2004 IEEE 5th Workshop on Signal Processing Advances in Wireless Communications

Performance Analysis of a Class of Clustered Wireless Networks

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Abstract— **In this paper, we derive performance expressions for the throughput and blocking probability for a class of wireless networks with a clustering protocol. The nodes are assumed stationary and establish connections with the master node according to a priority scheme that relates to their distances from the master node. The latter is selected randomly in a cell and data are transmitted within predefined sectors.**

I. INTRODUCTION

The design of wireless networks needs to cater to several performance measures such as (i) routing delay (ii) total throughput and (iii) blocking probability. There is a fundamental tradeoff among these measures, and it is essential to understand how they are intertwined as a function of the network protocol and architecture. There are several routing methods [1]–[2] and protocols [3]–[5] to consider and many new others to propose.

It is not yet fully understood how most protocols perform with respect to the design criteria of throughput, blocking probability, and packet delay, especially when the designer is given the flexibility to alter the number of available slots, the number of clusters, and the number of nodes per cluster. For example, based on a particular pattern of requests for connection by the nodes, design questions like the following may arise and deserve examination: How should the total number of slots scale with the number of nodes in order to guarantee a certain blocking probability? and what will be the order of throughput that can be attained for a specific scheduling and routing algorithm?

In this paper, we explore these issues for a cluster-based protocol architecture in which two nodes establish a connection with a probability that is inversely proportional to the distance between them. We will derive bounds on total throughput and blocking probability. Results on bandwidth allocation and mean hop length already appear in [6], [7].

II. ARCHITECTURE, CHANNEL AND PROTOCOL MODELS

The space is divided into $M+1$ virtual geographical cells, each containing N nodes with one additional node acting as a master node. The total number of nodes in the network is denoted by N_t, i.e., $N_t = (M+1) \times (N+1)$. A frequency slot is allocated to each node that wishes to communicate with the master node in a cell. We allow for frequency reuse across cells in a manner similar to that in mobile cellular systems. The master nodes perform important tasks like routing and congestion control for high data rate communications, as well as handling some data processing and network information for nodes connected to them. Being a master node is power consuming and hence the nodes are made to take turns as master nodes with equal probability, but with the constraint that there can be only one master node in a cell at any time. The nodes communicating in the same frequency slot in other cells cause interference with the chosen cell and this interference is measured in terms of the signal-to-interference ratio (SIR) defined as follows. The SIR for a node i at time k on an uplink channel is defined by

$$\gamma_i(k) = \frac{G_{ii}(k)p_i(k)}{\sum\limits_{j \in \mathcal{A}} G_{ij}(k)p_j(k) \ + \ \sigma^2} \tag{1}$$

where, for each time instant k, G_{ij} denotes the channel gain from the $j-$th node to the intended master node of the $i-$th node, p_i is the transmission power from the $i-$th node, and σ^2 is white Gaussian noise power at the receiver of the master node that node i is connected to. Moreover, $\mathcal{A}$ denotes the set of all nodes that are interfering with node i from all cells - see Fig. 1. We assume that the transmission power of each node at every instant satisfies $P_{\min} < p_i(k) < P_{\max}$. We use the model from [8] for the channel gain from the $i-$th node to its master node. In this model, G_{ii} has a lognormal distribution, i.e.,

$$G_{ii} = S_0 d_{ii}^{-\beta} 10^{\alpha/10} \tag{2}$$

where S_0 is a function of the carrier frequency, β is the path loss exponent (PLE), and d_{ii} is the distance of the master node from the node i. The value of β depends on the physical environment and varies between 2 and 6 (usually 4), while α is a zero mean Gaussian random variable with variance σ_α^2, which usually ranges between 6 and 12.

