SNR PERFORMANCE OF AN ADAPTIVE SIGMA DELTA MODULATOR

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ABSTRACT

In a previous study [1], we proposed an adaptive sigma delta modulator with improved dynamic range. The modulator adapts the step size of the quantizer from estimates of the quantizer input instead of the modulator input. In this study, we conduct an error variance analysis of the new modulator and derive expression for the SNR. The derived expression shows that the SNR is independent of the input signal strength, which supports the simulation results.

1. INTRODUCTION

Adaptive Sigma Delta Modulation (ASDM) attempts to increase the dynamic range of sigma delta modulators by scaling either the input signal or the step-size of the quantizer through an estimation of the input signal strength. This estimation can be done from the input signal itself or from the modulator output, and there have been several studies in the literature on this issue (e.g., [2]–[6]).

In a previous study [1], we proposed an adaptation scheme whereby the step-size of the quantizer is adapted based on analysis of the input signal to the quantizer. This structure was shown in [1, 7] to lead to significant improvement in the dynamic range of the modulator. In this paper we investigate the SNR performance of the modulation structure by conducting a variance analysis of the modulation error. The analysis will lead to a closed-form expression for the SNR and will show that the SNR is independent of the input signal power.

Figure 1 shows the basic structure of the proposed adaptive SDM, with a one bit quantizer. The modulation and demodulation blocks are shown in parts a and b, respectively. The adaptation scheme is shown in Figure 2.

The equations describing the behavior of this modulator are:

$$
\begin{aligned}
e_a(n) &= x(n) - v(n-1) && (1)\\
p(n) &= p(n-1) + e_a(n), \quad p(0) = 0 && (2)\\
y(n) &= \text{sign}\,[p(n)] && (3)\\
q(n) &= \text{sign}\,[|p(n)| - d(n-1)] && (4)\\
d(n) &= \alpha^{q(n)} d(n-1) && (5)\\
v(n) &= y(n)d(n). && (6)
\end{aligned}
$$

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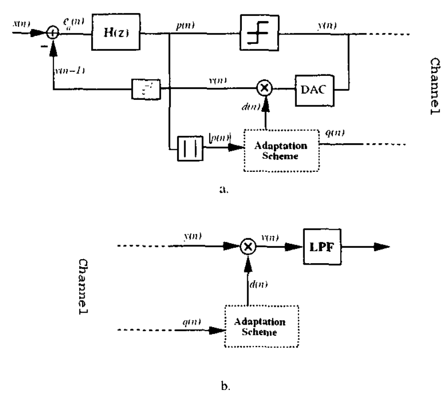

Figure 1: Block diagram of the proposed structure. a. Modulator b. Demodulator.

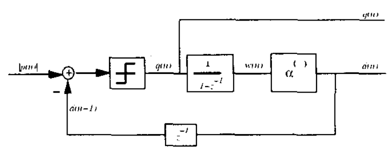

Figure 2: Adaptation scheme of the proposed modulator.

2. ANALYSIS OF THE NEW MODULATOR

The analysis of the proposed modulator is restricted to the case where H(z) is an integrator. The variance of the modulation error, $e_a(n)$, is computed, from which an expression for the SNR is derived.

2.1. Equivalent Structure of the Modulator

Taking the logarithm of both sides of equation (5) we get,

$$\log_\alpha(d(n)) = \log_\alpha(d(n-1)) + q(n). \quad (7)$$

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Using the fact that the logarithm is an increasing function, we can write

$$q(n) = \text{sign}\left[\log_\alpha(|p(n)|) - \log_\alpha(d(n-1))\right]. \quad (8)$$

Now let

$$x_d(n) \triangleq \log_\alpha(|p(n)|), \quad (9)$$

and

$$y_d(n) \triangleq \log_\alpha(d(n)). \quad (10)$$

From equations (7)-(10), we have

$$y_d(n) = y_d(n-1) + \text{sign}\left[x_d(n) - y_d(n-1)\right]. \quad (11)$$

This dynamic equation characterizes a delta modulator as illustrated in Figure 3 part a. Its linearized version is shown in part b. Therefore, we can redraw the adapter utilizing equations (9)-(11)

