ISCAS 2000 - IEEE International Symposium on Circuits and Systems, May 28-31, 2000, Geneva, Switzerland

# A UNIFIED APPROACH TO THE STEADY-STATE ANALYSIS OF QUANTIZED ADAPTIVE FILTERING ALGORITHMS

NABIL R. YOUSEF AND ALI H. SAYED

Electrical Engineering Department
University of California
Los Angeles, CA 90095

**ABSTRACT**

The steady-state performance of adaptive filters can significantly vary when they are implemented in finite precision arithmetic, which makes it vital to analyze their performance in a quantized environment. Such analyses can become difficult for adaptive algorithms with nonlinear update equations. This paper develops a new feedback approach to the steady-state analysis of quantized adaptive algorithms that bypasses many of the difficulties encountered in traditional approaches. In so doing, we not only re-derive several earlier results in the literature, but we often do so under weaker assumptions, in a more compact way, and we also obtain new results.

## 1. INTRODUCTION

This paper develops a new approach to the rounoff error analysis of adaptive filtering algorithms. The approach is based on showing how a generic quantized adaptive filter can be represented as a cascade of elementary sections, with each section consisting of a lossless system in the feedforward path and a feedback interconnection, with roundoff errors acting as disturbances to the system. By studying the energy flow through the cascade, we are able to establish a fundamental error variance relation. Using this relation, for quantized and infinite precision algorithms, we are able to extend the results of the infinite precision case to that of the quantized case with minimal calculations. We also derive new results.

Thus consider noisy measurements $\{d(i)\}$ that arise from the linear model

$$d(i) = \mathbf{u}_i \mathbf{w}^o + v(i), \tag{1}$$

where $\mathbf{w}^o$ is an unknown *column* vector of $N$ coefficients that we wish to estimate, $v(i)$ accounts for measurement noise and modeling errors, and $\mathbf{u}_i$ denotes a nonzero *row* input (regressor) vector. Many adaptive schemes have been developed in the literature for the estimation of $\mathbf{w}^o$ in different contexts (*e.g.*, echo cancelation, system identification, channel equalization). In this paper, we focus on the following general class of algorithms:

$$\mathbf{w}_{i+1} \;=\; \mathbf{w}_i + \mu\, \mathbf{u}_i^*\, \mathbf{F}(e(i)), \tag{2}$$

where $\mathbf{w}_i$ is an estimate for $\mathbf{w}^o$ at iteration $i$, $\mu$ is the step-size, and $\mathbf{F}(e(i))$ is a (linear or nonlinear) function of the so-called output estimation error, defined by

Table 1: *Examples for* $\mathbf{F}(e(i))$.

| ALGORITHM | $\mathbf{F}(e(i))$ |
|---|---|
| LMS | $e(i)$ |
| LMF | $e^3(i)$ |
| LMMN | $\delta e(i) + (1-\delta)e^3(i)$ |
| SA | $\text{sign}[e(i)]$ |

$$e(i) = d(i) - \mathbf{u}_i \mathbf{w}_i\,. \tag{3}$$

Different choices for $\mathbf{F}(e(i))$ result in different adaptive algorithms. For example, Tab. 1 defines $\mathbf{F}(e(i))$ for several famous special cases of (2). In the table, $\delta$ is a constant.[1]

An important performance measure for an adaptive filter is its steady-state mean-square-error (MSE), which is defined as

$$\text{MSE} = \lim_{i\to\infty} \mathrm{E}\,|e(i)|^2 \;=\; \lim_{i\to\infty} \mathrm{E}\,|v(i) + \mathbf{u}_i \tilde{\mathbf{w}}_i|^2\,,$$

where $\tilde{\mathbf{w}}_i = \mathbf{w}^o - \mathbf{w}_i$ denotes the weight error vector. Under the realistic assumption that (see, *e.g.*, [1]–[3]):

<u>A.1</u> The noise sequence $\{v(i)\}$ is iid and statistically independent of the regressor sequence $\{\mathbf{u}_i\}$,

we find that the MSE is equivalently given by

$$\text{MSE} = \sigma_v^2 + \lim_{i\to\infty} \mathrm{E}\,|\mathbf{u}_i \tilde{\mathbf{w}}_i|^2\,. \tag{4}$$

Now the standard way for evaluating (4), and which dominates most derivations in the literature, is the following. First, one assumes, in addition to A.1, that the regression vector $\mathbf{u}_i$ is independent of $\tilde{\mathbf{w}}_i$. Then the above MSE becomes

$$\text{MSE} = \sigma_v^2 + \lim_{i\to\infty} \mathrm{Tr}(\mathbf{R}\,\mathbf{C}_i), \tag{5}$$

where $\mathbf{C}_i = \mathrm{E}\,\tilde{\mathbf{w}}_i \tilde{\mathbf{w}}_i^*$ denotes the weight error covariance matrix and $\mathbf{R} = \mathrm{E}\,\mathbf{u}_i^* \mathbf{u}_i$ is the input covariance matrix. As is evident from (5), this method of computation requires the determination of the steady-state value of $\mathbf{C}_i$, say $\mathbf{C}_\infty$. In quantized environments, finding $\mathbf{C}_\infty$ is a burden, especially for adaptive schemes with nonlinear update equations. The following are the novel contributions of this work:

THIS WORK WAS PARTIALLY SUPPORTED BY THE NATIONAL SCIENCE FOUNDATION UNDER AWARD NUMBERS CCR-9732376 AND ECS-9820765.

[1] In this article we assume real-valued data for compactness.