III. ROUTING PROTOCOL

A packet is transmitted from a source to its final destination through intermediary master nodes. We assume that each node has a buffer of sufficient size to store routed packets. The operation of the nodes in any cell follows a periodic cycle. Each cycle starts with a set-up phase when a node is chosen as a master node. The set-up phase is followed by a transmission phase during which all nodes in a cell that want to communicate with the master node send their packets through available frequency slots. Figure 2 gives a schematic of how the time is split into cycles made of set-up and transmission phases. In the set-up phase, ev-

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This work was partially supported by NSF grants CCR-0208573 and ECS-0401188.

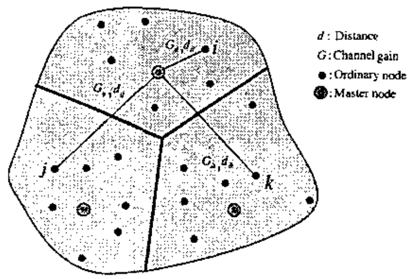

Fig. 1. *A schematic representation with three cells, three master nodes, and active and interfering nodes. The active node is node i and the interfering nodes are nodes j and k.*

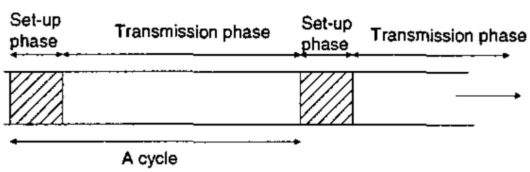

Fig. 2. *A time line showing a cell operation. One master node is chosen during the set-up phase and packets are transmitted during the transmission phase to the master node.*

ery node in a cell expresses its desire to be the master node with a probability that is equal to all nodes. When there is contention, all the nodes that express a desire to be the master node for that cell broadcast the number of cycles that have elapsed since they have been a master node. The node that has the least amount of interference is chosen as the master node. Once a particular node is chosen as a master node, it lets all other nodes know through a broadcast in that cell that it is the master node for the current cycle. Each node then gauges its distance from the master node to determine the probability with which it could establish a connection. This is done as follows.

Let $d_{\min}$ denote the smallest distance between the master node and its closest neighbor. As soon as the master node is chosen, the master node broadcasts its own coordinates and its variable $d_{\min}$ to all other nodes. Once this is done, every node i calculates the distance d_{ii} that separates it from the master node. Then each node i will try to connect to the master node with probability $(d_{\min}/d_{ii})^{1/\delta}$, where δ is a user defined design parameter that is between 0 and 1. Note that the smaller the value of δ, the higher the number of nodes that would express a desire to connect with the master node. At the end of this set-up phase, it is decided based on the number of available frequency slots, say Q, which nodes connect with the master node during the transmission phase. If the number of nodes that express a desire to connect with the master node is more than Q, then the master chooses Q nodes of those with highest probability of connection during the transmission phase.

In the transmission phase, the following routing decisions occur. If the intended final destination node for a packet is in a different cell, then the information is routed through the master node and a centralized base station through specific frequency slots that are separate from those that are available inside each cell. If the final destination node is within the same cell, then the following routing multi-hop algorithm is adopted. All possible routes out of a master node will only lie inside a sector of angle θ in the direction of the final destination, with the current relay node as the vertex. In a particular hop (when a packet is waiting at a relay node to be routed), if the next chosen master node is in any of the possible routes to the final destination, then the relay node routes the packets to the new master node. Otherwise, the packets wait in the buffer. If the packets move from the relay node to the new master node, then the following occurs. In the next cycle, when the node ceases its functions as master node, it waits till another master node is available in any of the possible routes from the current location to the final destination. Hence, a successful hop in the direction of the final destination may take several cycles. We assume that the nodes in a cell are uniformly distributed inside a circle of radius r. Only nodes that use the same frequency slot as a particular node i in other cells cause interference with node i. We assume that these interfering nodes are located within radius R from the master node i is connected to. We will also assume that each geographical cell is surrounded by M other cells within an area A_c and that these cells can cause interference. The following rules of connection apply in every geographical cell in the network during the transmission phase:

1. No two nodes are at the same distance from a master node in a cell. This assumption can be accomodated by assuming slight perturbations in the location of the nodes.
2. The probability that a node connects to a master node is inversely proportional to its distance from the master node as explained before. In other words, a node closer to the master node has a higher priority of connection than a node farther away.

