RLS-LAGUERRE LATTICE ADAPTIVE FILTERING: ERROR FEEDBACK AND ARRAY-BASED ALGORITHMS

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ABSTRACT

This paper develops lattice structures for RLS Laguerre adaptive filtering including error-feedback and array-based lattice versions. All structures are efficient in that their computational cost is proportional to the number of taps. Although these structures are theoretically equivalent, their performance can differ under practical considerations, such as finite-precision effects and regularization. Simulations are included to illustrate this point.

1. INTRODUCTION

In the work [1], the authors addressed the problem of deriving lattice (i.e., order recursive) RLS filters for Laguerre networks. One of the benefits of working with a Laguerre network is that fewer parameters can be used to model long impulse responses in a stable manner (see, e.g., [3, 4]). In such networks, however, successive regression vectors are not shifted versions of each other. Still, the authors showed in [1] that a more general form of data structure exists and that it can be exploited to derive a fast order-recursive filter. Figure 1 illustrates the Laguerre network, where

$$L_0(z) = \frac{\sqrt{1-a^2}}{1-az^{-1}} \text{ and } L(z) = \frac{z^{-1}-a}{1-az^{-1}}, \quad 0 < |a| < 1. \quad (1)$$

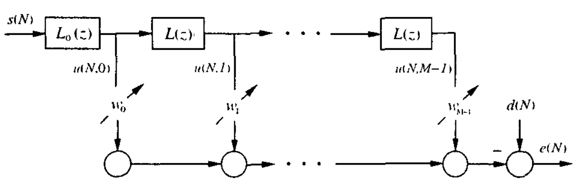

Figure 1: *A transversal Laguerre structure for adaptive filtering.*

Now, the RLS lattice algorithm that was derived in [1] appears in a form that is based on a-posteriori errors and it does not involve an error feedback mechanism. Although it is a common lattice form, several other equivalent forms can be derived. While all these implementations are theoretically equivalent, they tend to differ in performance under different operating conditions (such as in finite-precision implementations and regularization – see, e.g., [5, 6]). Such variants generally present good numerical properties and are better suited for fixed point implementations (some forms are better suited than others) and for very large scale integration. These facts motivate us to derive in this paper the related lattice variants for Laguerre structures.

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2. THE A-PRIORI ERROR FEEDBACK FILTER

Consider a data matrix $H_{M+2,N}$ (whose rows are $(M+2)$-dimensional) and partition it as

$$\begin{aligned} H_{M+2,N} &= \begin{bmatrix} H_{M+1,N} & x_{M+1,N} \end{bmatrix} \\ &= \begin{bmatrix} x_{0,N} & \bar{H}_{M,N} & x_{M+1,N} \end{bmatrix} \end{aligned}$$

The last column of $H_{M+2,N}$ is denoted by $x_{M+1,N}$ and its first column by $x_{0,N}$. Note that while in the case of regressors with shift structure, there exists a simple relation between $\bar{H}_{M,N}$ and $H_{M,N}$, this relation is less obvious in the Laguerre case (see [1]). Let further c_N denote the column vector

$$c_N \triangleq \sqrt{1-a^2} \begin{bmatrix} a^N & a^{N-1} & \cdots & a & 1 \end{bmatrix}^T$$

which is defined in terms of the pole a of each section $L(z)$, and let $y_N \triangleq \begin{bmatrix} d(0) & d(1) & \cdots & d(N) \end{bmatrix}^T$. Table 1 lists an order-recursive algorithm for updating the a-posteriori errors, and which arises from the problems of projecting (in a regularized manner) the vectors $\{x_{0,N}, x_{M+1,N}\}$ onto $\bar{H}_{M,N}$ and $H_{M+1,N}$ and similarly, the vectors $\{y_N, c_N\}$ onto $H_{M,N}$ (see [1]). These projection problems involve the following error vectors:

$$\begin{aligned} b_{M+1,N} &= x_{M+1,N} - H_{M+1,N} w^b_{M+1,N} \\ \bar{b}_{M,N} &= x_{M+1,N} - \bar{H}_{M,N} w^{\bar{b}}_{M,N} \\ f_{M+1,N} &= x_{0,N} - \bar{H}_{M,N} w^f_{M,N} \\ e_{M,N} &= y_N - H_{M,N} w_{M,N} \\ \bar{c}_{M,N} &= c_N - H_{M,N} w^c_{M,N} \end{aligned}$$

where, for example, $w^f_{M,N}$ is the solution to

$$\min_{w_M} [\, \mu\lambda^{N+1} \|w_M\|^2 + \|x_{0,N} - \bar{H}_{M,N} w_M\|^2) \,] \quad (2)$$

This problem projects the first column $x_{0,N}$ onto $\bar{H}_{M,N}$. Similarly we define $\{w^b_{M,N}, w^c_{M,N}, w_{M,N}\}$. The last entries of the above error vectors are denoted by

$$\{\beta_{M+1}(N), \bar{\beta}_{M+1}(N), \alpha_{M+1}(N), \epsilon_{M+1}(N), \bar{c}_{M+1}(N)\}.$$

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Initialization

For $M = 0$ to $M - 1$ set:

μ is a small positive number.

$$\delta_M(-1) = \rho_M(-1) = \tau_M(-1) = 0$$
$$\zeta^f_M(-1) = \zeta^b_M(-1) = \zeta^{\bar{b}}_M(-1) = \mu$$
$$\zeta^c_M(-1) = 1$$

For $N \geq 0$, repeat:

$$u(N) = au(N-1) + \sqrt{1-a^2}s(N)$$

$$\gamma_0(N) = 1 \qquad e_0(N) = d(N)$$
$$\bar{\gamma}_0(N) = 1 \qquad f_0(N) = u(N)$$
$$\tilde{c}_0(N) = \sqrt{1-a^2} \qquad b_0(N) = u(N)$$

For $M = 0$ to $M - 1$, repeat:

$$\tau_M(N) = a\tau_M(N-1) + \frac{\tilde{c}^*_M(N)b_M(N)}{\gamma_M(N)}$$
$$\zeta^c_M(N) = a^2\zeta^c_M(N-1) + \frac{|\tilde{c}_M(N)|^2}{\gamma_M(N)}$$
$$\kappa^{\bar{b}}_M(N) = \frac{\tau_M(N)}{\zeta^c_M(N)}$$
$$\bar{b}_M(N) = -\frac{1}{a}\left(b_M(N) - \kappa^{\bar{b}}_M(N)\tilde{c}_M(N)\right)$$

$$\zeta^f_M(N) = \zeta^f_M(N-1) + \frac{|f_M(N)|^2}{\bar{\gamma}_M(N)}$$
$$\zeta^b_M(N) = \zeta^b_M(N-1) + \frac{|b_M(N)|^2}{\gamma_M(N)}$$
$$\zeta^{\bar{b}}_M(N) = \zeta^{\bar{b}}_M(N-1) + \frac{|\bar{b}_M(N)|^2}{\bar{\gamma}_M(N)}$$

$$\delta_M(N) = \delta_M(N-1) + \frac{f^*_M(N)\bar{b}_M(N)}{\bar{\gamma}_M(N)}$$
$$\rho_M(N) = \rho_M(N-1) + \frac{e^*_M(N)b_M(N)}{\gamma_M(N)}$$

$$\gamma_{M+1}(N) = \gamma_M(N) - \frac{|b_M(N)|^2}{\zeta^b_M(N)}$$
$$\bar{\gamma}_{M+1}(N) = \bar{\gamma}_M(N) - \frac{|\bar{b}_M(N)|^2}{\zeta^{\bar{b}}_M(N)}$$

$$\kappa^c_M(N) = \frac{\tau^*_M(N)}{\zeta^b_M(N)} \qquad \kappa_M(N) = \frac{\rho^*_M(N)}{\zeta^b_M(N)}$$
$$\kappa^b_M(N) = \frac{\delta_M(N)}{\zeta^f_M(N)} \qquad \kappa^f_M(N) = \frac{\delta^*_M(N)}{\zeta^{\bar{b}}_M(N)}$$

