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Time-variant structured matrices: An application to instrumental variable methods

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ABSTRACT

We derive a recursive algorithm for the time-update of the triangular factors of non-Hermitian time-variant matrices with structure. These are matrices that undergo low-rank modifications as time progresses, special cases of which often arise in adaptive filtering and instrumental variable (IV) methods. A natural implementation of the algorithm is via two coupled triangular arrays of processing elements. We consider, in particular, an IV parameter estimation problem and show how the arrays collapse to a coupled parallelizable solution of the identification problem.

1. INTRODUCTION

The notion of displacement structure provides a natural framework for the solution of many problems in signal processing and mathematics. It represents a powerful and unifying tool for exploiting structure in numerous applications, as detailed in several recent surveys on the topic.[1,2,3] More recently, we have extended the concept of structured matrices to the time-variant setting[4,5,6] and shown that we can, as well, study matrices that undergo low-rank modifications as time progresses. Special examples often arise in adaptive filtering.[7,8,9] In this paper, we further extend our earlier results to the non-Hermitian case and exhibit an application to instrumental variable methods.[10,11]

We shall say that an $n \times n$ matrix $\mathbf{R}(t)$ has a time-variant Toeplitz-like displacement structure if the matrix difference $\nabla\mathbf{R}$ defined by

$$\nabla\mathbf{R} = \mathbf{R}(t) - \mathbf{F}(t)\mathbf{R}(t-\Delta)\mathbf{A}^*(t) \;\;,$$

has low rank, say $r(t)$ (usually $r(t) \ll n$), for some lower triangular $n \times n$ matrices $\mathbf{F}(t)$ and $\mathbf{A}(t)$, whose diagonal elements we shall denote by $\{f_i(t)\}_{i=0}^{n-1}$ and $\{a_i(t)\}_{i=0}^{n-1}$, respectively. The indices t and $(t-\Delta)$ denote two discrete-time instants that are Δ units apart. It follows from the low rank property that we can factor $\nabla\mathbf{R}$ and write

$$\mathbf{R}(t) - \mathbf{F}(t)\mathbf{R}(t-\Delta)\mathbf{A}^*(t) = \mathbf{G}(t)\mathbf{B}^*(t) \;\;, \tag{1}$$

where $\mathbf{G}(t)$ and $\mathbf{B}(t)$ are $n \times r(t)$ so-called generator matrices.

Special cases of the time-variant structure (1) often arise in adaptive filtering where one is usually faced with the task of computing a new estimate at time t upon the arrival of a new datum, given the old estimate at time $t-1$. In the standard recursive least-squares setting,[7,8,9] this problem reduces to solving normal equations where the coefficient matrix, say $\mathbf{\Phi}(t)$, varies with time as follows: $\mathbf{\Phi}(t) - \lambda\mathbf{\Phi}(t-1) = \mathbf{u}^*(t)\mathbf{u}(t)$, for some scalar $0 \ll \lambda \leq 1$ and row vector $\mathbf{u}(t)$. This is clearly a special case of (1) with $\mathbf{R}(t) = \mathbf{\Phi}(t)$, $\mathbf{F}(t) = \mathbf{A}(t) = \sqrt{\lambda}\mathbf{I}$, $r(t) = 1$, $\mathbf{G}(t) = \mathbf{B}(t) = \mathbf{u}^*(t)$, and $\Delta = 1$. In this case, $\mathbf{\Phi}(t)$ and $\lambda\mathbf{\Phi}(t-1)$ differ by a rank one "update" matrix. In other problems, such as the block RLS formulation or the sliding window case, the matrices $\mathbf{\Phi}(t)$ and $\lambda\mathbf{\Phi}(t-1)$ differ by a higher-order rank update, where the single column $\mathbf{u}^*(t)$ is replaced by a matrix with multiple columns. Yet another manifestation of the non-Hermitian structure (1) arises in instrumental variable methods and will be considered in a Section 4.

We begin by stating and proving a readily established matrix result that will play an important role in the derivation of the array algorithms of this paper. For this purpose, we first review the generalized SVD (GSVD)[12,13] of two matrices: given $n \times m$ $(n \leq m)$ matrices $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{AB}^*$ is full rank, there exist $n \times n$ unitary matrices $\mathbf{U}$ and $\mathbf{V}$ ($\mathbf{UU}^* = \mathbf{I}_n = \mathbf{VV}^*$), an $m \times m$ invertible matrix $\mathbf{X}$, and an $n \times n$ diagonal matrix $\mathbf{\Sigma}$ with nonnegative entries such that ($\mathbf{I}_n$ denotes the $n \times n$ identity matrix)

$$\mathbf{A} = \mathbf{U}\begin{bmatrix} \mathbf{I}_n & \mathbf{0} \end{bmatrix}\mathbf{X}\,, \quad \mathbf{B}^* = \mathbf{X}^{-1}\begin{bmatrix} \mathbf{\Sigma} \\ \mathbf{0} \end{bmatrix}\mathbf{V}^*\,.$$

Lemma 1.1 *Consider four $n \times m$ $(n \leq m)$ matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$. If $\mathbf{AC}^* = \mathbf{BD}^*$ is full rank then there exist $m \times m$ matrices $\mathbf{\Theta}$ and $\mathbf{\Gamma}$ such that $\mathbf{\Theta\Gamma}^* = \mathbf{I}_m$, $\mathbf{A} = \mathbf{B\Theta}$, and $\mathbf{C} = \mathbf{D\Gamma}$.*

Proof: Introduce the GSVD's of $\mathbf{A}$ and $\mathbf{C}$, and $\mathbf{B}$ and $\mathbf{D}$, viz.,

$$\mathbf{A} = \mathbf{U_A}\begin{bmatrix} \mathbf{I}_n & \mathbf{0} \end{bmatrix}\mathbf{X}\,, \quad \mathbf{C}^* = \mathbf{X}^{-1}\begin{bmatrix} \mathbf{\Sigma_A} \\ \mathbf{0} \end{bmatrix}\mathbf{V_A^*}\,,$$

$$\mathbf{B} = \mathbf{U_B}\begin{bmatrix} \mathbf{I}_n & \mathbf{0} \end{bmatrix}\mathbf{Y}\,, \quad \mathbf{D}^* = \mathbf{Y}^{-1}\begin{bmatrix} \mathbf{\Sigma_B} \\ \mathbf{0} \end{bmatrix}\mathbf{V_B^*}\,,$$

where $\mathbf{U_A}, \mathbf{U_B}, \mathbf{V_A}$, and $\mathbf{V_B}$ are $n \times n$ unitary matrices, and $\mathbf{\Sigma_A}$ and $\mathbf{\Sigma_B}$ are $n \times n$ diagonal matrices with nonnegative entries. It follows from $\mathbf{AC}^* = \mathbf{BD}^*$ and the full rank property that we can choose $\mathbf{\Sigma_A} = \mathbf{\Sigma_B}$, $\mathbf{U_A} = \mathbf{U_B}$, and $\mathbf{V_A} = \mathbf{V_B}$. Let $\mathbf{\Theta} = \mathbf{Y}^{-1}\mathbf{X}$ and $\mathbf{\Gamma} = \mathbf{Y}^*\mathbf{X}^{-*}$. It then follows that $\mathbf{\Theta\Gamma}^* = \mathbf{I}_m$, $\mathbf{B\Theta} = \mathbf{A}$, and $\mathbf{D\Gamma} = \mathbf{C}$.

