 = \begin{bmatrix} u_{00} & v_{00} \\ u_{10} & v_{10} \\ u_{20} & v_{20} \\ \vdots & \vdots \end{bmatrix} = \begin{bmatrix} \mathbf{u}_0 & \mathbf{v}_0 \end{bmatrix},$$

where  $\mathbf{u}_0$  and  $\mathbf{v}_0$  are to be determined. Let  $\Theta(z)$  denote the associated cascade obtained after  $n$  recursive steps of (32),

$$\Theta(z) = \Theta_0(z)\Theta_1(z)\dots\Theta_{n-1}(z), \quad \Theta_i(z) = \Theta_i \begin{bmatrix} \frac{z-\alpha_i}{1-\alpha_i^*z} & 0 \\ 0 & 1 \end{bmatrix}.$$

If we define  $p_n(z) = (z - \alpha_0)(z - \alpha_1)\dots(z - \alpha_{n-1})$ , then it follows from (30) that  $P_0(z)G\Theta(z) = \mathbf{O}(p_n(z))$ . This is due to the fact that the generator recursion (32) introduces one zero row at each step. Therefore,  $P_0(\alpha_i)G\Theta(\alpha_i) = \mathbf{0}$  for  $i = 0, 1, \dots, n - 1$ , or equivalently,

$$s(\alpha_i) = \frac{P_0(\alpha_i)\mathbf{v}_0}{P_0(\alpha_i)\mathbf{u}_0} \quad \text{where} \quad s(z) = -\frac{\Theta_{12}(z)}{\Theta_{22}(z)}.$$

This shows that in order to solve the Nevanlinna-Pick problem, using the array interpretation (32), we have to choose the initial column vectors  $\mathbf{u}_0$  and  $\mathbf{v}_0$  such that the ratio (of scalars)

$P_0(\alpha_i)\mathbf{v}_0/P_0(\alpha_i)\mathbf{u}_0$  is equal to  $\beta_i$ . If we choose  $\mathbf{u}_0$  as in the Carathéodory-Fejér case, say  $\mathbf{u}_0 = \begin{bmatrix} 1 & 0 & 0 & \dots \end{bmatrix}^{\mathbf{T}}$ , then  $P_0(\alpha_i)\mathbf{u}_0 = 1$  and we must choose  $\mathbf{v}_0$  such that  $P_0(\alpha_i)\mathbf{v}_0 = \beta_i$ . This is equivalent to saying that the entries of the column vector  $\mathbf{v}_0$  must be the Newton-series coefficients computed from the given  $\beta_i$ 's (recall the definition of the Newton series expansion (31) and the divided difference recursion), which we denote by  $\{\phi_i\}_{i=0}^{n-1}$ , viz.,  $\mathbf{v}_0 = \begin{bmatrix} \phi_0 & \phi_1 & \phi_2 & \dots \end{bmatrix}^{\mathbf{T}}$ . This explains why the solution to the Nevanlinna-Pick problem using (30) or (32) has an initial step that converts the given interpolation data  $(\alpha_i, \beta_i)$  to another set of data  $(\alpha_i, \phi_i)$  before starting the recursive algorithm, which corresponds to computing the so-called Fenyves arrays [DGK81, LA83, Ack91]. The construction in (4) avoids this initial step and constructs  $G$  directly from the interpolation data.

Using the bidiagonal  $F$  we can readily verify (from Theorem 2.1) that the scalar Nevanlinna-Pick problem is solvable if, and only if,  $|\gamma_i| < 1$  for  $i = 0, 1, \dots, n-1$ . This establishes a clear connection between the two conditions: positive-definiteness of (28), which corresponds to defining a displacement equation (5) with a diagonal  $F$  as in (4), and the boundedness condition  $\{|\gamma_i| < 1\}_{i=0}^{n-1}$ , which corresponds to defining a displacement equation (5) with the leading  $n \times n$  submatrix of the bidiagonal  $F$  in (33).

## 6 CONCLUDING REMARKS

We described a computationally-oriented solution to analytic rational interpolation problems by exploiting the notion of displacement structure and an associated generalized Schur algorithm. In particular, we showed how to obtain a simple array algorithm that is composed of a sequence of rotations and multiplications by “Blaschke” matrices, and emphasized the role played by the underlying (transmission line) cascade system and its blocking properties.

We further remark that the recursive algorithm can also be used to solve two-sided interpolation problems (see, *e.g.*, [Say92, Chapter 3]) as well as unconstrained rational interpolation problems, where the major concern is the minimality degree of the interpolant rather than its analyticity [BSK94]. Further extensions of the approach of the present paper to the time-variant setting are also possible and lead to a recursive solution of a general time-variant Hermite-Fejér problem (see, *e.g.*, [SCK94, CSK94, Say92] for an argument along the lines of this paper).

In this paper we have focused on interpolation with respect to points in the unit disc; there is a corresponding half-plane theory that can be studied in the same fashion. The natural displacement

equation then has the (Lyapunov) form  $FR + RF^* = GJG^*$ . Recursive factorization algorithms for  $R$  defined by this equation and more general ones of the form  $\Omega R \Delta^* - FRA^* = GJB^*$  can be found in [LAK86, KS91, Say92].

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