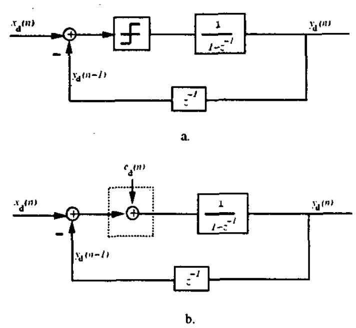

Figure 3: A Delta Modulator. a. Typical b. Linearized

as shown in Figure 4. The adaptation block together with the quantizer of the modulator now look like a log-PCM [8], except that the PCM block is replaced by a delta modulator. The delta modulator

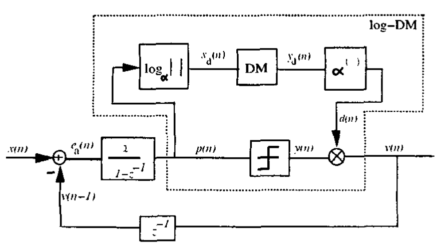

Figure 4: Equivalent form of the ASDM.

can be linearized by replacing its quantizer by an additive quantization noise $e_d(n)$, as shown in Figure 3b. The noise $e_d(n)$ is assumed to be uniformly distributed in an interval $[-\Delta, \Delta]$ (usually $\Delta = 1$ for single bit DM). Then, $y_d(n)$ is given by

$$y_d(n) = x_d(n) + e_d(n). \quad (12)$$

Using equation (10), we have

$$d(n) = \alpha^{x_d(n)+e_d(n)}. \quad (13)$$

Substituting the expression for $x_d(n)$ from equation (9), we get

$$d(n) = \alpha^{\log_\alpha(|p(n)|)+e_d(n)} = |p(n)|\alpha^{e_d(n)}. \quad (14)$$

Substituting back into (6), we have

$$v(n) = p(n)\alpha^{e_d(n)}. \quad (15)$$

Finally, if we denote

$$K(n) \triangleq \alpha^{e_d(n)}, \quad (16)$$

then we obtain

$$\boxed{v(n) = p(n)K(n)} \quad (17)$$

This result shows that we can model the adapter and quantizer in the main loop of Figure 1a as a time varying gain $K(n)$. Figure 5 shows the resulting equivalent structure of our ASDM. Since the distribution of the random error signal $e_d(n)$ is known, the distribution of the variable gain $K(n)$ can be defined. Our further analysis is based on the basic assumptions:

1. All random processes are stationary.
2. The variable gain $K(n)$ is independent of everything else.

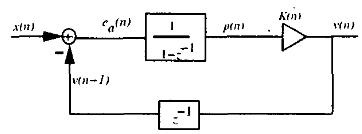

Figure 5: The ASDM as a linear time variant (LTV) system.

2.2. Variance Analysis

In this section we evaluate the variance of the error signal, $e_a(n)$, which can be shown to be zero mean. From the equivalent structure shown in Figure 4, we can write

$$p^2(n) = \left(1 - K(n-1)\right)^2 p^2(n-1) + x^2(n) +$$
$$+2\left(1 - K(n-1)\right)p(n-1)x(n).$$

Based on the independence and stationarity assumptions, the second moment of $p(n)$ is

$$E\{p^2(n)\} = E\left\{\left(1 - K(n-1)\right)^2\right\}E\{p^2(n-1)\Big\} +$$
$$+E\{x^2(n)\} + 2E\left\{\left(1 - K(n-1)\right)\right\}E\{p(n-1)x(n)\Big\}.$$

Therefore,

$$E_{p^2} = (1 - 2E_K + E_{K^2})E_{p^2} + E_{x^2} + 2(1 - E_K)E_{px} \quad (18)$$

where the notation E_{p^2} and E_{px} refers to $E\{p^2(n)\}$ and $E\{p(n-1)x(n)\}$. The term E_{px} can be shown to be equal to (the proof is omitted for brevity):

$$E_{px} = \frac{1}{E_K}E_{x^2} \quad (19)$$

where $E_K = E\{K(n)\}$ and $E_{x^2} = E\{x^2(n)\}$. Substituting back into equation (18) and collecting terms, we get