■

2. THE ARRAY EQUATIONS

We assume from now on that the matrix $\mathbf{R}(t)$ is strongly regular, viz., that all its principal leading submatrices are invertible. This guarantees the existence of lower and upper triangular factors, $\bar{\mathbf{L}}(t)$ and $\bar{\mathbf{U}}(t)$, respectively, such that

$$\mathbf{R}(t) = \bar{\mathbf{L}}(t)\bar{\mathbf{U}}(t).$$

[If desired, the triangular factors can be made unique by requiring, for example, that the diagonal entries of one of them, say $\bar{\mathbf{L}}(t)$, be equal to unity.]. We shall denote the nonzero parts of the columns of $\bar{\mathbf{L}}(t)$ and the rows of $\bar{\mathbf{U}}(t)$ by $\{\bar{\mathbf{l}}_i(t)\}_{i=0}^{n-1}$ and $\{\bar{\mathbf{u}}_i(t)\}_{i=0}^{n-1}$, respectively. That is, $\bar{\mathbf{l}}_i(t)$ is a column vector and $\bar{\mathbf{u}}_i(t)$ is a row vector. We shall also, and without loss of generality, write $(t-1)$ instead of $(t-\Delta)$.

We first show how to time-update the triangular factors $\bar{\mathbf{L}}(\cdot)$ and $\bar{\mathbf{U}}(\cdot)$. It follows from (1) that we can write

$$\underbrace{\begin{bmatrix} \bar{\mathbf{L}}(t) & \mathbf{0} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} \bar{\mathbf{U}}(t) \\ \mathbf{0} \end{bmatrix}}_{\mathbf{C}^*} = \underbrace{\begin{bmatrix} \mathbf{F}(t)\bar{\mathbf{L}}(t-1) & \mathbf{G}(t) \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} \bar{\mathbf{U}}(t-1)\mathbf{A}^*(t) \\ \mathbf{B}^*(t) \end{bmatrix}}_{\mathbf{D}^*}.$$

This expression fits into the statement of Lemma 1.1, where we have indicated the corresponding $\{\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}\}$. Hence, there exist (rotation) matrices $\mathbf{\Theta}(t)$ and $\mathbf{\Gamma}(t)$ such that $\mathbf{\Theta}(t)\mathbf{\Gamma}^*(t) = \mathbf{I}$ and

$$\begin{aligned} \begin{bmatrix} \bar{\mathbf{L}}(t) & \mathbf{0} \end{bmatrix} &= \begin{bmatrix} \mathbf{F}(t)\bar{\mathbf{L}}(t-1) & \mathbf{G}(t) \end{bmatrix} \mathbf{\Theta}(t) , \\ \begin{bmatrix} \bar{\mathbf{U}}^*(t) & \mathbf{0} \end{bmatrix} &= \begin{bmatrix} \mathbf{A}(t)\bar{\mathbf{U}}^*(t-1) & \mathbf{B}(t) \end{bmatrix} \mathbf{\Gamma}(t) . \end{aligned} \tag{2}$$

In other words, $\mathbf{\Theta}(t)$ and $\mathbf{\Gamma}(t)$ are (coupled) rotation matrices that produce the block zero entries in the postarrays on the left-hand side of the above expressions. Each one of them takes a prearray of numbers of the form

$$\begin{bmatrix} x & & & x & x & x \\ x & x & & x & x & x \\ x & x & x & x & x & x \end{bmatrix}$$

and transforms it to a postarray of numbers of the form

$$\begin{bmatrix} x & & & 0 & 0 & 0 \\ x & x & & 0 & 0 & 0 \\ x & x & x & 0 & 0 & 0 \end{bmatrix}.$$

We say that (2) involves two array equations: one for the update of the lower triangular factor $\bar{\mathbf{L}}(t)$ and the other for the update of the upper triangular factor $\bar{\mathbf{U}}(t)$. The two arrays are coupled via the rotation matrices that need to satisfy the condition $\mathbf{\Theta}(t)\mathbf{\Gamma}^*(t) = \mathbf{I}$ (i.e., $\mathbf{\Gamma}^*(t)$ is the inverse of $\mathbf{\Theta}(t)$).

The array transformations in (2) can also be implemented as a sequence of rotations, say $\{\mathbf{\Theta}_0(t), \mathbf{\Theta}_1(t), ..\}$ and $\{\mathbf{\Gamma}_0(t), \mathbf{\Gamma}_1(t), ..\}$ that satisfy $\mathbf{\Theta}_i(t)\mathbf{\Gamma}_i^*(t) = \mathbf{I}$ and produce the block zeros in the postarrays by introducing one zero (row) at a time, such as

$$\begin{bmatrix} x & & & x & x & x \\ x & x & & x & x & x \\ x & x & x & x & x & x \end{bmatrix} \xrightarrow{\mathbf{\Theta}_0(t)} \begin{bmatrix} x & & & 0 & 0 & 0 \\ x & x & & x & x & x \\ x & x & x & x & x & x \end{bmatrix} \xrightarrow{\mathbf{\Theta}_1(t)} \begin{bmatrix} x & & & 0 & 0 & 0 \\ x & x & & 0 & 0 & 0 \\ x & x & x & x & x & x \end{bmatrix} \cdots$$

We thus know how to time-update the triangular factors $\bar{\mathbf{L}}(\cdot)$ and $\bar{\mathbf{U}}(\cdot)$: just form the prearrays in (2) and then choose convenient (coupled) rotation matrices $\mathbf{\Theta}(t)$ and $\mathbf{\Gamma}(t)$ that would produce the desired zero patterns in the postarrays. Explicit expressions for such coupled rotations are not relevant at this stage of our argument and will, therefore, be postponed to Section 5. We now move to a closer analysis of the quantities that are propagated by this procedure.

