DISPLACEMENT STRUCTURE AND RATIONAL INTERPOLATION THEORY*

Ali H. Sayed† and Thomas Kailath
Information Systems Laboratory, Stanford University, CA 94305, USA.

ABSTRACT

We derive square-root based algorithms for structured matrices and discuss potential applications to interpolation problems. The mathematical machinery used here is based on a standard Gaussian elimination technique and on simple results from matrix and linear system theory.

Key Words: Structured matrices, triangular factorization, interpolation problems, J-lossless systems, transmission lines.

1. INTRODUCTION

The displacement structure concept provides a powerful and unifying tool for exploiting the inherent structure in diverse problems in signal processing and mathematics. In this paper, we describe an efficient recursive (square-root or array) algorithm for factoring the structured matrices that arise in these applications, and emphasize how naturally transmission-line structures arise in this context. We stress the fact that transmission-line cascades have useful physical characteristics such as causality, energy conservation, and blocking properties. The first two have been exploited previously. The blocking property, viz., that signals propagating through the cascade at certain frequencies get annihilated, is here exploited to derive efficient recursive solutions to general rational interpolation problems, which arise in many applications in circuit and system theory.

In our opinion, our major contribution is an elementary, self-contained, and computationally oriented development of the subject. This is not at all to detract from all the other rich mathematical insights provided by the many earlier contributions to this deep subject. The main feature of our derivation is that it requires very little mathematical background, being based largely on combining a simple Gaussian elimination step with displacement structure.

We shall use the notation $\mathcal{H}_A^k(z)$ to refer to the block-Toeplitz upper-triangular matrix whose top row is given by $\begin{bmatrix} A(z) & \frac{1}{1!}A^{(1)}(z) & \frac{1}{2!}A^{(2)}(z) & \ldots & \frac{1}{(k-1)!}A^{(k-1)}(z) \end{bmatrix}$, where $A(z)$ is a rational matrix function analytic at $z, k \geq 1$ is a positive integer, and $A^{(i)}(z)$ denotes the i^{th} derivative at z. We also denote by $e_i = \begin{bmatrix} \mathbf{0}_{1\times i} & 1 & \mathbf{0} \end{bmatrix}$ the i^{th} basis vector of the $n-$dimensional space of complex numbers $\mathbf{C}^{1\times n}$. The symbol $*$ stands for Hermitian conjugation (complex conjugation for scalars).

---

*This work was supported in part by the Air Force Office of Scientific Research, Air Force Systems Command under Contract AFOSR91-0060 and by the Army Research Office under contract DAAL03-89-K-0109.

†The work of this author was also supported by a fellowship from Fundação de Amparo à Pesquisa do Estado de São Paulo while on leave from the Department of Electrical Engineering, Escola Politécnica da Universidade de São Paulo, Brazil. He is currently with the Department of Electrical and Computer Engineering, University of California, Santa Barbara, CA 93106, USA.

2. AN ARRAY ALGORITHM FOR STRUCTURED MATRICES

We start with a positive-definite Hermitian matrix R that has low displacement rank, say r, with respect to the displacement operation $R - FRF^*$. That is, R satisfies an equation of the form $R - FRF^* = GJG^*$, for some $n \times r$ so-called generator matrix G and a signature matrix $J = (I_p \oplus -I_q)$. The diagonal entries of the (stable) lower triangular matrix F will be denoted by $\{f_i\}_{i=0}^{n-1}$ ($|f_i| < 1$). The positive-definiteness of R guarantees the existence of a unique (lower triangular) Cholesky factor $\bar{L}$ such that $R = \bar{L}\bar{L}^*$, and we shall denote the *nonzero* parts of the columns of $\bar{L}$ by $\{\bar{l}_i\}_{i=0}^{n-1}$. It readily follows from the displacement equation that we can write

$$\begin{bmatrix} \bar{L} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \bar{L} & \mathbf{0} \end{bmatrix}^* = \begin{bmatrix} F\bar{L} & G \end{bmatrix} \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & J \end{bmatrix} \begin{bmatrix} F\bar{L} & G \end{bmatrix}^* .$$