$$\tilde{c}_{M+1}(N) = \tilde{c}_M(N) - \kappa^c_M(N)b_M(N)$$
$$e_{M+1}(N) = e_M(N) - \kappa_M(N)b_M(N)$$
$$b_{M+1}(N) = \bar{b}_M(N) - \kappa^b_M(N)f_M(N)$$
$$f_{M+1}(N) = f_M(N) - \kappa^f_M(N)\bar{b}_M(N)$$

Alternative order-updates:

$$\zeta^f_{M+1}(N) = \zeta^f_M(N) - \frac{|\delta_M(N)|^2}{\zeta^{\bar{b}}_M(N)}$$
$$\zeta^b_{M+1}(N) = \zeta^{\bar{b}}_M(N) - \frac{|\delta_M(N)|^2}{\zeta^f_M(N)}$$
$$\zeta^c_{M+1}(N) = \zeta^c_M(N) - \frac{|\tau_M(N)|^2}{\zeta^b_M(N)}$$
$$\zeta^{\bar{b}}_M(N) = \zeta^b_M(N) - \frac{|\tau_M(N)|^2}{\zeta^c_M(N)}$$
$$\zeta_{M+1}(N) = \zeta_M(N) - \frac{|\rho_M(N)|^2}{\zeta^b_M(N)}$$
$$\gamma_{M+1}(N) = \bar{\gamma}_M(N) - \frac{|f_M(N)|^2}{\zeta^f_M(N)}$$

Table 1: *The a posteriori Laguerre lattice filter.*

2.1. A-Priori Estimation Errors

The a-priori, as opposed to a-posteriori, estimation errors can be defined similarly. Introduce

$$\begin{aligned}
\beta_{M+1,N} &= x_{M+1,N} - H_{M+1,N}w^b_{M+1,N-1} && (3)\\
\bar{\beta}_{M,N} &= x_{M+1,N} - \bar{H}_{M,N}w^b_{M,N-1} && (4)\\
\alpha_{M+1,N} &= x_{0,N} - \bar{H}_{M,N}w^f_{M,N-1} &&\\
\epsilon_{M,N} &= y_N - H_{M,N}w_{M,N-1} &&
\end{aligned}$$

where $w^f_{M,N-1}$, for example, is the solution to a problem similar to (2) with all $N's$ replaced by $N-1$. Comparing the expressions for the a-priori error vectors with the expressions above for the a-posteriori error vectors, the only differences lie in the use of the prior weight vector estimates. Similarly, the last entries of the a-priori error vectors are denoted by

$$\{\beta_{M+1}(N), \bar{\beta}_M(N), \alpha_{M+1}(N), \epsilon_M(N)\}$$

By following the same argument as in Sec. III of [1], it can be verified that these errors satisfy similar order-update relations, which are listed in Table 2. The expression for $\nu_{M+1}(N)$ also has an additional factor a.

2.2. Time-Updates for the Reflection Coefficients

Unlike conventional derivations of error-feedback lattice algorithms for tapped-delay line structures, we shall obtain time-update relations for the reflection coefficients in a more direct way, by exploiting the fact that these coefficients can be regarded as solutions to least-squares problem of first order [2]. For instance, $\kappa_M(N)$ in Table 1 is defined as

$$\kappa_M(N) = (\mu + b'^*_{M,N}b'_{M,N})^{-1}b'^*_{M,N}e'_{M,N} \qquad (5)$$

where the quantities $\{b'_{M,N}, e'_{M,N}\}$ are vectors of angle normalized prediction errors. For instance,

$$b'_{M,N} \triangleq \left[\begin{array}{cccc} \frac{b_M(0)}{\sqrt{\gamma_M(0)}} & \frac{b_M(1)}{\sqrt{\gamma_M(1)}} & \cdots & \frac{b_M(N)}{\sqrt{\gamma_M(N)}} \end{array}\right]^T$$

