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FAST ALGORITHMS FOR GENERALIZED DISPLACEMENT STRUCTURES*

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ABSTRACT

We introduce a generalized displacement structure that includes a variety of previously studied cases, and leads to a unified approach to fast $O(n^2)$ algorithms for the triangular factorization of structured matrices such as Toeplitz-like and Hankel-like structures. The algorithm is derived through a recursive state-space procedure and can be used to solve a variety of other problems. For instance, we show that the state-space derivation allows us to extend previous results on the cascade decomposition of lossless and J-lossless rational matrices to a larger class of rational matrices. Also new recursive solutions to certain interpolation and $H^{\infty}-$control problems are obtained.

Key Words: Structured matrices, triangular factorization, state-space recursion, J-lossless systems, interpolation, $H^{\infty}-$control.

1. INTRODUCTION

In this paper we introduce a generalized displacement structure of the form

$$\Omega R \Delta^* - FRA^* = GJB^* \tag{1}$$

where the symbol * denotes complex conjugation, Ω, Δ, F and A are $n \times n$ lower triangular matrices, G and B are $n \times r$ $(r \leq n)$ generator matrices and J is an $r \times r$ matrix satisfying $J^2 = I$. The matrix R is said to have displacement rank r and the matrices Ω, Δ, F and A are called displacement operators. There are examples where the appropriate use of all four matrices $(\Omega,\ \Delta,\ F,\ A)$ reduces the displacement rank. We shall obtain the triangular factorization of R by embedding (1) into two discrete-time systems that satisfy a generalized notion of J-losslessness, and then performing the cascade decomposition of the corresponding transfer matrices. This approach is a generalization of a recursive (state-space) embedding technique introduced recently by Lev-Ari and Kailath [1]. The derived recursions can be applied to solve a variety of problems [2, 3, 4, 5] *e.g.* fast O(n^2) algorithms for the triangular and orthogonal factorization of strongly regular structured matrices

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and their inverses, cascade decomposition of systems satisfying a generalized notion of J-losslessness and recursive state-space solution to interpolation and H^∞-control problems. The results can be also extended to deal with the case of structured matrices with arbitrary rank profile. In this paper we shall only discuss the factorization of R and the cascade decomposition of the embedded transfer matrices. Brief mention will be made of the interpolation and $H^\infty-$control problems. For brevity we shall only state the main results; the details and proofs can be found in [2].

2. A GENERALIZED STATE SPACE APPROACH

We begin by noting conditions for the existence of a unique solution R of (1). Let ω_i, δ_i, f_i and a_i $(i = 0, 1, \ldots, n-1)$ designate the diagonal elements of the $n \times n$ lower triangular matrices Ω, Δ, F and A respectively.

Lemma 1 (Uniqueness of R) *The matrix equation* $\Omega R \Delta^* - FRA^* = X$ *has a unique solution* R *for any matrix* X *if, and only if,* $(\omega_j \delta_i^* - f_j a_i^*) \neq 0$ *for all* i, j.

In the sequel we assume that uniqueness of R is guaranteed and that Ω and Δ are invertible. However the recursions are still valid when either Ω or Δ is singular, and the nonuniqueness of R need not be a problem in some cases [2, 6]. The embedding procedure is guaranteed by the following result.

Theorem 1 (Generalized Embedding) *Given* $\Omega R \Delta^* - FRA^* = GJB^*$ *and* R *invertible, we can find unique matrices* $\bar{H}_{r\times n}$, $\bar{K}_{r\times r}$, $\bar{C}_{r\times n}$ *and* $\bar{D}_{r\times r}$ *such that :*