$$(2E_K - E_{K^2})E_{p^2} = \frac{2 - E_K}{E_K} E_{x^2}. \qquad (20)$$

Finally, we have

$$E_{p^2} = \Psi E_{x^2}, \qquad (21)$$

with

$$\Psi = \frac{2 - E_K}{E_K(2E_K - E_{K^2})}. \qquad (22)$$

Also, from the relation between $v(n)$ and $p(n)$, we get

$$E_{v^2} = E_{K^2} E_{p^2} = \Psi E_{K^2} E_{x^2} \qquad (23)$$

To obtain an expression for the error variance $\sigma^2_{e_a}$, we make use of the fact that (the proof is also omitted for brevity)

$$\sigma^2_{e_a} = \sigma^2_v - \sigma^2_x. \qquad (24)$$

Since the means of $v(n)$ and $x(n)$ are equal (recall that the mean of $e_a(n)$ is zero), we get

$$\sigma^2_{e_a} = E_{v^2} - E_{x^2} \qquad (25)$$

so that

$$\sigma^2_{e_a} = (\Psi E_{K^2} - 1)E_{x^2}. \qquad (26)$$

If the input is zero mean, then

$$\sigma^2_{e_a} = (\Psi E_{K^2} - 1)\sigma^2_x. \qquad (27)$$

The first and second moments of $K(n)$ can be shown to be

$$\begin{aligned} E_K &\triangleq E\{K\} = \frac{\zeta}{2\Delta}\left(e^{\Delta/\zeta} - e^{-\Delta/\zeta}\right) \\ E_{K^2} &\triangleq E\{K^2\} = \frac{\zeta}{4\Delta}\left(e^{2\Delta/\zeta} - e^{-2\Delta/\zeta}\right), \end{aligned}$$

where $\zeta = \log_\alpha e$.

2.3. SNR Computation

At the receiver side, the signal $v(n)$ is filtered using a low-pass filter with cut-off frequency f_c, which is chosen to be equal to the input signal band frequency f_B. The modulation error is computed by comparing the filtered signal to the input signal $x(n)$, as shown in Figure 6a.

Since $f_c = f_B$, we can introduce an identical filter at the summer end connected to the input $x(n)$ in Figure 6a assuming ideal filtering. Using linearity, the two filters are moved after the summer resulting in the equivalent form shown in Figure 6b. This form is useful in computing the SNR performance of the modulator as follows.

The SNR is defined as the ratio between the input signal variance and the filtered error variance $e_f(n)$, shown in Figure 6, i.e.,

$$SNR = \frac{\sigma^2_x}{\sigma^2_{e_f}}. \qquad (28)$$

The variance of the filtered error $e_f(n)$ is computed by integrating the spectrum of the error $e_a(n)$ over the input signal

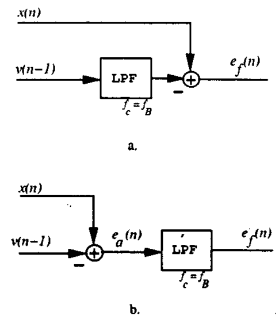

Figure 6: Modulation error. Two equivalent forms.

band. To do so, we need to obtain the autocorrelation sequence of $e_a(n)$ as follows. Since $e_a(n) = x(n) - v(n-1)$ and $v(n) = K(n)p(n)$, then the dynamics of the error $e_a(n)$ is given by

$$e_a(n) = x(n) - K(n-1)p(n-1). \qquad (29)$$

¿From equation (2), we can write

$$p(n) = \sum_{i=0}^{n} e_a(i). \qquad (30)$$

Substituting back into equation (29), we get

$$e_a(n) = x(n) - K(n-1) \sum_{i=0}^{i=n-1} e_a(i). \qquad (31)$$

Multiplying both sides by $e_a(m)$, $m < n$, yields

$$e_a(n)e_a(m) = x(n)e_a(m) - K(n-1) \sum_{i=0}^{i=n-1} e_a(i)e_a(m). \qquad (32)$$

The expected value of the term $e_a(n)e_a(m)$ is

$$\begin{aligned} E\{e_a(n)e_a(m)\} = E\{x(n)e_a(m)\} - E\{K(n-1)\} \\ E\left\{\sum_{i=0}^{i=n-1} e_a(i)e_a(m)\right\}. \end{aligned}$$