where $\gamma_M(N)$ is a conversion factor. Equation (5) can be interpreted as the regularized least-squares solution of a first-order least-squares problem, namely that of projecting (in a regularized manner) the vector $e'_{M,N}$ onto the vector $b'_{M,N}$. This simple observation shows that $\kappa_M(N)$ can be time-updated via a standard RLS recursion of the form:

$$\kappa_M(N) \quad = \quad \kappa_M(N-1) + \frac{\beta^*_M(N)\gamma_M(N)}{\zeta^b_M(N)}\epsilon_{M+1}(N)$$

In a similar fashion, we can justify the time-update for the remaining reflection coefficients, by simply using their corresponding RLS recursions. Figure 2 illustrates the resulting RLS-Laguerre lattice structure that is based on a priori errors. The corresponding recursions are listed in Table 2.

3. THE ARRAY-BASED LATTICE ALGORITHM

We can also derive another equivalent lattice form, albeit one that is described in terms of compact arrays. This form involves only

Initialization

For $M = 0$ to $M-1$ set:

μ is a small positive number.

$$\kappa_M^b(-1) = \kappa_M^{\bar{b}}(-1) = \kappa_M^f(-1) = \kappa_M^c(-1) = \kappa_M(-1) = 0$$
$$\zeta_M^f(-1) = \zeta_M^b(-1) = \zeta_M^{\bar{b}}(-1) = \mu$$
$$\zeta_M^c(-1) = 1$$

For $N \geq 0$, repeat:

$$u(N) = au(N-1) + \sqrt{1-a^2}\,s(N)$$

$$\gamma_0(N) = 1 \qquad \epsilon_0(N) = d(N)$$
$$\bar{\gamma}_0(N) = 1 \qquad \alpha_0(N) = u(N)$$
$$\nu_0(N) = \sqrt{1-a^2} \qquad \beta_0(N) = u(N)$$

For $M = 0$ to $M-1$, repeat:

$$\zeta_M^c(N) = a^2\zeta_M^c(N-1) + |\nu_M(N)|^2\gamma_M(N)$$
$$\bar{\beta}_M(N) = -a\beta_M(N) + \kappa_M^{\bar{b}}(N-1)\nu_M(N)$$
$$\kappa_M^{\bar{b}}(N) = \frac{1}{a}\left(\kappa_M^{\bar{b}}(N-1) - \frac{\nu_M(N)\gamma_M(N)}{\zeta_M^c(N)}\bar{\beta}_M(N)\right)$$

$$\zeta_M^{\bar{b}}(N) = \zeta_M^{\bar{b}}(N-1) + |\bar{\beta}_M(N)|^2\bar{\gamma}_M(N)$$
$$\zeta_M^f(N) = \zeta_M^f(N-1) + |\alpha_M(N)|^2\bar{\gamma}_M(N)$$
$$\zeta_M^b(N) = \zeta_M^b(N-1) + |\beta_M(N)|^2\gamma_M(N)$$

$$\nu_{M+1}(N) = \nu_M(N) - a\kappa_M^c(N-1)\beta_M(N)$$
$$\epsilon_{M+1}(N) = \epsilon_M(N) - \kappa_M(N-1)\beta_M(N)$$
$$\beta_{M+1}(N) = \bar{\beta}_M(N) - \kappa_M^b(N-1)\alpha_M(N)$$
$$\alpha_{M+1}(N) = \alpha_M(N) - \kappa_M^f(N-1)\bar{\beta}_M(N)$$

$$\kappa_M^c(N) = a\kappa_M^c(N-1) + \frac{\beta_M^*(N)\gamma_M(N)}{\zeta_M^b(N)}\nu_{M+1}(N)$$
$$\kappa_M^f(N) = \kappa_M^f(N-1) + \frac{\bar{\beta}_M^*(N)\bar{\gamma}_M(N)}{\zeta_M^{\bar{b}}(N)}\alpha_{M+1}(N)$$
$$\kappa_M^b(N) = \kappa_M^b(N-1) + \frac{\beta_M^*(N)\gamma_M(N)}{\zeta_M^f(N)}\beta_{M+1}(N)$$
$$\kappa_M(N) = \kappa_M(N-1) + \frac{\beta_M^*(N)\gamma_M(N)}{\zeta_M^b(N)}\epsilon_{M+1}(N)$$