1. *Defining* $\bar{T}(z)$ *and* $\bar{W}(z)$ *as the transfer matrices of the discrete-time systems* $\begin{bmatrix} \Omega^{-1}F & \Omega^{-1}G & \bar{H} & \bar{K} \end{bmatrix}$ *and* $\begin{bmatrix} \Delta^{-1}A & \Delta^{-1}B & \bar{C} & \bar{D} \end{bmatrix}$ *respectively,* i.e. $\bar{T}(z) = \bar{K} + \bar{H}(z\Omega - F)^{-1}G$ *and* $\bar{W}(z) = \bar{D} + \bar{C}(z\Delta - A)^{-1}B$ *. Then*

$$\begin{bmatrix} F & G \\ \bar{H} & \bar{K} \end{bmatrix} \begin{bmatrix} R & \mathbf{0} \\ \mathbf{0} & J \end{bmatrix} \begin{bmatrix} A & B \\ \bar{C} & \bar{D} \end{bmatrix}^* = \begin{bmatrix} \Omega R \Delta^* & \mathbf{0} \\ \mathbf{0} & J \end{bmatrix} \tag{2}$$

   and $\bar{T}(\tau) = \bar{W}(\mu) = I$, *where* τ *and* μ *are points chosen from the complex plane and satisfying* $\tau\mu^* = 1$. *We call* $\begin{bmatrix} F & G & \bar{H} & \bar{K} \end{bmatrix}$ *and* $\begin{bmatrix} A & B & \bar{C} & \bar{D} \end{bmatrix}$ *a discrete-time embedding of (1).*

2. *All possible discrete-time embeddings* $\begin{bmatrix} F & G & H & K \end{bmatrix}$ *and* $\begin{bmatrix} A & B & C & D \end{bmatrix}$ *satisfying a relation of the form (2) are given by* $\begin{bmatrix} F & G & U\bar{H} & U\bar{K} \end{bmatrix}$ *and* $\begin{bmatrix} A & B & V\bar{C} & V\bar{D} \end{bmatrix}$, *where* U *and* V *are constant matrices satisfying* $UJV^* = J$. *Moreover,* $T(z)JW^*(w) = J$ on $zw^* = 1$ *and*

$$T(z) = U\left\{I - (z-\tau)JB^*(\Delta^* - \tau A^*)^{-1}R^{-1}(z\Omega - F)^{-1}G\right\} \tag{3}$$

$$W(z) = V\left\{I - (z-\mu)J^*G^*(\Omega^* - \mu F^*)^{-1}R^{-*}(z\Delta - A)^{-1}B\right\} \tag{4}$$

$$T(z)JW^*(w) = J - (zw^* - 1)H(z\Omega - F)^{-1}\Omega R \Delta^*(w\Delta - A)^{-*}C^* \tag{5}$$

   where $T(z) = K + H(z\Omega - F)^{-1}G$ *and* $W(z) = D + C(z\Delta - A)^{-1}B$.

The generalized embedding theorem is valid for matrices Ω, Δ, F and A that are not necessarily lower triangular; the triangularity of the displacement operators is needed for the derivation of a recursive algorithm. Our next step is to show that the cascade decomposition of the two discrete-time systems $\begin{bmatrix} \Omega^{-1}F & \Omega^{-1}G & H & K \end{bmatrix}$ and $\begin{bmatrix} \Delta^{-1}A & \Delta^{-1}B & C & D \end{bmatrix}$ leads to the triangular factorization of R as well.

3. TRIANGULAR FACTORIZATION

A strongly regular $n \times n$ matrix (namely one with nonzero leading principal minors) R admits a triangular factorization of the form $R = LDU$, where L is a lower-triangular matrix with unit diagonal elements, U is an upper-triangular matrix with unit diagonal elements and $D =$ diagonal $\{\sigma_0, \sigma_1, \ldots, \sigma_{n-1}\}$:

$$L = \begin{bmatrix} \tilde{l}_0 & \tilde{l}_1 & \ldots & \tilde{l}_{n-1} \end{bmatrix}, \quad U = \begin{bmatrix} \tilde{u}_0 & \tilde{u}_1 & \ldots & \tilde{u}_{n-1} \end{bmatrix}^* \tag{6}$$