Before proceeding, we introduce the following definitions.

Definition: **(Ordered Chain)** An ordered chain is a set of real numbers where its $i-$th element is less than its $j-$th element if $i > j$ (i.e., it is a set of decreasing real numbers).

Definition: **(Prioritized Cell)** A prioritized cell is a cell in which the set of all probabilities that a source-master node pair becomes active forms an ordered chain.

Definition: **(α-Prioritized Cell)** An α-prioritized cell is a prioritized cell in which the $(k+1)-$th element in the set of probabilities is α times the $k-$th element for some positive $\alpha < 1$.

Note that every cell in the model that we have described is a prioritized cell. We can also assume from the two rules of connection described before that there exists an $0 < \alpha < 1$ such that every cell in our system is an α-prioritized cell. Note that α can be arbitrarily close to 1, in which case a high density of nodes that express a desire for connection can be allowed. In the sequel, we let d_o denote the smallest distance between any two nodes in the entire network.

IV. Main Results

Consider now a cell having N nodes, with one node serving as a master node during any transmission phase. With the above described architecture, channel, and protocol models for the wireless network, we set out to derive some useful performance measures. Consider the transmission of packets in a particular frequency slot from a node i to a master node in the same cell. We shall assume that the SIR has reached a steady-state value, γ, and that it is the same for all frequency slots and for all master nodes. Then the achievable rate R_b (bits/sample) in this slot is given by Shannon's formula:

$$R_b = \frac{1}{2}\log(1+\gamma)$$

It follows that the total achievable rate in a cell over all frequency slots Q is

$$R_c = Q \times R_b$$

and the total achievable rate over the network with $M+1$ cells is

$$R_n = (M+1) \times R_c$$

Let $\bar{d}$ be the average distance travelled by a packet in a frequency slot within a cell (clearly, $\bar{d}$ is bounded by the diameter of the cell, $2r$). Then the total achievable rate in a cell, measured in bit-meter/sample is given by

$$R_d = R_c \times \bar{d}$$

We also define the blocking probability in a cell as $\mathrm{Prob}(\bar{Z} > Q)$, where $\bar{Z}$ is the average number of nodes that express a desire to connect with a master node. We now summarize the results that are derived in the paper.

Result 1 [Slots and Blocking Probability] For an α-prioritized cell with $N+1$ nodes, the number of frequency slots sufficient to ensure a maximum blocking probability of $1/\nu$ is

$$Q = \min\left\{N, \sqrt{2N\ln\nu} + \frac{1-\alpha^{N+1}}{1-\alpha}\right\} \qquad (3)$$

where $\nu \geq 1$. $\diamond$

The above result is illustrated in Figure 3 for $\alpha = 0.9$. It can be seen that, the higher the number of frequency slots, the smaller the blocking probability.

Result 2 [Connection Requests] For an α-prioritized cell with N nodes, and for N arbitrarily large, there exists an N_o dependent on α such that for all $N > N_o$, the number

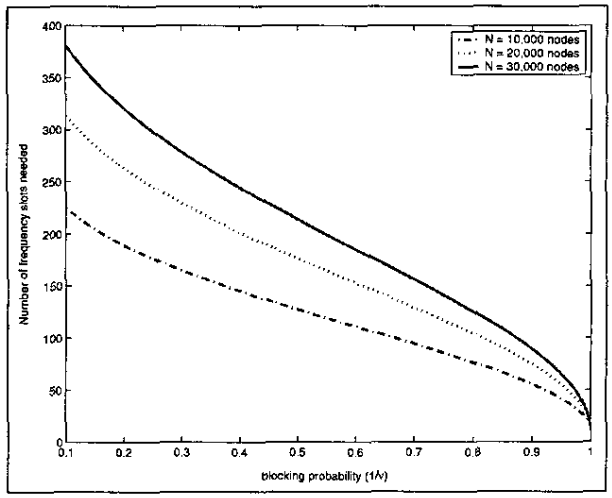

Fig. 3. *The plot shows how the number of frequency slots and blocking probability scale with each other.*