$$\gamma_{M+1}(N) = \gamma_M(N) - \frac{|b_M(N)|^2}{\zeta_M^b(N)}$$
$$\bar{\gamma}_{M+1}(N) = \bar{\gamma}_M(N) - \frac{|\bar{b}_M(N)|^2}{\zeta_M^{\bar{b}}(N)}$$

Table 2: *Error-feedback Laguerre lattice filter based on a priori errors.*

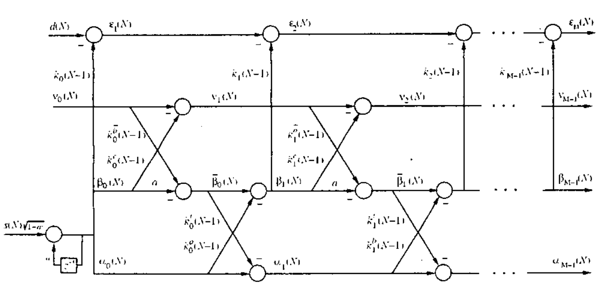

Figure 2: *Laguerre lattice network based on a-priori errors with error feedback.*

orthogonal rotations and tends to exhibit good numerical properties. For this purpose, we first need to define the following quantities:

$$q_M^b(N) \triangleq \delta_M(N)/\zeta_M^{\bar{b}/2}(N), \quad q_M^f(N) \triangleq \delta_M^*(N)/\zeta_M^{f/2}(N)$$
$$q_M^{\bar{b}}(N) \triangleq \tau_M(N)/\zeta_M^{c/2}(N), \quad q_M^c(N) \triangleq \tau_M^*(N)/\zeta_M^{\bar{b}/2}(N).$$

The second step is to rewrite all the recursions in Table 2 in terms of these quantities, and in terms of the angle normalized prediction errors $\{b'_M(N), e'_M(N), \bar{c}'_M(N), \bar{b}'_M(N)\}$ defined before, e.g.,

$$\tau_M(N) = a\tau_M(N-1) + \bar{c}'^*_M(N)b'_M(N) \qquad (6)$$
$$\zeta_M^c(N) = a^2\zeta_M^c(N-1) + |\bar{c}'_M(N)|^2 \qquad (7)$$
$$\zeta_M^b(N) = \zeta_M^b(N-1) + |b'_M(N)|^2 \qquad (8)$$
$$\zeta_M^{\bar{b}}(N) = \zeta_M^b(N) - |q_M^{\bar{b}}(N)|^2 \qquad (9)$$

The third step is to implement a unitary transformation matrix Θ that lower triangularizes the following pre-array of numbers:

$$\begin{bmatrix} a\zeta_M^{c/2}(N-1) & \bar{c}'^*_M(N) \\ q_M^{\bar{b}*}(N-1) & b'^*_M(N) \end{bmatrix}\Theta = \begin{bmatrix} m & 0 \\ n & p \end{bmatrix}$$

for some $\{m, n, p\}$. Squaring both sides of the above equation, we get, using equations (7)-(8),

$$m = \zeta_M^{c/2}(N), \quad n = q_M^{\bar{b}*}(N), \quad p = \bar{b}'^*_M(N)\ .$$

Proceeding similary we can derive three additional array transformations, all of which are listed in Table 4. The resulting algorithm is also represented schematically in Fig. 3. The matrices $\{\Theta_M^{\bar{b}}(N), \Theta_M^c(N), \Theta_M^f(N), \Theta_M^b(N)\}$ are 2×2 unitary transformations that introduce the zero entries in the post-arrays at the desired locations.