The columns $\tilde{l}_i$ and $\tilde{u}_i$ are of the form

$$\tilde{l}_i = \begin{bmatrix} 0 & \ldots & 0 & 1 & l_i^T \end{bmatrix}^T \quad \text{and} \quad \tilde{u}_i = \begin{bmatrix} 0 & \ldots & 0 & 1 & u_i^T \end{bmatrix}^T \tag{7}$$

The recursive triangularization procedure can be carried out as follows:

$$\tilde{R}_i = \begin{bmatrix} 1 & \mathbf{0} \\ l_i & I \end{bmatrix} \begin{bmatrix} \sigma_i & \mathbf{0} \\ \mathbf{0} & \tilde{R}_{i+1} \end{bmatrix} \begin{bmatrix} 1 & \mathbf{0} \\ u_i & I \end{bmatrix}^* \quad \text{with} \quad \tilde{R}_0 = R \tag{8}$$

$\tilde{R}_{i+1}$ is the Schur complement of $\tilde{R}_i$ with respect to σ_i. We are interested in decomposing $T(z)$ and $W(z)$ into a cascade of first order sections:

$$T(z) = T_{n-1}(z)\ldots T_1(z)T_0(z) \quad \text{and} \quad W(z) = W_{n-1}(z)\ldots W_1(z)W_0(z) \tag{9}$$

where $T_i(z)$ and $W_i(z)$ satisfy the generalized J-losslessness property $T_i(z)JW_i^*(w) = J$ on $zw^* = 1$, and are of the form $T_i(z) = k_i + h_i(z\omega_i - f_i)^{-1}g_i$ and $W_i(z) = d_i + c_i(z\delta_i - a_i)^{-1}b_i$. This decomposition generalizes the results of Potapov [7] and Genin et al. [8].

Using the generalized embedding theorem and some straightforward manipulations [2] we can show that this decomposition can be achieved through the following recursive procedure:

$$\begin{bmatrix} \Omega_i^{-1}F_i & \Omega_i^{-1}G_i \\ H_i & K_i \end{bmatrix} = T_i \begin{bmatrix} 1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Omega_{i+1}^{-1}F_{i+1} & \Omega_{i+1}^{-1}G_{i+1} \\ \mathbf{0} & H_{i+1} & K_{i+1} \end{bmatrix} \begin{bmatrix} \omega_i^{-1}f_i & \mathbf{0} & \omega_i^{-1}g_i \\ \mathbf{0} & I_{n-i-1} & \mathbf{0} \\ h_i & \mathbf{0} & k_i \end{bmatrix} T_i^{-1}$$

$$\begin{bmatrix} \Delta_i^{-1}A_i & \Delta_i^{-1}B_i \\ C_i & D_i \end{bmatrix} = Q_i \begin{bmatrix} 1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Delta_{i+1}^{-1}A_{i+1} & \Delta_{i+1}^{-1}B_{i+1} \\ \mathbf{0} & C_{i+1} & D_{i+1} \end{bmatrix} \begin{bmatrix} \delta_i^{-1}a_i & \mathbf{0} & \delta_i^{-1}b_i \\ \mathbf{0} & I_{n-i-1} & \mathbf{0} \\ c_i & \mathbf{0} & d_i \end{bmatrix} Q_i^{-1}$$

where

$$T_i = \begin{bmatrix} 1 & \mathbf{0} & \mathbf{0} \\ l_i & I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I \end{bmatrix} \quad \text{and} \quad Q_i = \begin{bmatrix} 1 & \mathbf{0} & \mathbf{0} \\ u_i & I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I \end{bmatrix} \tag{10}$$

$$\Omega_i = \begin{bmatrix} \omega_i & \mathbf{0} \\ ? & \Omega_{i+1} \end{bmatrix} \quad \text{and} \quad \Delta_i = \begin{bmatrix} \delta_i & \mathbf{0} \\ ? & \Delta_{i+1} \end{bmatrix} \tag{11}$$