of nodes Z_N that express a desire to connect to the master node in a cell will be almost surely bounded by

$$Z_N \leq \min\left\{N, N\sqrt{2\ln(1/\alpha)} + \frac{1}{1-\alpha}\right\} \qquad (4)$$

$\diamond$

For example, let $\alpha = 0.95$ for a cell. Then there exists an N_o such that if the cell has $N > N_o$ nodes, then the number of nodes that express a desire for connection is almost surely bounded by $0.3203N + 20$ (i.e., 32 percent of N).

Result 3 [Throughput] Consider a network of N_t nodes split into many α-prioritized cells. Then, for a cell with Q slots,

$$R_d \leq R_b \times Q \times \left\{\frac{P_{\max} S_0'/\gamma}{\frac{S_0'(R^{1-\beta} - r^{1-\beta})}{(R-r)(1-\beta)} M P_{\min} + \sigma^2}\right\}^{1/\beta} \qquad (5)$$

where $S_0' = e^{\frac{(\sigma_\alpha \ln 10/10)^2}{2}} S_0$. $\diamond$

We now establish the above results.

V. Analysis

A. Available Slots and Blocking Probability

Let us order the nodes in a cell from $i = 1$ to $i = N$. Let B_i denote the event that node i expresses a desire to connect with the master node. Without loss of generality, we can assume that the $\{B_i\}$ are independent. Let $\mathcal{B}_N$ denote the sigma algebra formed by the events $\{B_1, ...B_N\}$. Let

$$Z_N = \sum_{i=1}^{N} I(B_i)$$

where $I(.)$ is the indicator function; it is equal to one when event B_i occurs and zero otherwise. The variable Z_N denotes the total number of nodes that express a desire to connect with the master node; its value is a function of N. In this section, we derive an almost sure bound on Z_N as $N \to \infty$. We also give an expression for the number of frequency slots that are required in a cell to achieve a blocking probability of utmost $1/\nu$. Let

$$Z_k = \sum_{i=1}^{i=k} I(B_i)$$

denote the number of nodes that express a desire to connect with the master node if there were k nodes in the cell apart from the master node. We first recall the following lemma.

Lemma 1 (Azuma's Inequality [9]) Suppose $\{Y_0 = 0, Y_1, Y_2, Y_3, ...\}$ is a martingale sequence such that for each k, $|Y_k - Y_{k-1}| \leq c_k$, where c_k may depend on k. Then, for all $k \geq 1$ and for any $\mu > 0$,

$$P(Y_k \geq \mu) \leq \exp\left\{ -\frac{\mu^2}{2\sum_{j=1}^{k} c_j^2} \right\}$$

$\diamond$

Motivated by the discussion in [10, p. 396], we now introduce a martingale sequence Y_k and use the above lemma to obtain a bound on Z_N. Let

$$Y_k = Z_k - \sum_{i=1}^{k} P(B_i)$$

with $Y_0 = 0$. It can be easily seen that Y_k is a martingale. This is because

$$\begin{aligned} E[Y_{k+1}|Y_k] &= E[Y_{k+1}|\mathcal{B}_k] \\ &= E\left[\left(\sum_{i=1}^{k+1} I(B_i) - \sum_{i=1}^{k+1} P(B_i)\right) |\mathcal{B}_k\right] \\ &= Y_k \end{aligned}$$