This implies there exists an $(I \oplus J)-$unitary matrix $\mathbf{\Gamma}$ such that $\begin{bmatrix} \bar{L} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} F\bar{L} & G \end{bmatrix} \mathbf{\Gamma}$. The transformation $\mathbf{\Gamma}$ can be achieved in many different ways, in particular through a sequence of elementary transformations, say $\mathbf{\Gamma}_0, \mathbf{\Gamma}_1, \mathbf{\Gamma}_2, ..$, that produce the block zero in the postarray by introducing *one zero row at a time*: the first transformation $\mathbf{\Gamma}_0$ annihilates the first row of G and produces a new matrix G_1,

$$\begin{bmatrix} F\bar{L} & G \end{bmatrix} \mathbf{\Gamma}_0 = \begin{bmatrix} \bar{l}_0 & \begin{matrix} \mathbf{0} \\ F_1\bar{L}_1 \end{matrix} & \begin{matrix} \mathbf{0} \\ G_1 \end{matrix} \end{bmatrix} ,$$

where F_1 and $\bar{L}_1$ are the submatrices obtained after deleting the first row and column of F and $\bar{L}$, respectively. The second transformation $\mathbf{\Gamma}_1$ annihilates the first row of G_1 and produces a new matrix G_2,

$$\begin{bmatrix} F_1\bar{L}_1 & G_1 \end{bmatrix} \mathbf{\Gamma}_1 = \begin{bmatrix} \bar{l}_1 & \begin{matrix} \mathbf{0} \\ F_2\bar{L}_2 \end{matrix} & \begin{matrix} \mathbf{0} \\ G_2 \end{matrix} \end{bmatrix} ,$$

and so on. Following this line of thought, it is rather straightforward to prove the following result. [We shall, for brevity, omit several details here and in subsequent sections. The interested reader may consult [4, 5] and the monographs [1]-[3], and the references therein, for more details and for a discussion of prior work in the literature].

Theorem 2.1 *The successive Schur complements of R are also structured, viz., they satisfy displacement equations of the form $R_i - F_iR_iF_i^* = G_iJG_i^*$, where the generator matrices G_i satisfy the recursive construction: start with $F_0 = F, G_0 = G$, and repeat*

$$\begin{bmatrix} \mathbf{0} \\ G_{i+1} \end{bmatrix} = G_i\Theta_i \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix} + \Phi_i G_i \Theta_i \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} , \; \Phi_i = (I_{n-i} - f_i^* F_i)^{-1}(F_i - f_i I_{n-i}) \; , \quad (1)$$

where F_i is the submatrix obtained after deleting the first row and column of F_{i-1}, and Θ_i is an arbitrary $J-$unitary matrix that reduces the first row of G_i (denoted by g_i) to the form $g_i\Theta_i = \begin{bmatrix} \delta_i & \mathbf{0} \end{bmatrix}$. The columns of $\bar{L}$ are given by $\bar{l}_i = \sqrt{1-|f_i|^2}\,(I_{n-i} - f_i^*F_i)^{-1}G_i\Theta_i e_0^{\mathbf{T}}$.

Expression (1) shows that G_{i+1} is obtained as follows: choose a convenient rotation Θ_i and apply it to G_i, keep the last columns of $G_i\Theta_i$ unchanged, and multiply the first column of $G_i\Theta_i$ by the "Blaschke" matrix Φ_i. Moreover, let d_i denote the $(0,0)$ entry of R_i ($d_i = |\delta_i|^2/(1-|f_i|^2)$). It then follows that the expressions for l_i ($= \bar{l}_i d_i^{1/2}$) and G_i can be grouped together into the following revealing form:

$$\begin{bmatrix} l_i & \begin{matrix} \mathbf{0} \\ G_{i+1} \end{matrix} \end{bmatrix} = \begin{bmatrix} F_il_i & G_i \end{bmatrix} \begin{bmatrix} f_i^* & \frac{\delta_i}{d_i}\begin{bmatrix} 1 & \mathbf{0} \end{bmatrix} \\ \Theta_i \begin{bmatrix} \delta_i \\ \mathbf{0} \end{bmatrix} & \Theta_i \begin{bmatrix} -f_i & \mathbf{0} \\ \mathbf{0} & I_{r-1} \end{bmatrix} \end{bmatrix} .$$

This shows that each step of the generator recursion involves a first-order state-space system that appears on the right-hand-side of the above expression. Let $\Theta_i(z)$ denote its $r \times r$ transfer matrix (with inputs from the left). It follows, upon simplification, that $\Theta_i(z) = \Theta_i(B_i(z) \oplus I_{r-1})$, where $B_i(z) = (z - f_i)/(1 - zf_i^*)$. Each such section is clearly $J-$lossless since $\Theta_i(z)$ is analytic in $|z| < 1$ due to $|f_i| < 1$, and $\Theta_i(z)J\Theta_i^*(z) = J$ on $|z| = 1$ because $B_i(z)$ is a Blaschke factor and Θ_i is $J-$unitary. Furthermore, each $\Theta_i(z)$ also has an important "blocking" property that will be very relevant to the solution of interpolation problems. It is evident that $g_i\Theta_i(f_i) = \mathbf{0}$ since $g_i\Theta_i = \begin{bmatrix} \delta_i & \mathbf{0} \end{bmatrix}$ and $B_i(f_i) = 0$.