The letter ? stands for irrelevant entries. More specifically we are led to the following recursive algorithm:

Recursive Algorithm: *The triangular factorization of R and the cascade decomposition of $T(z)$ and $W(z)$ can be carried out recursively as follows*:

- Begin with $F_0 = F,\ G_0 = G,\ A_0 = A,\ B_0 = B,\ \Omega_0 = \Omega$ and $\Delta_0 = \Delta$.
- For $i = 0$ to $n-1$, compute recursively:

1. Find $f_i,\ F_{i+1},\ a_i,\ A_{i+1},\ \omega_i,\ \Omega_{i+1}$ and $\delta_i,\ \Delta_{i+1}$:

$$F_i = \begin{bmatrix} f_i & \mathbf{0} \\ ? & F_{i+1} \end{bmatrix}, \quad A_i = \begin{bmatrix} a_i & \mathbf{0} \\ ? & A_{i+1} \end{bmatrix} \tag{12}$$

$$\Omega_i = \begin{bmatrix} \omega_i & \mathbf{0} \\ ? & \Omega_{i+1} \end{bmatrix}, \quad \Delta_i = \begin{bmatrix} \delta_i & \mathbf{0} \\ ? & \Delta_{i+1} \end{bmatrix} \tag{13}$$

2. Choose τ_i and μ_i such that $\tau_i \mu_i^* = 1$
3. Set

$$\Phi_i = \left(\frac{a_i^* - \mu_i^* \delta_i^*}{\omega_i - \mu_i^* f_i} \right) (f_i \Omega_i - \omega_i F_i)(\delta_i^* \Omega_i - a_i^* F_i)^{-1} \tag{14}$$

$$\Gamma_i = \left(\frac{f_i^* - \tau_i^* \omega_i^*}{\delta_i - \tau_i^* a_i} \right) (a_i \Delta_i - \delta_i A_i)(\omega_i^* \Delta_i - f_i^* A_i)^{-1} \tag{15}$$

4. Set $g_i =$ first row of G_i and $b_i =$ first row of B_i
5. Compute the triangular factors $l_i,\ u_i$ and σ_i :

$$\sigma_i = \frac{g_i J b_i^*}{\omega_i \delta_i^* - f_i a_i^*}, \quad \begin{bmatrix} 1 \\ l_i \end{bmatrix} = \frac{1}{\sigma_i} (\delta_i^* \Omega_i - a_i^* F_i)^{-1} G_i J b_i^* \tag{16}$$

$$\begin{bmatrix} 1 \\ u_i \end{bmatrix} = \frac{1}{\sigma_i^*} (\omega_i^* \Delta_i - f_i^* A_i)^{-1} B_i J^* g_i^* \tag{17}$$

6. Compute G_{i+1} and B_{i+1} :

$$\begin{bmatrix} \mathbf{0} \\ G_{i+1} \end{bmatrix} = \left[G_i + (\Phi_i - I) G_i \frac{J b_i^* g_i}{g_i J b_i^*} \right] \bar{U}_i^{-1} \tag{18}$$

$$\begin{bmatrix} \mathbf{0} \\ B_{i+1} \end{bmatrix} = \left[B_i + (\Gamma_i - I) B_i \frac{J^* g_i^* b_i}{b_i J^* g_i^*} \right] \bar{V}_i^{-1} \tag{19}$$

where $\bar{U}_i^{-1}$ and $\bar{V}_i^{-1}$ are constant matrices satisfying $\bar{U}_i^{-1} J \bar{V}_i^{-*} = J$. We can choose, for example, $\bar{U}_i = \bar{V}_i = I$.