3. STRUCTURED SCHUR COMPLEMENTS

The annihilation of each row in the prearrays (2) can be achieved in several ways and we do not pretend to exhaust all the possibilities here. We shall instead focus on the general theme. So let $\mathbf{\Theta}_0(t)$ and $\mathbf{\Gamma}_0(t)$ be two coupled rotation matrices ($\mathbf{\Theta}_0(t)\mathbf{\Gamma}_0^*(t) = \mathbf{I}$) that procude a top zero row in each of the postarrays of (2). This is achieved by pivoting with the top entry of each of the first columns of $\mathbf{F}(t)\bar{\mathbf{L}}(t-1)$ and $\mathbf{A}(t)\bar{\mathbf{U}}^*(t-1)$; thus leaving the other columns of $\mathbf{F}(t)\bar{\mathbf{L}}(t-1)$ and $\mathbf{A}(t)\bar{\mathbf{U}}^*(t-1)$ unaltered, viz.,

$$
\begin{aligned}
\begin{bmatrix} \mathbf{F}(t)\bar{\mathbf{L}}(t-1) & \mathbf{G}(t) \end{bmatrix} \mathbf{\Theta}_0(t) &= \begin{bmatrix} \bar{\mathbf{l}}_0(t) & \begin{matrix} \mathbf{0} \\ \mathbf{F}_1(t)\bar{\mathbf{L}}_1(t-1) \end{matrix} & \begin{matrix} \mathbf{0} \\ \mathbf{G}_1(t) \end{matrix} \end{bmatrix}, \\
\begin{bmatrix} \mathbf{A}(t)\bar{\mathbf{U}}^*(t-1) & \mathbf{B}(t) \end{bmatrix} \mathbf{\Gamma}_0(t) &= \begin{bmatrix} \bar{\mathbf{u}}_0^*(t) & \begin{matrix} \mathbf{0} \\ \mathbf{A}_1(t)\bar{\mathbf{U}}_1^*(t-1) \end{matrix} & \begin{matrix} \mathbf{0} \\ \mathbf{B}_1(t) \end{matrix} \end{bmatrix},
\end{aligned} \tag{3}
$$

where $\mathbf{F}_1(t), \mathbf{A}_1(t), \bar{\mathbf{L}}_1(t-1)$, and $\bar{\mathbf{U}}_1^*(t-1)$ are the submatrices obtained after deleting the first row and column of $\mathbf{F}(t), \mathbf{A}(t), \bar{\mathbf{L}}(t-1)$, and $\bar{\mathbf{U}}^*(t-1)$, respectively. The first columns in each of the postarrays in (3) have to be $\bar{\mathbf{l}}_0(t)$ and $\bar{\mathbf{u}}_0^*(t)$ by virtue of the identity (2). The symbols $\mathbf{G}_1(t)$ and $\mathbf{B}_1(t)$ are used to denote the resulting nonzero rows of the postarrays after the application of the first transformation $\{\mathbf{\Theta}_0(t), \mathbf{\Gamma}_0(t)\}$. They will be further annihilated by applying subsequent transformations $\{\mathbf{\Theta}_i(t), \mathbf{\Gamma}_i(t)\}_{i\geq 1}$, as detailed ahead. But let us, for the moment, check the significance of the matrices $\mathbf{G}_1(t)$ and $\mathbf{B}_1(t)$. Multiplying the arrays in (3) leads to the equality,

$$
\begin{bmatrix} \mathbf{F}(t)\bar{\mathbf{L}}(t-1) & \mathbf{G}(t) \end{bmatrix} \underbrace{\mathbf{\Theta}_0(t)\mathbf{\Gamma}_0^*(t)}_{\mathbf{I}} \begin{bmatrix} \mathbf{A}(t)\bar{\mathbf{U}}^*(t-1) & \mathbf{B}(t) \end{bmatrix}^* =
$$

$$
\begin{bmatrix} \bar{\mathbf{l}}_0(t) & \begin{matrix} \mathbf{0} \\ \mathbf{F}_1(t)\bar{\mathbf{L}}_1(t-1) \end{matrix} & \begin{matrix} \mathbf{0} \\ \mathbf{G}_1(t) \end{matrix} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{u}}_0^*(t) & \begin{matrix} \mathbf{0} \\ \mathbf{A}_1(t)\bar{\mathbf{U}}_1^*(t-1) \end{matrix} & \begin{matrix} \mathbf{0} \\ \mathbf{B}_1(t) \end{matrix} \end{bmatrix}^*,
$$

or, equivalently,

$$
\underbrace{\mathbf{F}(t)\mathbf{R}(t-1)\mathbf{A}^*(t) + \mathbf{G}(t)\mathbf{B}^*(t)}_{\mathbf{R}(t)} =
$$

$$
\bar{\mathbf{l}}_0(t)\bar{\mathbf{u}}_0(t) + \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{F}_1(t)\bar{\mathbf{L}}_1(t-1)\bar{\mathbf{U}}_1(t-1)\mathbf{A}_1^*(t) + \mathbf{G}_1(t)\mathbf{B}_1^*(t) \end{bmatrix}. \tag{4}
$$

Let $\mathbf{R}_1(t)$ denote the Schur complement of $\mathbf{R}(t)$ with respect to its $(0,0)$ entry. That is, $\mathbf{R}_1(t)$ is the $(n-1)\times(n-1)$ matrix obtained by subtracting from $\mathbf{R}(t)$ the outer product $\bar{\mathbf{l}}_0(t)\bar{\mathbf{u}}_0(t)$,

$$
\begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_1(t) \end{bmatrix} = \mathbf{R}(t) - \bar{\mathbf{l}}_0(t)\bar{\mathbf{u}}_0(t) . \tag{5}
$$

It is well known[3,12] that the triangular factors of $\mathbf{R}_1(t)$ are obtained from those of $\mathbf{R}(t)$ by simply deleting the first row and column of $\bar{\mathbf{L}}(t)$ and $\bar{\mathbf{U}}^*(t)$, viz.,

$$
\mathbf{R}_1(t) = \bar{\mathbf{L}}_1(t)\bar{\mathbf{U}}_1(t) .
$$

It then follows from (4) that

$$
\mathbf{R}(t) - \bar{\mathbf{l}}_0(t)\bar{\mathbf{u}}_0(t) = \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{F}_1(t)\mathbf{R}_1(t-1)\mathbf{A}_1^*(t) + \mathbf{G}_1(t)\mathbf{B}_1^*(t) \end{bmatrix},
$$

and, using (5), we conclude that

$$\mathbf{R}_1(t) - \mathbf{F}_1(t)\mathbf{R}_1(t-1)\mathbf{A}_1^*(t) = \mathbf{G}_1(t)\mathbf{B}_1^*(t) \ .$$

This shows that the matrices $\mathbf{G}_1(t)$ and $\mathbf{B}_1(t)$ are simply generator matrices for the first Schur complement of $\mathbf{R}(t)$, with respect to the displacement operation $\mathbf{R}_1(t) - \mathbf{F}_1(t)\mathbf{R}_1(t-1)\mathbf{A}_1^*(t)$.