Since the input signal $x(n)$ is independent of the previous errors $e_a(m), m < n$, then

$$E\{x(n)e_a(m)\} = 0. \qquad (33)$$

Thus, equation (32) becomes

$$E\{e_a(n)e_a(m)\} = -E\{K(n-1)\} \sum_{i=0}^{i=n-1} E\{e_a(i)e_a(m)\}. \qquad (34)$$

Since the process is stationary, the autocorrelation sequence of the error $e_a(n)$, denoted by $r_e(\cdot)$, is

$$r_e(n-m) = -E_K \sum_{i=0}^{i=n-1} r_e(i-m). \qquad (35)$$

In matrix form, we can write

$$A\Gamma = -\mathbf{1}\sigma_{e_a}^2 \quad (36)$$

where

$$A = \begin{pmatrix} 1+\beta & 1 & 1 & .. & 1 \\ 2 & 1+\beta & 1 & .. & 1 \\ 2 & 2 & 1+\beta & .. & 1 \\ . & . & . & .. & . \\ . & . & . & .. & . \\ 2 & 2 & 2 & .. & 1+\beta \end{pmatrix}; \quad (M \times M) \quad (37)$$

$$\mathbf{1} = [1 \;\; 1 \;\; 1 \;\; \ldots \;\; 1]^T; \quad (M \times 1) \quad (38)$$

$$\beta = \frac{1}{E_K};$$

and Γ is the vector containing the autocorrelation sequence r_e. Solving for Γ, we get

$$\Gamma = -A^{-1}\mathbf{1}\sigma_{e_a}^2. \quad (39)$$

By definition, the spectrum of the error $e_a(n)$ is

$$S_e(w) = \sum_{m=-\infty}^{\infty} r_e(m)e^{-jw}. \quad (40)$$

The sequence r_e is real and symmetric. Therefore

$$S_e(w) = 2\sum_{m=1}^{\infty} r_e(m)\cos(wm) + r_e(0). \quad (41)$$

Finally, the variance of the filtered error $e_f(n)$ is obtained by integrating the spectrum of $e_a(n)$ over the input frequency band w_B, i.e.,

$$\sigma_{e_f}^2 = \frac{1}{2\pi}\int_{-w_B}^{w_B} S_e(w)dw. \quad (42)$$

The following result can now be proven (the proof is omitted for brevity).

Lemma *Based on the assumption that*

$$r_e(m) = 0 \quad for \quad m > M,$$

then

$$\sigma_{e_f}^2 = 2f_B\left(1 - 2S_c^T A^{-1}\mathbf{1}\right)\sigma_{e_a}^2. \quad (43)$$

where

$$S_c^T = \left[\frac{\sin(w_B)}{w_B}\;\frac{\sin(2w_B)}{2w_B}\;\ldots\;\frac{\sin(Mw_B)}{Mw_B}\right]$$

and $w_B = 2\pi f_B$.

◇

If we substitute equation (27) into (43), we get

$$\sigma_{e_f}^2 = 2f_B\left(1 - 2S_c^T A^{-1}\mathbf{1}\right)(\Psi E_{K^2} - 1)\sigma_x^2 \quad (44)$$

so that

$$\boxed{SNR = \frac{R}{(1-2S_c^T A^{-1}\mathbf{1})(\Psi E_{K^2}-1)}} \quad (45)$$

where R is the oversampling ratio (OSR). The theoretical SNR is clearly independent of the input variance. Figure 7 shows a comparison between the theoretical and simulated SNR. The figure shows a close relation between them for input amplitudes as far down as -90dB.

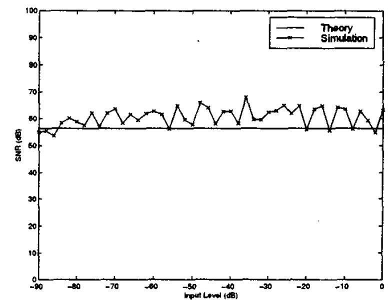

Figure 7: Comparison between the theoretical and simulated SNR.

3. CONCLUSION

The performance of the adaptive sigma delta modulator of [1] has been analyzed. The modulator is shown to be equivalent to a linear time-variant system with randomly varying gain. An expression for the SNR has been derived, and the result is shown to be independent of the input strength.

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