- The state-space realizations of the first order sections $T_i(z)$ and $W_i(z)$ follow from:

$$h_i = \bar{U}_i \left(\frac{1}{\sigma_i \omega_i} \frac{\tau_i \omega_i - f_i}{(\delta_i^* - \tau_i a_i^*)} J b_i^* \right) \quad \text{and} \quad k_i = \bar{U}_i \left(I - \frac{1}{\sigma_i \omega_i} \frac{J b_i^* g_i}{(\delta_i^* - \tau_i a_i^*)} \right) \tag{20}$$

$$c_i = \bar{V}_i \left(\frac{1}{\sigma_i^* \delta_i} \frac{\mu_i \delta_i - a_i}{(\omega_i^* - \mu_i f_i^*)} J^* g_i^* \right) \quad \text{and} \quad d_i = \bar{V}_i \left(I - \frac{1}{\sigma_i^* \delta_i} \frac{J^* g_i^* b_i}{(\omega_i^* - \mu_i f_i^*)} \right) \tag{21}$$

4. THE SCHUR ALGORITHM

The recursions presented here include some important special cases where fast algorithms have been derived, *e.g.* Toeplitz-like and Hankel-like structures and structured matrices with Toeplitz displacement operators. For brevity, we only show here how to reduce the general recursions to the classical Schur algorithm. For this purpose we consider the special case $\Omega = \Delta = I$, $F = A = Z$, $G = B = \begin{bmatrix} u_0 & v_0 \end{bmatrix}$, $J =$ diagonal $\{1, -1\}$, and we assume that the solution of $R - ZRZ^* = GJG^*$ is Hermitian positive definite. Here u_0 and v_0 are $n \times 1$ column vectors and Z is the lower triangular shift matrix with ones on the first subdiagonal. We choose $\tau_i = 1$ and write $G_i = \begin{bmatrix} u_i & v_i \end{bmatrix}$ where u_i and v_i are $(n-i) \times 1$ column vectors. Let $g_i = \begin{bmatrix} u_{i0} & v_{i0} \end{bmatrix}$ be the first row of G_i. The generator recursion (18) reduces to

$$\begin{bmatrix} \mathbf{0} \\ G_{i+1} \end{bmatrix} = \left[G_i + (Z - I) G_i \frac{J g_i^* g_i}{g_i J g_i^*} \right] \bar{U}_i^{-1} \tag{22}$$

where $\bar{U}_i^{-1}$ is any J-unitary matrix. Choose

$$\bar{U}_i^{-1} = \frac{1}{\sqrt{1 - |k_i^2|}} \begin{bmatrix} 1 & -k_i \\ -k_i^* & 1 \end{bmatrix} \quad \text{where} \quad k_i = \frac{v_{i0}}{u_{i0}}$$

and observe that $\sigma_i = |u_{i0}|^2 - |v_{i0}|^2$ and hence $|k_i| < 1$ because $R > 0 \iff \sigma_i > 0 \iff |k_i| < 1$. Moreover $\bar{U}_i^{-1}$ is such that $g_i \bar{U}_i^{-1} = \sigma_i^{\frac{1}{2}} \begin{bmatrix} 1 & 0 \end{bmatrix}$. Assume that all matrices have been extended to semi-infinite dimensions and introduce $G_i(z) \stackrel{\text{def}}{=} \begin{bmatrix} 1 & z & z^2 & \ldots \end{bmatrix} G_i = \begin{bmatrix} u_i(z) & v_i(z) \end{bmatrix}$. $G_i(z)$ is called the generating function of G_i. Equation (22) can then be written in the form :

$$z G_{i+1}(z) = G_i(z) \frac{1}{\sqrt{1 - |k_i|^2}} \begin{bmatrix} 1 & -k_i \\ -k_i^* & 1 \end{bmatrix} \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix} \tag{23}$$

which is the linearized version of Schur's recursion [9, 10]. If we define $\bar{v}_i(z) = z^i v_i(z)$, $\bar{u}_i(z) = z^i u_i(z)$ and $f_i(z) = \frac{\bar{v}_i(z)}{\bar{u}_i(z)}$ then (23) reduces to the classical Schur recursion [11]:

$$f_{i+1}(z) = \frac{1}{z} \frac{f_i(z) - k_i}{1 - k_i^* f_i(z)} \quad \text{with} \quad k_i = f_i(0) \tag{24}$$