The procedure can now be continued by annihilating the top rows of $\mathbf{G}_1(t)$ and $\mathbf{B}_1(t)$ as follows

$$\begin{aligned}
\begin{bmatrix} \mathbf{F}_1(t)\bar{\mathbf{L}}_1(t-1) & \mathbf{G}_1(t) \end{bmatrix} \mathbf{\Theta}_1(t) &= \begin{bmatrix} \bar{\mathbf{l}}_1(t) & \begin{matrix} \mathbf{0} \\ \mathbf{F}_2(t)\bar{\mathbf{L}}_2(t-1) \end{matrix} & \begin{matrix} \mathbf{0} \\ \mathbf{G}_2(t) \end{matrix} \end{bmatrix} , \\
\begin{bmatrix} \mathbf{A}_1(t)\bar{\mathbf{U}}_1^*(t-1) & \mathbf{B}_1(t) \end{bmatrix} \mathbf{\Gamma}_1(t) &= \begin{bmatrix} \bar{\mathbf{u}}_1^*(t) & \begin{matrix} \mathbf{0} \\ \mathbf{A}_2(t)\bar{\mathbf{U}}_2^*(t-1) \end{matrix} & \begin{matrix} \mathbf{0} \\ \mathbf{B}_2(t) \end{matrix} \end{bmatrix} ,
\end{aligned}$$

where $\mathbf{F}_2(t), \mathbf{A}_2(t), \bar{\mathbf{L}}_2(t-1)$, and $\bar{\mathbf{U}}_2^*(t-1)$ are the submatrices obtained after deleting the first row and column of $\mathbf{F}_1(t), \mathbf{A}_1(t), \bar{\mathbf{L}}_1(t-1)$, and $\bar{\mathbf{U}}_1^*(t-1)$, respectively. The first columns in each of the above postarrays have to be $\bar{\mathbf{l}}_1(t)$ and $\bar{\mathbf{u}}_1^*(t)$ by virtue of the identity (2). The resulting matrices $\mathbf{G}_2(t)$ and $\mathbf{B}_2(t)$ can then be shown, by following the exact same arguments as above, to be generator matrices for the Schur complement of $\mathbf{R}(t)$ with respect to its leading 2×2 submatrix, denoted by $\mathbf{R}_2(t)$,

$$\begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_2(t) \end{bmatrix} = \mathbf{R}_1(t) - \bar{\mathbf{l}}_1(t)\bar{\mathbf{u}}_1(t) \ .$$

That is,

$$\mathbf{R}_2(t) - \mathbf{F}_2(t)\mathbf{R}_2(t-1)\mathbf{A}_2^*(t) = \mathbf{G}_2(t)\mathbf{B}_2^*(t) \ .$$

We are thus led to the following algorithm.

Algorithm 3.1 *The triangular factors, $\bar{\mathbf{L}}(t)$ and $\bar{\mathbf{U}}(t)$, of an $n \times n$ strongly regular matrix $\mathbf{R}(t)$ with time-variant Toeplitz-like structure as in (1), can be time-updated as follows: start with $\mathbf{F}_0(t) = \mathbf{F}(t)$, $\mathbf{A}_0(t) = \mathbf{A}(t)$, $\mathbf{G}_0(t) = \mathbf{G}(t)$, $\mathbf{B}_0(t) = \mathbf{B}(t)$, $\bar{\mathbf{L}}_0(t-1) = \bar{\mathbf{L}}(t-1)$, $\bar{\mathbf{U}}_0(t-1) = \bar{\mathbf{U}}(t-1)$, and repeat for $i = 0, 1, \ldots, n-1$:*

$$\begin{aligned}
\begin{bmatrix} \mathbf{F}_i(t)\bar{\mathbf{L}}_i(t-1) & \mathbf{G}_i(t) \end{bmatrix} \mathbf{\Theta}_i(t) &= \begin{bmatrix} \bar{\mathbf{l}}_i(t) & \begin{matrix} \mathbf{0} \\ \mathbf{F}_{i+1}(t)\bar{\mathbf{L}}_{i+1}(t-1) \end{matrix} & \begin{matrix} \mathbf{0} \\ \mathbf{G}_{i+1}(t) \end{matrix} \end{bmatrix} , \\
\begin{bmatrix} \mathbf{A}_i(t)\bar{\mathbf{U}}_i^*(t-1) & \mathbf{B}_i(t) \end{bmatrix} \mathbf{\Gamma}_i(t) &= \begin{bmatrix} \bar{\mathbf{u}}_i^*(t) & \begin{matrix} \mathbf{0} \\ \mathbf{A}_{i+1}(t)\bar{\mathbf{U}}_{i+1}^*(t-1) \end{matrix} & \begin{matrix} \mathbf{0} \\ \mathbf{B}_{i+1}(t) \end{matrix} \end{bmatrix} ,
\end{aligned} \tag{6}$$

where $\mathbf{\Theta}_i(t)$ and $\mathbf{\Gamma}_i(t)$ are coupled rotation matrices that satisfy $\mathbf{\Theta}_i(t)\mathbf{\Gamma}_i^*(t) = \mathbf{I}$ and are chosen so as to introduce the zero rows in the $(1,3)$ entries of the postarrays. It further follows that

$$\mathbf{R}_{i+1}(t) - \mathbf{F}_{i+1}(t)\mathbf{R}_{i+1}(t-1)\mathbf{A}_{i+1}^*(t) = \mathbf{G}_{i+1}(t)\mathbf{B}_{i+1}^*(t).$$

It is thus clear that the sequence of transformations $\{\mathbf{\Theta}_i(t), \mathbf{\Gamma}_i(t)\}$ are chosen so as to annihilate the top rows in the $(1,3)$ entries of the postarrays (6). Each such transformation then produces one column and one row of the triangular factors of $\mathbf{R}(t)$. A sequence of k ($k < n$) transformations

$\{\mathbf{\Theta}_i(t), \mathbf{\Gamma}_i(t)\}$ will, consequently, produce the first k columns of $\bar{\mathbf{L}}(t)$ and the first k rows of $\mathbf{U}(t)$. For example,

$$\left[\begin{array}{cc} \mathbf{F}(t)\bar{\mathbf{L}}(t-1) & \mathbf{G}(t) \end{array} \right] \stackrel{\mathbf{\Theta}_0(t),\ldots,\mathbf{\Theta}_{k-1}(t)}{\longrightarrow} \left[\begin{array}{ccccc} & & & \mathbf{0} & \mathbf{0} \\ & & 0 & & \\ & & & \vdots & \vdots \\ \bar{\mathbf{l}}_0(t) & \ldots & \vdots & & \\ & & & \mathbf{0} & \mathbf{0} \\ & & 0 & & \\ & & & \mathbf{0} & \mathbf{0} \\ & & \bar{\mathbf{l}}_{k-1}(t) & & \\ & & & \mathbf{F}_k\bar{\mathbf{L}}_k(t-1) & \mathbf{G}_k(t) \end{array} \right] .$$