3. RATIONAL ANALYTIC INTERPOLATION PROBLEMS

It follows from the above discussion that the array algorithm allows us to construct first order sections $\Theta_i(z)$ that have local zero-directions determined by the first rows of the successive generators G_i, viz., $g_0\Theta_0(f_0) = \mathbf{0}$, $g_1\Theta_1(f_1) = \mathbf{0}$, etc. But what about the entire $J-$lossless cascade $\mathbf{\Theta}(z) = \Theta_0(z)\Theta_1(z)\ldots\Theta_{n-1}(z)$? The point is that proper choices of $\{F, G\}$ will force global blocking properties on $\mathbf{\Theta}(z)$, thus solving interpolation problems.

To show this, and in order to convey the main ideas, let us consider the special case of the well-known tangential Nevanlinna-Pick (NP) problem: given n complex points $\{f_i\}_{i=0}^{n-1}$ inside the open unit disc ($|z| < 1$) and n pairs of row vectors (u_i, v_i), it is required to describe all $p \times q$ rational Schur-type matrix functions $S(z)$ (*i.e.,* $S(z)$ is analytic and strictly bounded by unity in $|z| < 1$) that satisfy the interpolation conditions $u_iS(f_i) = v_i$. The first step in the recursive solution consists in constructing three matrices F, G, and J directly from the interpolation data: F = diagonal $\{f_0, \ldots, f_{n-1}\}$, the i^{th} row of G is $\begin{bmatrix} u_i & v_i \end{bmatrix}$, and $J = (I_p \oplus -I_q)$. These matrices *implicitly* define a displacement equation of the form $R - FRF^* = GJG^*$, where the matrix R is not explicitly known. In fact, our solution does not require knowledge of R. It only exploits the fact that a structured matrix R is implicitly defined by the available interpolation data. It can be also shown that the NP problem is solvable iff R is positive-definite. We shall, for brevity, omit the relevant details here and instead concentrate on showing how the array algorithm, when applied to F and G, leads to a solution of the NP problem. Indeed, we shall show soon that, because of the local blocking properties, we obtain that the rows of G are zero-directions for the entire cascade, viz., $\begin{bmatrix} u_i & v_i \end{bmatrix}\mathbf{\Theta}(f_i) = \mathbf{0}$. Hence, if we partition $\mathbf{\Theta}(z)$ accordingly with J we get $u_i\mathbf{\Theta}_{12}(f_i) + v_i\mathbf{\Theta}_{22}(f_i) = \mathbf{0}$, which implies that the Schur function $S = -\mathbf{\Theta}_{12}\mathbf{\Theta}_{22}^{-1}$ satisfies $u_iS(f_i) = v_i$ and is thus a solution to the NP problem. An obvious explanation for this is the following: when the rows of G are fed through the cascade, they produce the zero directions $\{g_i\}$ at the inputs of the corresponding sections $\{\Theta_i(z)\}$ at the appropriate 'frequencies' $\{f_i\}$, thus annihilating the entire cascade. More specifically, first note that $\begin{bmatrix} u_0 & v_0 \end{bmatrix}\Theta_0(f_0) = \mathbf{0}$. This annihilates the first section and consequently the entire cascade at f_0. Now using (1), and the fact that Φ_0 = diagonal $\{0, \frac{f_1 - f_0}{1 - f_1 f_0^*}, \frac{f_2 - f_0}{1 - f_2 f_0^*}, \ldots, \}$, we conclude that the first row of G_1 is given by $g_1 = \begin{bmatrix} u_1 & v_1 \end{bmatrix}\Theta_0(\alpha_1)$. That is, g_1 is the result of passing $\begin{bmatrix} u_1 & v_1 \end{bmatrix}$ through $\Theta_0(f_1)$. But $g_1\Theta_1(f_1) = \mathbf{0}$, so that $\begin{bmatrix} u_1 & v_1 \end{bmatrix}\mathbf{\Theta}(f_1) = \mathbf{0}$. In general we get $\begin{bmatrix} u_i & v_i \end{bmatrix}\mathbf{\Theta}(f_i) = \mathbf{0}$. Note that the matrix R does not enter the calculations.

The array algorithm however allows us to solve more general interpolation problems of the Hermite-Fejér (HF) type as stated ahead. This is obtained by defining appropriate matrices F with general Jordan structures rather than in diagonal form as in the NP case.