Moreover, it can be seen that $|Y_k - Y_{k-1}| \leq 1$. Now applying Azuma's inequality with $\mu = \sqrt{2k \ln \nu}$, we get

$$P\left\{Y_k \geq \sqrt{2k \ln \nu}\right\} \leq \frac{1}{\nu}, \quad k \geq 1 \tag{6}$$

or, equivalently,

$$P\left\{ \sum_{i=1}^{k} I(B_i) - \sum_{i=1}^{k} P(B_i) \geq \sqrt{2k \ln \nu} \right\} \leq \frac{1}{\nu}, \quad k \geq 1 \tag{7}$$

Noting that the cell is an α-prioritized cell, and using the fact that the $\{B_i\}$ are independent, we have

$$\sum_{i=1}^{k} P(B_i) = \frac{1 - \alpha^{k+1}}{1 - \alpha} \tag{8}$$

Substituting (8) into (7), we get

$$P\left\{ Z_k \geq \sqrt{2k \ln \nu} + \frac{1 - \alpha^{k+1}}{1 - \alpha} \right\} \leq \frac{1}{\nu}$$

Hence, considering the fact that there are N nodes in a cell apart from the master node,

$$P\left\{ Z_N \geq \sqrt{2N \ln \nu} + \frac{1 - \alpha^{N+1}}{1 - \alpha} \right\} \leq \frac{1}{\nu}$$

This expression establishes Result 1. It indicates that $\min\{N, \sqrt{2N \ln \nu} + \frac{1-\alpha^{N+1}}{1-\alpha}\}$ frequency slots are sufficient to ensure a blocking probability of utmost $1/\nu$.

B. Connection Requests

A related question of interest is how Z_N scales as $N \to \infty$. Consider again Azuma's inequality but choose now $\mu = k\sqrt{2\ln(1/\alpha)}$. Then

$$P\left(Y_k \geq k\sqrt{2\ln(1/\alpha)}\right) \leq \alpha^k, \quad k \geq 1 \tag{9}$$

Summing over k, we get

$$\sum_{k=0}^{\infty} P\left\{Y_k > k\sqrt{-2\ln \alpha}\right\} \leq \sum_{k=0}^{\infty} \alpha^k < \frac{1}{1 - \alpha} < \infty \tag{10}$$

Now from the Borel Cantelli Lemma [10, p.228], we conclude that the event $\{Y_k \geq k\sqrt{-2\ln \alpha}\}$ cannot occur infinitely often. Then, for k sufficiently large, say $k \geq N_o$, we have $Y_k \leq k\sqrt{2\ln(1/\alpha)}$ a.s. Hence, for k sufficiently large,

$$Z_k \leq k\sqrt{2\ln(1/\alpha)} + \frac{1}{1 - \alpha} \text{ a.s} \tag{11}$$

When the number of nodes in a cell is N, we get

$$Z_N \leq N\sqrt{2\ln(1/\alpha)} + \frac{1}{1 - \alpha} \text{ a.s} \tag{12}$$

which establishes Result 2.

C. Throughput

Now that we have a bound on the number of nodes that desire connection, we can examine how the number of slots, blocking probability, and the rate in a cell scale with respect to each other. We assume a uniform distribution for d_{ij}, the distance from the j-th interfering node to the master node of i, in the range of r to R, and assume that nodes farther than R do not interfere. Then

$$f_{d_{ij}}(d) = 1/(R - r), \qquad r \leq d \leq R$$

where $f_{d_{ij}}(d)$ is the probability density function of d_{ij}. Now using $E(G_{ij}|d_{ij}) = S_0' d_{ij}^{-\beta}$ we get

$$\begin{aligned} E(G_{ij}) &= \int_r^R E(G_{ij}|d_{ij} = r) f_{d_{ij}}(d) dr \\ &= \frac{S_0'(R^{1-\beta} - r^{1-\beta})}{(R - r)(1 - \beta)} \end{aligned}$$