The term $\mathbf{F}_k\bar{\mathbf{L}}_k(t-1)$ is common to both the pre- and post-arrays. If we denote the first k columns of $\bar{\mathbf{L}}(t)$ by $\bar{\mathbf{L}}_{0:k-1}(t)$, and the sequence of first k rotations by $\mathbf{\Theta}_{0:k-1}(t)$, then the above expression can be rewritten more compactly in the form

$$\left[\begin{array}{cc} \mathbf{F}(t)\bar{\mathbf{L}}_{0:k-1}(t-1) & \mathbf{G}(t) \end{array} \right] \stackrel{\mathbf{\Theta}_{0:k-1}(t)}{\longrightarrow} \left[\begin{array}{cc} & \mathbf{0} \\ \bar{\mathbf{L}}_{0:k-1}(t) & \vdots \\ & \mathbf{0} \\ & \mathbf{G}_k(t) \end{array} \right] . \tag{7}$$

Similarly, if we denote the first k rows of $\bar{\mathbf{U}}(t)$ by $\bar{\mathbf{U}}_{0:k-1}(t)$ and the first k rotations by $\mathbf{\Gamma}_{0:k-1}(t)$, then the update for $\mathbf{B}(t)$ can be written as

$$\left[\begin{array}{cc} \mathbf{A}(t)\bar{\mathbf{U}}^*_{0:k-1}(t-1) & \mathbf{B}(t) \end{array} \right] \stackrel{\mathbf{\Gamma}_{0:k-1}(t)}{\longrightarrow} \left[\begin{array}{cc} & \mathbf{0} \\ \bar{\mathbf{U}}^*_{0:k-1}(t) & \vdots \\ & \mathbf{0} \\ & \mathbf{B}_k(t) \end{array} \right] . \tag{8}$$

The rotation $\mathbf{\Theta}_{0:k-1}(t)$ annihilates the top k rows of $\mathbf{G}(t)$, while the rotation $\mathbf{\Gamma}_{0:k-1}(t)$ annihilates the top k rows of $\mathbf{B}(t)$. The resulting $\mathbf{G}_k(t)$ and $\mathbf{B}_k(t)$ will then be generator matrices for the Schur complement of $\mathbf{R}(t)$ with respect to its leading $k \times k$ submatrix.

4. AN APPLICATION TO INSTRUMENTAL VARIABLE METHODS

In instrumental variable (IV) methods[10,11] one is often faced with the task of solving a linear system of equations of the form

$$\mathbf{\Phi}(t)\mathbf{w}(t) = \mathbf{s}(t) \ , \tag{9}$$

where $\mathbf{\Phi}(t)$ is $n \times n$, $\mathbf{s}(t)$ is $n \times 1$, and $\mathbf{w}(t)$ is an $n \times 1$ unknown weight vector. It often happens that the quantities $\mathbf{\Phi}(t)$ and $\mathbf{s}(t)$ satisfy time-updates of the form

$$\mathbf{\Phi}(t) - \lambda\mathbf{\Phi}(t-1) = \mathbf{m}^*(t)\mathbf{v}(t) \ , \ \ \mathbf{s}(t) - \lambda\mathbf{s}(t-1) = d(t)\mathbf{m}^*(t) \ , \tag{10}$$

where λ is a positive scalar ($0 \ll \lambda \leq 1$), $\mathbf{v}(t)$ and $\mathbf{m}(t)$ are row vectors,

$$\mathbf{v}(t) = \left[\begin{array}{cccc} v_1(t) & v_2(t) & \ldots & v_n(t) \end{array} \right] \ , \ \ \mathbf{m}(t) = \left[\begin{array}{cccc} m_1(t) & m_2(t) & \ldots & m_n(t) \end{array} \right] \ ,$$

and $d(t)$ is a scalar. In the IV applications, we are interested in solving (9) for successive time instants, say $t, t+1, t+2, \ldots$, by employing a computationally efficient procedure that should exploit the existing low-rank update structure of both $\mathbf{\Phi}(t)$ and $\mathbf{s}(t)$, as evidenced by (10). We shall see that it is indeed possible to efficiently compute the successive weight vectors, $\mathbf{w}(t) \rightarrow \mathbf{w}(t+1) \rightarrow \mathbf{w}(t+2) \rightarrow, \ldots$, without the need to solve afresh the equations (9) at each time instant t. A major feature of our solution is that it completely avoids a back-substitution step.

To begin with, we observe that $\mathbf{\Phi}(t)$ is itself a time-variant structured matrix since its time-update in (10) is a special case of (1) with $\mathbf{F}(t) = \mathbf{A}(t) = \sqrt{\lambda}\ \mathbf{I}_n$, $\mathbf{G}(t) = \mathbf{m}^*(t)$, $\mathbf{B}(t) = \mathbf{v}^*(t)$, and $\Delta = 1$. We, therefore, conclude that its triangular factors, denoted by $\bar{\mathbf{L}}_\phi(t)$ and $\bar{\mathbf{U}}_\phi(t)$, can be time-updated via the following arrays (recall (2))

$$\begin{aligned} \begin{bmatrix} \sqrt{\lambda}\bar{\mathbf{L}}_\phi(t-1) & \mathbf{m}^*(t) \end{bmatrix} \mathbf{\Theta}_\phi(t) &= \begin{bmatrix} \bar{\mathbf{L}}_\phi(t) & \mathbf{0} \end{bmatrix}, \\ \begin{bmatrix} \sqrt{\lambda}\bar{\mathbf{U}}^*_\phi(t-1) & \mathbf{v}^*(t) \end{bmatrix} \mathbf{\Gamma}_\phi(t) &= \begin{bmatrix} \bar{\mathbf{U}}^*_\phi(t) & \mathbf{0} \end{bmatrix}, \end{aligned} \tag{11}$$

where $\{\mathbf{\Theta}_\phi(t), \mathbf{\Gamma}(t)_\phi\}$ are coupled rotations ($\mathbf{\Theta}_\phi(t)\mathbf{\Gamma}^*_\phi(t) = \mathbf{I}_n$) that produce the block zeros in the postarrays.

Once the triangular factors of $\mathbf{\Phi}(\cdot)$ have been time-updated, from time $(t-1)$ to time t, then one way to determine the weight vector $\mathbf{w}(t)$ is to solve the following triangular system of linear equations, say via back-substitution,

$$\bar{\mathbf{U}}_\phi(t)\mathbf{w}(t) = \bar{\mathbf{L}}_\phi^{-1}(t)\mathbf{s}(t).$$

This however, does not yield a parallelizable algorithm. We now show how to apply the result of Algorithm 3.1 in order to obtain an alternative procedure for the extraction of the weight vector $\mathbf{w}(t)$ so as to avoid the need for a backsubstitution step. The analogue of this procedure for adaptive problems is developed in[4,5,9].