4. SIMULATIONS

Table 3 sumarizes the computational cost of these algorithms for a least-squares problem of order M. We see that some forms are more costly in terms of multiplications while others are more costly in terms of divisions.

Algorithm	Mult.	Div.	Add.	$\sqrt{\cdot}$
Standard a posteriori	$17M+2$	$14M$	$14M+1$	-
A priori err. Feedback.	$24M+2$	$7M$	$16M+1$	-
Array lattice	$48M$	$8M$	$24M$	$4M$

Table 3: *Comparison of the computational cost of the different Laguerre lattice algorithms for filters of order M.*

Although all the Laguerre lattice variants studied here are theoretically equivalent, they differ in computational cost and perhaps more importantly in robustness to finite-precision effects. For example, for 15 bits and a small regularization factor μ, the array lattice algorithm seems to exhibit the best performance among all lattice variants. The other algorithms can breakdown due to division by small numbers (especially for longer filters). We observed this behavior in simulations. The break-down is a consequence

Initialization

For $M = 0$ to $M-1$ set:

μ is a small positive number.

$$q_M^f(-1) = q_M^b(-1) = q_M^c(-1) = q_M^d(-1) = \kappa_M^{\bar{b}}(-1) = 0$$
$$\zeta_M^{f/2}(-1) = \zeta_M^{b/2}(-1) = \zeta_M^{\bar{b}/2}(-1) = \sqrt{\mu}$$
$$\zeta_M^c(-1) = 1$$

For $N \geq 0$, repeat:

$$u(N) = au(N-1) + \sqrt{1-a^2}\,s(N)$$

$$\gamma_0^{1/2}(N) = 1$$
$$e_0'(N) = d(N)$$
$$f_0'(N) = b_0'(N) = u(N)$$
$$\bar{c}_0'(N) = \sqrt{1-a^2}$$

For $M = 0$ to $M-1$, repeat:

$$\begin{bmatrix} a\zeta_M^{c/2}(N-1) & \bar{c}'^*_M(N) \\ q_M^{\bar{b}*}(N-1) & b'^*_M(N) \end{bmatrix} \Theta_M^{\bar{b}}(N) = \begin{bmatrix} \zeta_M^{c/2}(N) & 0 \\ q_M^{\bar{b}*}(N) & \bar{b}'^*_M(N) \end{bmatrix}$$

$$\begin{bmatrix} \zeta_M^{b/2}(N-1) & b'^*_M(N) \\ aq_M^{c*}(N-1) & \bar{c}'^*_M(N) \\ q_M^{d*}(N-1) & e'^*_M(N) \\ 0 & \gamma_M^{1/2}(N) \end{bmatrix} \Theta_M^c(N) = \begin{bmatrix} \zeta_M^{b/2}(N) & 0 \\ q_M^{c*}(N) & \bar{c}'^*_M(N) \\ q_M^{d*}(N) & e'^*_{M+1}(N) \\ \times & \gamma_{M+1}^{1/2}(N) \end{bmatrix}$$

$$\begin{bmatrix} \zeta_M^{f/2}(N-1) & f'^*_M(N) \\ q_M^{f*}(N-1) & \bar{b}'^*_M(N) \end{bmatrix} \Theta_M^f(N) = \begin{bmatrix} \zeta_M^{f/2}(N) & 0 \\ q_M^{f*}(N) & b'^*_{M+1}(N) \end{bmatrix}$$

$$\begin{bmatrix} \zeta_M^{\bar{b}/2}(N-1) & \bar{b}'^*_M(N) \\ q_M^{b*}(N-1) & f'^*_M(N) \end{bmatrix} \Theta_M^b(N) = \begin{bmatrix} \zeta_M^{\bar{b}/2}(N) & 0 \\ q_M^{b*}(N) & f'^*_{M+1}(N) \end{bmatrix}$$