So consider m points $\{\alpha_i\}_{i=0}^{m-1}$ inside the open unit disc and associate with each point α_i a positive integer $r_i \geq 1$ and two row vectors $\mathbf{a}_i$ and $\mathbf{b}_i$ partitioned as follows: $\mathbf{a}_i = \begin{bmatrix} u_1^{(i)} & \ldots & u_{r_i}^{(i)} \end{bmatrix}$, $\mathbf{b}_i = \begin{bmatrix} v_1^{(i)} & \ldots & v_{r_i}^{(i)} \end{bmatrix}$, where $u_j^{(i)}$ and $v_j^{(i)}$ are $1 \times p$ and $1 \times q$ row vectors, respectively. If an interpolating point α_i is repeated (say, $\alpha_i = \alpha_{i+1} = \ldots =$

α_{i+j}), we shall then assume that $\{u_1^{(i)}, \ldots, u_1^{(i+j)}\}$ are linearly independent. This rules out degenerate cases [5]. The tangential Hermite-Fejér problem is then to determine all rational $p \times q$ Schur-type functions $S(z)$ that satisfy $\mathbf{b}_i = \mathbf{a}_i \mathcal{H}_S^{r_i}(\alpha_i)$. As in the NP case, we construct three matrices F, G, and J: we associate with each α_i a Jordan block $\bar{F}_i$ of size $r_i \times r_i$, with $\alpha_i's$ on the main diagonal and $1's$ on the first subdiagonal, and two matrices $U_i = \begin{bmatrix} u_1^{(i)\mathbf{T}} & \ldots & u_{r_i}^{(i)\mathbf{T}} \end{bmatrix}^{\mathbf{T}}$, $V_i = \begin{bmatrix} v_1^{(i)\mathbf{T}} & \ldots & v_{r_i}^{(i)\mathbf{T}} \end{bmatrix}^{\mathbf{T}}$. Then

$$F = \text{ diagonal } \{\bar{F}_0, \bar{F}_1, \ldots, \bar{F}_{m-1}\}\ , \quad G = \begin{bmatrix} U_0^{\mathbf{T}} & \ldots & U_{m-1}^{\mathbf{T}} \\ V_0^{\mathbf{T}} & \ldots & V_{m-1}^{\mathbf{T}} \end{bmatrix}^{\mathbf{T}}, \quad J = (I_p \oplus -I_q).$$

We further associate with the Hermite-Fejér problem the displacement equation $R - FRF^* = GJG^*$. It can also be shown that a solution exists iff R is positive-definite. Moreover, and following a similar argument as in the NP case, we can verify that the rows of G are also zero directions for the cascade $\mathbf{\Theta}(z)$, which now has zeros with multiplicities at the $\alpha_i's$. More precisely, if we again partition the cascade $\mathbf{\Theta}(z)$ accordingly with J, then the Schur function $S = -\mathbf{\Theta}_{12}\mathbf{\Theta}_{22}^{-1}$ turns out to satisfy the HF conditions $\mathbf{b}_i = \mathbf{a}_i \mathcal{H}_S^{r_i}(\alpha_i)$. Moreover, it can be further shown that all solutions are parametrized in terms of a linear fractional transformation based on $\mathbf{\Theta}(z)$ and on an arbitrary Schur matrix function K: $S(z) = -\left[\mathbf{\Theta}_{11}(z)K(z) + \mathbf{\Theta}_{12}(z)\right]\left[\mathbf{\Theta}_{21}(z)K(z) + \mathbf{\Theta}_{22}(z)\right]^{-1}$.

4. CONCLUDING REMARKS.

To conclude this brief exposition we stress the fact that the derivation given above is exclusively carried out in the matrix domain, working with matrix quantities only, and not in the function domain. This feature allows us to extend the matrix-based derivation rather smoothly to the time-variant setting, and to solve general time-variant interpolation problems.

We also add that a non Hermitian version of the array algorithm is possible and leads to two cascade structures with intrinsic interpolation properties [5], which can be advantageously used in the solution of several types of unconstrained interpolation problems such as Padé approximation, Lagrange interpolation, and others.

ACKNOLWEDGMENT

These results are based on joint works with Profs. T. Constantinescu and H. Lev-Ari, and with T. Boros.

REFERENCES

[1] H. Dym (1989). *J-Contractive Matrix Functions, Reproducing Kernel Hilbert Spaces and Interpolation*, vol. 71. CBMS, AMS, Providence, RI.

[2] J. A. Ball, I. Gohberg, and L. Rodman (1990). *Interpolation of Rational Matrix Functions*, vol. 45 of *Operator Theory: Advances and Applications.* Birkhäuser, Basel.

[3] C. Foias and A. E. Frazho (1990). *The Commutant Lifting Approach to Interpolation Problems*, vol. 44 of *Operator Theory: Advances and Applications*, Birkhäuser, Basel.

[4] A. H. Sayed, T. Constantinescu, and T. Kailath (1994). Square-root algorithms for structured matrices, interpolation, and completion problems. To appear in *IMA volumes in Mathematics and Its Applications*, Springer-Verlag.

[5] A. H. Sayed (1992). *Displacement Structure in Signal Processing and Mathematics.* PhD thesis, Stanford University, Stanford, CA.