The main point is to start by expressing $\mathbf{w}(t)$ as a Schur complement in a suitably-defined block matrix, and then to properly exploit the structure of this matrix. So consider the following $(2n \times (n+1))$ extended matrix

$$\mathbf{R}(t) = \begin{bmatrix} \mathbf{\Phi}(t) & \mathbf{s}(t) \\ \mathbf{I}_n & \mathbf{0} \end{bmatrix},$$

and note that the Schur complement of $\mathbf{\Phi}(t)$ in $\mathbf{R}(t)$ is $-\mathbf{\Phi}^{-1}(t)\mathbf{s}(t)$, which is equal to (minus) the desired weight vector $\mathbf{w}(t)$. This already suggests the following route: if we can show how to efficiently go from the Schur complement at time $t-1$ to the Schur complement at time t, then we obtain an efficient procedure for going from $\mathbf{w}(t-1)$ to $\mathbf{w}(t)$. But we claim that this is precisely what is provided by Algorithm 3.1, as we now further elaborate.

We shall, for convenience, redefine $\mathbf{R}(t)$ as a $(2n \times 2n)$ square matrix,

$$\mathbf{R}(t) = \begin{bmatrix} \mathbf{\Phi}(t) & \mathbf{s}(t) & \mathbf{0} \\ \mathbf{I}_n & \mathbf{0} & \mathbf{0} \end{bmatrix}, \tag{12}$$

where the Schur complement of $\mathbf{\Phi}(t)$ is now $\begin{bmatrix} -\mathbf{w}(t) & \mathbf{0} \end{bmatrix}$, which still completely identifies $\mathbf{w}(t)$. Now, relation (10) implies that the matrix $\mathbf{R}(t)$ in (12) is a time-variant Toeplitz-like matrix. Indeed, it follows from (10) that

$$\mathbf{R}(t) - \begin{bmatrix} \lambda\mathbf{I}_n & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_n \end{bmatrix} \mathbf{R}(t-1) = \begin{bmatrix} \mathbf{m}^*(t) \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{v}(t) & d(t) & \mathbf{0} \end{bmatrix},$$

which is a special case of (1) with $\mathbf{A}(t) = \mathbf{I}_{2n}, \mathbf{F}(t) = (\lambda\mathbf{I}_n \oplus \mathbf{I}_n)$, $\Delta = 1$, and

$$\mathbf{G}(t) = \begin{bmatrix} \mathbf{m}^*(t) \\ \mathbf{0} \end{bmatrix}, \quad \mathbf{B}(t) = \begin{bmatrix} \mathbf{v}^*(t) \\ d(t) \\ \mathbf{0} \end{bmatrix}.$$

We are interested in the first n Schur complementation steps of $\mathbf{R}(t)$, i.e., in the first n columns and rows of its triangular factors, denoted by $\bar{\mathbf{L}}(t)$ and $\bar{\mathbf{U}}(t)$. But since $\mathbf{\Phi}(t)$ is the leading $n \times n$ submatrix of $\mathbf{R}(t)$, it readily follows that $\bar{\mathbf{L}}(t)$ and $\bar{\mathbf{U}}(t)$ have the following forms [Here we invoke the fact that the triangular factorization of a matrix $\mathbf{R}$ obeys a nested property: the triangular factors of a leading submatrix of $\mathbf{R}$ are themselves leading submatrices of the triangular factors of $\mathbf{R}$.]:

$$\bar{\mathbf{L}}(t) = \begin{bmatrix} \bar{\mathbf{L}}_\phi(t) & \mathbf{0} \\ \mathbf{X} & ? \end{bmatrix}, \quad \bar{\mathbf{U}}(t) = \begin{bmatrix} \bar{\mathbf{U}}_\phi(t) & \mathbf{Y} \\ \mathbf{0} & ? \end{bmatrix},$$

for some matrices $\mathbf{X}$ and $\mathbf{Y}$ to be specified ahead, and where the symbol ? stands for irrelevant entries. By comparing entries on both sides of the equality

$$\bar{\mathbf{L}}(t)\bar{\mathbf{U}}(t) = \begin{bmatrix} \mathbf{\Phi}(t) & \mathbf{s}(t) & \mathbf{0} \\ \mathbf{I}_n & \mathbf{0} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{L}}_\phi(t) & \mathbf{0} \\ \mathbf{X} & ? \end{bmatrix} \begin{bmatrix} \bar{\mathbf{U}}_\phi(t) & \mathbf{Y} \\ \mathbf{0} & ? \end{bmatrix},$$

we readily conclude that we must have

$$\mathbf{X} = \bar{\mathbf{U}}_\phi^{-1}(t), \quad \mathbf{Y} = \begin{bmatrix} \bar{\mathbf{L}}_\phi^{-1}(t)\mathbf{s}(t) & \mathbf{0} \end{bmatrix}.$$

That is, the first n columns of the triangular factor $\bar{\mathbf{L}}(t)$ and the first n rows of the triangular factor $\bar{\mathbf{U}}(t)$ are completely determined by $\bar{\mathbf{L}}_\phi(t)$, $\bar{\mathbf{U}}_\phi(t)$ and $\mathbf{s}(t)$, and are equal to

$$\begin{bmatrix} \bar{\mathbf{L}}_\phi(t) \\ \bar{\mathbf{U}}_\phi^{-1}(t) \end{bmatrix}, \quad \begin{bmatrix} \bar{\mathbf{U}}_\phi(t) & \bar{\mathbf{L}}_\phi^{-1}(t)\mathbf{s}(t) & \mathbf{0} \end{bmatrix},$$

respectively. But recall that we are interested in computing the Schur complement of $\mathbf{\Phi}(t)$ in $\mathbf{R}(t)$. Hence, we only need to apply the first n recursive steps of Algorithm 3.1 to $\mathbf{R}(t)$, which is $2n \times 2n$. Using (7) and (8), we conclude that the following arrays can be used to update the first n columns and rows of the triangular factors of $\mathbf{R}(t)$,

$$\begin{bmatrix} \lambda\,\bar{\mathbf{L}}_\phi(t-1) & \mathbf{m}^*(t) \\ \\ \bar{\mathbf{U}}_\phi^{-1}(t-1) & \mathbf{0} \end{bmatrix} \mathbf{\Theta}_{0:n-1}(t) \;=\; \begin{bmatrix} \bar{\mathbf{L}}_\phi(t) & \mathbf{0} \\ \\ \bar{\mathbf{U}}_\phi^{-1}(t) & \mathbf{g}_n(t) \end{bmatrix},$$