Table 4: *The array-based Laguerre lattice filter.*

of the fact that, in the absence of regularization, the initial least-squares problems become rank-deficient. In the array form, no regularization is needed. In Fig. 4 we compare the performance of the different lattice implementations for the same filter order and quantized at 15 bits, when an impulsive disturbance is introduced at $N = 200$. A similar simulation example was performed in [6], in order to illustrate the recovery of the MSE convergence under a sudden nonstationarity. We used zero regularization for the array form ($\mu = 0$) and $\mu = 0.1$ for the other versions. For a 10 bit-wordlength we observed that the standard lattice form breaks down after the impulsive disturbance. The error feedback form can still recover its final MSE, while the array form achieves a higher MSE value. For 15 bits wordlength, the array form returns to its final MSE faster than the a priori error feedback form, while the standard lattice form achieves a higher final MSE.

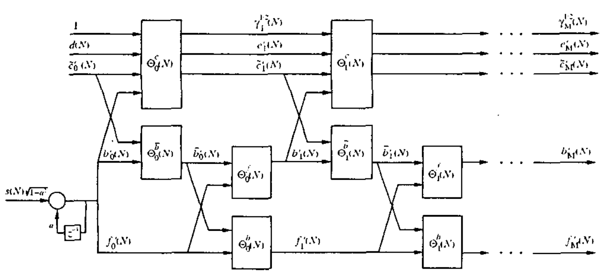

Figure 3: *Schematic representation of the array-based Laguerre lattice filter.*

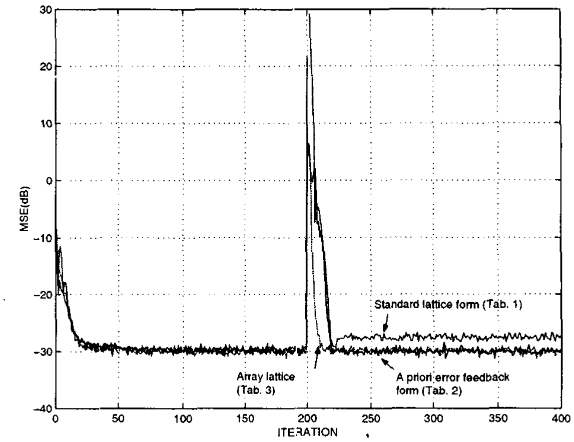

Figure 4: *MSE decay of the various Laguerre lattice forms under finite precision.*

We have also noticed that for high order Laguerre lattices filters, the regularization factor has to be high in order to avoid breakdown of the algorithm (in a 15 bit quantization). The array form has shown the best performance in this sense, which is independent of the regularization.

5. CONCLUDING REMARKS

In this article we described several lattice (order-recursive) variants for Laguerre adaptive filtering. In [7] we describe several fixed-order variants as well.

6. REFERENCES

[1] R. Merched and A. H. Sayed, "Order-Recursive RLS Laguerre Lattice Adaptive Filtering," *IEEE Transactions on Signal Processing*, vol. 48, no. 11, pp. 3000-3010, Nov. 2000.

[2] A. H. Sayed and T. Kailath, "A state-space approach to adaptive RLS filtering," *IEEE Signal Processing Magazine*, vol. 11, no. 3, pp. 18-60, Jul. 1994.

[3] B. Wahlberg, "System Identification using Laguerre models," *IEEE Trans, Automat. Control.*, vol. 36, pp. 551-562, May. 1991.

[4] J. W. Davidson and D. D. Falconer, "Reduced complexity echo cancelation using orthonormal functions," *IEEE Trans. on Circuits Syst.*, vol. 38, no. 1, pp. 20-28, Jan. 1991.

[5] J. Cioffi, "The fast adaptive rotor RLS algorithm," *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. 38, pp. 631-653, 1990.

[6] B. Yang and J. F. Bohme, "Rotation-based RLS algorithms: Unified derivations, numerical properties, and parallel implementations," *IEEE Transactions on Signal Processing*, vol. 40, no. 5, pp. 1151-1167, May 1992.

[7] R. Merched and A. H. Sayed, "Orthonormal realization of fast fixed-order RLS adaptive filters," *Proc. ICASSP*, Salt Lake City, Utah, May 2001.