Now let γ be the SIR level at the master node of a cell. From (1), we have

$$d_{ii}^{\beta}\left\{\sum_{j\in A}G_{ij}(k)p_j(k)\ +\ \sigma^2\right\} = p_i(k)S_0 10^{\alpha/10}/\gamma \tag{13}$$

Taking expectations of both sides of the above equation, we get

$$\begin{aligned} E(d_{ii}^{\beta})E\left\{\sum_{j\in A}G_{ij}(k)p_j(k)\ +\ \sigma^2\right\} \\ = E\left\{p_i(k)S_0 10^{\alpha/10}/\gamma\right\} \end{aligned} \tag{14}$$

which gives

$$E(d_{ii}^{\beta}) \le \frac{P_{\max}S_0'/\gamma}{\frac{S_0'(R^{1-\beta}-r^{1-\beta})}{(R-r)(1-\beta)}MP_{\min}+\sigma^2} \tag{15}$$

Using Jensen's inequality [10, p. 159], we get

$$\bar{d} \triangleq E(d_{ii}) \le \left\{\frac{P_{\max}S_0'/\gamma}{\frac{S_0'(R^{1-\beta}-r^{1-\beta})}{(R-r)(1-\beta)}MP_{\min}+\sigma^2}\right\}^{1/\beta} \tag{16}$$

In other words, this result shows that in order to enforce an SIR level of γ, the distance of node i to its master node must satisfy on average the bound in (16). Moreover, for a cell with Q slots,

$$R_d \le R_b \times Q \times \left\{\frac{P_{\max}S_0'/\gamma}{\frac{S_0'(R^{1-\beta}-r^{1-\beta})}{(R-r)(1-\beta)}MP_{\min}+\sigma^2}\right\}^{1/\beta} \tag{17}$$

which establishes Result 3. If, as suggested by Result 1, we choose $Q = (\sqrt{2N\ln\nu}+\frac{1-\alpha^N}{1-\alpha})$, then a blocking probability of $1/\nu$ is attained and the rate is bounded by

$$\begin{aligned} R_d \le R_b \times &\left\{\sqrt{2N\ln\nu}+\frac{1-\alpha^N}{1-\alpha}\right\} \\ &\times\left\{\frac{P_{\max}S_0'/\gamma}{\frac{S_0'(R^{1-\beta}-r^{1-\beta})}{(R-r)(1-\beta)}MP_{\min}+\sigma^2}\right\}^{1/\beta} \end{aligned} \tag{18}$$

VI. Simulations

In order to illustrate the performance of the proposed protocol, we simulate a cell with N nodes independently and uniformly distributed in an area. Figure 4 shows the number of slots and the resulting blocking probability for $N = 100$ nodes for both theory (using equation (3)) and practice.

VII. Conclusions

In this paper, we analyzed a class of wireless networks where nodes are allowed to become active for connection with a certain probability. We derived performance measure expressions in terms of blocking probability and total achievable rate for such a class of networks.

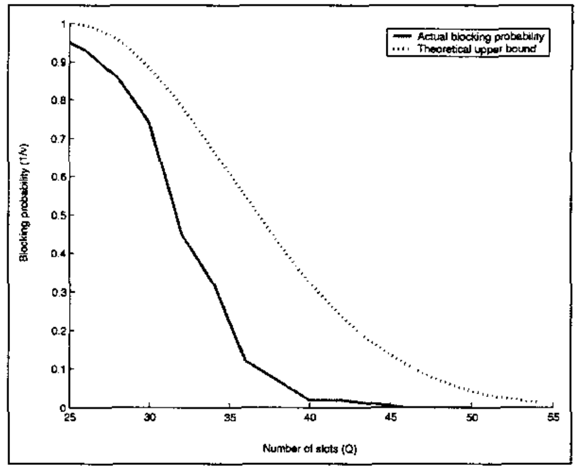

Fig. 4. *Number of slots and blocking probability for $N = 100$.*

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