$$\begin{bmatrix} \bar{\mathbf{U}}_\phi^*(t-1) & \mathbf{v}^*(t) \\ \\ \mathbf{s}^*(t-1)\bar{\mathbf{L}}_\phi^{-*}(t-1) & d^*(t) \\ \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{\Gamma}_{0:n-1}(t) \;=\; \begin{bmatrix} \bar{\mathbf{U}}_\phi(t) & \mathbf{0} \\ \\ \mathbf{s}^*(t)\bar{\mathbf{L}}_\phi^{-*}(t) & b_n(t) \\ \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \qquad (13)$$

where we have designated the resulting entries in the postarrays by $\mathbf{g}_n(t)$ and $\begin{bmatrix} b_n(t) & \mathbf{0} \end{bmatrix}^{\mathbf{T}}$, and where the rotations $\{\mathbf{\Theta}_{0:n-1}(t), \mathbf{\Gamma}_{0:n-1}(t)\}$ satisfy $\mathbf{\Theta}_{0:n-1}(t)\mathbf{\Gamma}^*_{0:n-1}(t) = \mathbf{I}$ and are chosen so as to produce the zero entries in the $(1,2)$ positions of the postarrays. It further follows from the result of Algorithm 3.1 that $\mathbf{g}_n(t)$ and $\begin{bmatrix} b_n(t) & \mathbf{0} \end{bmatrix}^{\mathbf{T}}$ are generators for the n^{th} Schur complement of $\mathbf{R}(t)$, which is nothing but $\begin{bmatrix} -\mathbf{w}(t) & \mathbf{0} \end{bmatrix}$. That is,

$$\begin{bmatrix} -\mathbf{w}(t) & \mathbf{0} \end{bmatrix} - \begin{bmatrix} -\mathbf{w}(t-1) & \mathbf{0} \end{bmatrix} = \mathbf{g}_n(t) \begin{bmatrix} b_n^*(t) & \mathbf{0} \end{bmatrix} ,$$

or, equivalently,

$$\mathbf{w}(t) = \mathbf{w}(t-1) - \mathbf{g}_n(t) b_n^*(t) \; . \tag{14}$$

We have thus derived a simple time-update relation for the weight-vector: given $\mathbf{w}(t-1)$, all we need to do is to form the prearrays in (13) (the last zero entries on both sides of the second array can clearly be ignored); apply rotations $\{\mathbf{\Theta}_{0:n-1}(t), \mathbf{\Gamma}_{0:n-1}(t)\}$ that annihilate the $(1,2)$ block entries in the postarrays; and use the resulting quantities $\mathbf{g}_n(t)$ and $b_n(t)$ to update $\mathbf{w}(t-1)$ to $\mathbf{w}(t)$ as in (14). This procedure completely avoids the need for a backsubstitution step.

4.1 A Remark on the Rotation Matrices.

The rotation matrices $\{\mathbf{\Theta}_{0:n-1}(t), \mathbf{\Gamma}_{0:n-1}(t)\}$ that are needed in (13) can be easily related to those required in (11), viz., $\{\mathbf{\Theta}_\phi(t), \mathbf{\Gamma}_\phi(t)\}$. Indeed, it is straightforward to check that

$$\mathbf{\Theta}_{0:n-1}(t) = \begin{bmatrix} \frac{1}{\sqrt{\lambda}}\mathbf{I} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \mathbf{\Theta}_\phi(t) \; , \quad \mathbf{\Gamma}_{0:n-1}(t) = \begin{bmatrix} \sqrt{\lambda}\,\mathbf{I} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \mathbf{\Gamma}_\phi(t) \; .$$

If we replace these expressions for $\{\mathbf{\Theta}_{0:n-1}(t), \mathbf{\Gamma}_{0:n-1}(t)\}$ into (13) we obtain the following equivalent array equations for the time-update of the weight vector:

$$\begin{bmatrix} \sqrt{\lambda}\,\bar{\mathbf{L}}_\phi(t-1) & \mathbf{m}^*(t) \\ \frac{1}{\sqrt{\lambda}}\,\bar{\mathbf{U}}_\phi^{-1}(t-1) & \mathbf{0} \end{bmatrix} \mathbf{\Theta}_\phi(t) \;=\; \begin{bmatrix} \bar{\mathbf{L}}_\phi(t) & \mathbf{0} \\ \bar{\mathbf{U}}_\phi^{-1}(t) & \mathbf{g}_n(t) \end{bmatrix} ,$$

$$\begin{bmatrix} \sqrt{\lambda}\,\bar{\mathbf{U}}_\phi^*(t-1) & \mathbf{v}^*(t) \\ \sqrt{\lambda}\,\mathbf{s}^*(t-1)\bar{\mathbf{L}}_\phi^{-*}(t-1) & d^*(t) \end{bmatrix} \mathbf{\Gamma}_\phi(t) \;=\; \begin{bmatrix} \bar{\mathbf{U}}_\phi(t) & \mathbf{0} \\ \mathbf{s}^*(t)\bar{\mathbf{L}}_\phi^{-*}(t) & b_n(t) \end{bmatrix} . \tag{15}$$

Algorithm 4.1.1 *The solution of the linear equations (9) that arise in the recursive instrumental variable problem can be time-updated by using the array equations (15), where* $\{\mathbf{\Theta}_\phi(t), \mathbf{\Gamma}_\phi(t)\}$ *are any* $n \times n$ *coupled matrices* ($\mathbf{\Theta}_\phi(t)\mathbf{\Gamma}_\phi^*(t) = \mathbf{I}_n$) *that produce the zero blocks in the* $(1,2)$ *entries of the postarrays. Moreover,* $\mathbf{w}(t) = \mathbf{w}(t-1) - \mathbf{g}_n(t) b_n^*(t)$.

The array equations (15) admit a coupled parallelizable implementation as depicted in Figure 1 for the special case $n = 3$. The figure consists of three triangular arrays and one linear array: the top-left triangular array receives the input entries $\{m_1^*(t), m_2^*(t), m_3^*(t)\}$ and rotates them in conjunction with $\sqrt{\lambda}\,\bar{\mathbf{L}}_\phi(t-1)$ via $\mathbf{\Theta}_\phi(t)$. This yields $\bar{\mathbf{L}}_\phi(t)$ that is stored in the same array for the next time instant. The rotation $\mathbf{\Theta}_\phi(t)$ is also used to rotate the zero vector along with $\frac{1}{\sqrt{\lambda}}\,\bar{\mathbf{U}}_\phi^{-1}(t-1)$ in the bottom-left triangular array; thus yielding $\mathbf{g}_3(t)$ and $\bar{\mathbf{U}}_\phi^{-1}(t)$, which is stored in the lower array for the next time instant. Similarly, the top-right triangular array receives the input entries $\{v_1^*(t), v_2^*(t), v_3^*(t)\}$ and rotates them in conjunction with $\sqrt{\lambda}\,\bar{\mathbf{U}}_\phi^*(t-1)$ via $\mathbf{\Gamma}_\phi(t)$. This yields $\bar{\mathbf{U}}_\phi(t)$ that is stored in the array. Finally, the linear array receives $d^*(t)$ and rotates it in conjunction with $\sqrt{\lambda}\,\mathbf{s}^*(t-1)\bar{\mathbf{L}}_\phi^{-*}(t-1)$ via the same rotation $\mathbf{\Gamma}_\phi(t)$ to yield the updated quantity $\mathbf{s}^*(t)\bar{\mathbf{L}}_\phi^{-*}(t)$ and $b_3(t)$. The product of the terms $\mathbf{g}_3(t)$ and $b_3^*(t)$ is then subtracted from $\mathbf{w}(t-1)$ to yield $\mathbf{w}(t)$, which is stored in the right-most cells of the lower array.