5. INTERPOLATION AND H^∞−CONTROL PROBLEMS

The classical Schur algorithm [9, 11] is an efficient recursive procedure for the triangular factorization of structured (Quasi-Toeplitz) matrices of the form $R - ZRZ^* = GJG^*$, where G is an $n \times 2$ generator matrix; this result connects the factorization of Quasi-Toeplitz matrices to the solution of the Caratheodory interpolation problem. On the other hand, we have noted that the state-space approach allows us to obtain recursive procedures for the triangular factorization of more general structures; it is natural to ask whether these procedures can be applied to the solution of interpolation problems. The answer is positive. For example, the solution of the the tangential Nevanlinna-Pick problem can be obtained by applying the recursive triangularization procedure to $R - FRF^* = GJG^*$, for suitably chosen

F and G. This leads to a cascade state-space realization of the solution. Moreover, note that our recursions are very general and can be applied to non-Hermitian structures as well. In these cases we are solving new nonsymmetric interpolation problems [3].
A second class of applications for these results includes the solution of classical $H^{\infty}-$ control problems. It is well known that many of these problems admit an interpolation formulation. We get an explicit cascade state-space realization of the solution in terms of J-lossless first-order sections [4].

6. CONCLUSION

We introduced a generalized displacement structure and derived a recursive procedure that computes the triangular factors of the matrix. We also showed that the recursions lead to a cascade factorization of the transfer matrices $T(z)$ and $W(z)$ obtained in the embedding procedure, in terms of first order sections with a generalized J-losslessness property. Extensions to interpolation problems were also noted.

REFERENCES

[1] Lev-Ari, H. and T. Kailath (1990). State-space approach to factorization of lossless transfer functions and structured matrices. Submitted to *Linear Algebra and Its Applications.*

[2] Kailath, T. and A.H. Sayed (1991). Fast algorithms for generalized displacement structures. Preprint.

[3] Sayed, A.H. and T. Kailath (1991). A recursive state-space approach to classical and to new nonsymmetric interpolation problems. Preprint.

[4] Sayed, A.H. and T. Kailath (1991). On the Nehari problem and structured matrices. Preprint.

[5] Chun, J. and T. Kailath (1991). Generalized displacement structure for block-Toeplitz, Toeplitz-block, and Toeplitz-derived matrices, *Numerical Linear Algebra, Digital Signal Processing and Parallel Algorithms,* eds. G.H. Golub and P. Van Dooren, Springer-Verlag, Berlin, 215-236.

[6] Chun, J. and T. Kailath (1990). Displacement structure for Hankel- and Vandermonde-like matrices. *Signal Processing, part I: Signal Processing Theory*, eds. L. Auslander, T. Kailath and S. Mitter, Springer-Verlag, NY, 37-58.

[7] Potapov, Y. (1960). The multiplicative structure of J-contractive matrix functions. *Amer. Math. Soc. Transl.*, ser.2, vol.15, 131-244.

[8] Genin, Y., P.V. Dooren, T. Kailath, J. Delosme and M. Morf (1983). On $\Sigma-$lossless transfer functions and related questions. *Linear Algebra and its Applications,* 50, 251-275.

[9] Lev-Ari, H. and T. Kailath (1986). Triangular factorization of structured hermitian matrices. *I.Schur Methods in Operator Theory and Signal Processing,* (ed.) I. Gohberg, Birkhauser, Boston, vol.18, 301-324.

[10] Kailath, T. (1987). Signal processing applications of some moment problems. *Moments in Mathematics,* (ed.) H. Landau, Amer. Math.Soc., Providence, vol.37, 71-109.

[11] Schur, I. (1986). On power series which are bounded in the interior of the unit circle. *I.Schur Methods in Operator Theory and Signal Processing,* (ed.) I. Gohberg, Birkhauser, Boston, vol. 18, 31-88 (English translation).