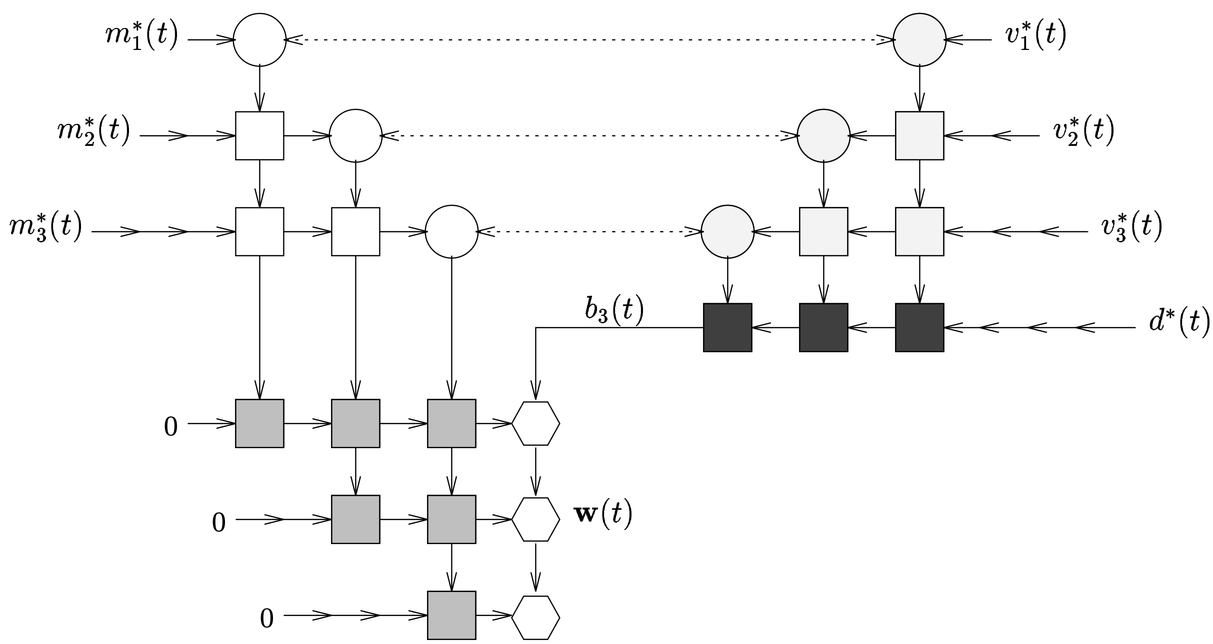

Figure 1: *A coupled parallelizable array for instrumental-variable parameter estimation.*

5. EXAMPLES OF COUPLED ROTATIONS

To conclude the paper, we now exhibit examples of elementary 2×2 coupled rotations that satisfy

$$\mathbf{\Theta}\mathbf{\Gamma}^* = \mathbf{I_2}\ , \tag{16}$$

and which can be used to implement the rotation operations referred to in the earlier sections. Other choices for the coupled rotations (e.g., of Householder type) are also possible but we shall limit ourselves here to the case of elementary rotations for illustrative purposes.

Consider two row vectors $\mathbf{x} = \begin{bmatrix} a & b \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} c & d \end{bmatrix}$, and suppose that we are interested in determining two matrices $\mathbf{\Theta}$ and $\mathbf{\Gamma}$ that are coupled as above and such that they perform the transformations

$$\begin{bmatrix} a & b \end{bmatrix}\mathbf{\Theta} = \begin{bmatrix} \alpha & 0 \end{bmatrix} \quad , \quad \begin{bmatrix} c & d \end{bmatrix}\mathbf{\Gamma} = \begin{bmatrix} \beta & 0 \end{bmatrix}. \tag{17}$$

It follows from (16) that the scalars α and β satisfy the equality $\alpha\beta^* = ac^* + bd^*$. We further assume that $ac^* + bd^* \neq 0$. This is a specialization of the full rank condition of Lemma 1.1 to the present case. Also, the strong regularity assumption throughout the paper guarantees that this condition will hold all through.

Expressions for $\mathbf{\Theta}$ and $\mathbf{\Gamma}$ that achieve (17) can be chosen as follows:

(i) If $a \neq 0$ and $c \neq 0$ then define $\gamma = b/a$, $\lambda = d/c$ and write

$$\mathbf{\Theta} = \frac{1}{1+\gamma\lambda^*}\begin{bmatrix} 1 & \gamma \\ \lambda^* & -1 \end{bmatrix}, \quad \mathbf{\Gamma} = \begin{bmatrix} 1 & \lambda \\ \gamma^* & -1 \end{bmatrix}.$$

(ii) If $a = 0$ and $c \neq 0$ then d is necessarily nonzero because of the condition $ac^* + bd^* \neq 0$. We can, therefore, choose $\mathbf{\Theta}$ and $\mathbf{\Gamma}$ to be the following so-called elimination matrices:

$$\mathbf{\Theta} = \begin{bmatrix} \alpha^* & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{\Gamma} = \begin{bmatrix} 0 & 1 \\ 1 & -\alpha \end{bmatrix}, \quad \text{where} \quad \alpha = \frac{c}{d}.$$

(iii) If $a \neq 0$ and $c = 0$ then b is necessarily nonzero, and we choose

$$\mathbf{\Theta} = \begin{bmatrix} 0 & 1 \\ 1 & -\alpha \end{bmatrix}, \quad \mathbf{\Gamma} = \begin{bmatrix} \alpha^* & 1 \\ 1 & 0 \end{bmatrix}, \quad \text{where} \quad \alpha = \frac{a}{b}.$$

6. CONCLUDING REMARKS

We extended the notion of displacement structure to the time-variant setting and used it to obtain a recursive time-update algorithm for the triangular factors of matrices with structure. We further studied a particular application in instrumental-variable methods, and showed how to avoid the back-substitution step that is often needed in solving for the weight vector. This was achieved by developing a parallelizable solution that time-updates the weight vector via array equations. Further applications to problems in adaptive filtering are considered in[4,5].

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