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1. We develop a feedback approach for evaluating the MSE of a large class of adaptive schemes when implemented in finite precision. This approach bypasses the need for working directly with $\mathbf{C}_i$ or with its limiting value, and it extends the results in [4] to the finite precision case.
2. The approach further establishes the significant conclusion that the finite precision analysis of an adaptive scheme can be obtained almost by inspection from the results in the infinite precision case for a large class of algorithms. In contrast, analyses for both cases have always been carried out separately in the literature.
3. The feedback approach not only allows us to re-derive several earlier results in literature in a unified manner, but it does so with less effort and often under weaker assumptions.
4. The approach also allows us to derive several new results, especially for adaptive filters with nonlinear updates for which approaches that require $\mathbf{C}_i$ are not easily applicable.

## 2. A MATHEMATICAL MODEL

Figure 1 shows the quantized model used in the paper, and which is widely used in the context of finite precision analyses of adaptive algorithms (see, e.g., [5]–[8]). In this figure, $Q[x]$ denotes the fixed point quantization of the value $x$, and the superscript $q$ distinguishes quantized quantities from infinite precision quantities. Throughout the paper, rounding quantization is considered. It is also assumed that the saturation thresholds of the quantizers are properly chosen such that saturation errors are negligible. Thus, only rounding errors are considered. The variance $\sigma^2$ of the rounding error is related to the quantizer saturation threshold $L$ according to

$$\sigma^2 = \frac{2^{-2B} L^2}{12},$$

where it is assumed that the quantizer uses $B$ bits in addition to a sign bit. The values of $B$ and $L$ considered for quantization of the data ($\mathbf{u}_i$, $d(i)$, and $y(i)$) will be denoted by $B_d$ and $L_d$ and the ones considered for quantization of the filter coefficients will be denoted by $B_c$ and $L_c$. The corresponding values of $\sigma^2$ will be denoted by $\sigma_d^2$ and $\sigma_c^2$, respectively. We can write

$$d^q(i) = d(i) + \beta(i), \; y^q(i) = \mathbf{u}_i^q \mathbf{w}_i^q + \gamma(i), \tag{6}$$

where $\beta(i)$ is the system output quantization error with variance $\sigma_d^2$, and $\gamma(i)$ is the quantization error that occurs in computing the term $\mathbf{u}_i^q \mathbf{w}_i^q$. The variance of $\gamma(i)$, $\sigma_\gamma^2$, depends on the procedure by which $y^q(i)$ is computed. If all $N$ products involved in $\mathbf{u}_i^q \mathbf{w}_i^q$ are computed with high precision, summed, and the final result is quantized to $B_d$ bits, then $\sigma_\gamma^2$ is approximately equal to $\sigma_d^2$. If each one of the $N$ products is quantized to, say $B_y$ bits, and the sum is then quantized to $B_d$ bits with $B_y$ being significantly greater than $B_d$, $\sigma_\gamma^2$ is equal to $\sigma_d^2 + N\sigma_y^2$. The quantized estimation error $e^q(i)$ is given from (6) by

$$e^q(i) = d^q(i) - y^q(i) = e(i) + \xi(i), \tag{7}$$

where $\xi(i) = \beta(i) - \gamma(i)$. Obviously, $\xi(i)$ is a zero-mean sequence with variance $\sigma_\xi^2 = \sigma_d^2 + \sigma_\gamma^2$. In general, the quantized error function $\mathbf{F}^q(e^q(i))$ can be written as

$$\mathbf{F}^q(e^q(i)) = \mathbf{F}(e^q(i)) + \eta(i), \tag{8}$$

where $\eta(i)$ is the error in calculating $\mathbf{F}^q(e^q(i))$ from $e^q(i)$. The variance of $\eta(i)$ depends on the adaptive algorithm used. For example, $\sigma_\eta^2$ is equal to 0, $\sigma_d^2$, $2\sigma_d^2$, and 0 for the LMS, LMF, LMMN, and SA, respectively.

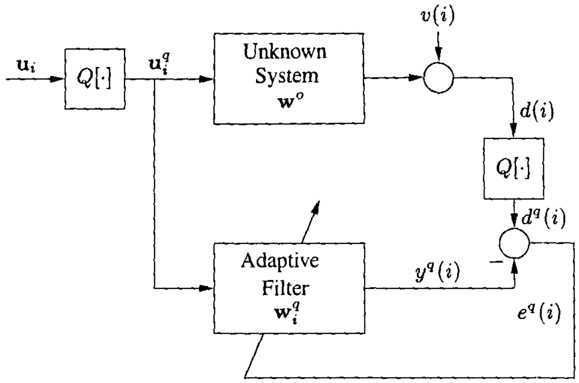


Figure 1: *Quantization model.*

Taking the above quantizations into consideration, the adaptive algorithm recursion (2) becomes

$$\begin{aligned} \mathbf{w}_{i+1}^q &= \mathbf{w}_i^q + Q\left[\mu\, \mathbf{u}_i^{q*}\, \mathbf{F}^q(e^q(i))\right] \\ &= \mathbf{w}_i^q + \mu\, \mathbf{u}_i^{q*}\, \mathbf{F}^q(e^q(i)) - \mathbf{m}_i, \end{aligned} \tag{9}$$

where $\mathbf{m}_i$ is a vector of multiplication quantization errors in the update term $\mu \mathbf{u}_i^{q*} \mathbf{F}^q(e^q(i))$, each entry of which has variance $\sigma_c^2$. The weight error vector is now defined as

$$\tilde{\mathbf{w}}_i = \mathbf{w}^o - \mathbf{w}_i^q. \tag{10}$$

## 3. QUANTIZED ENERGY RELATION

We start by defining the following so-called a-priori and a-posteriori estimation errors,

$$e_a(i) = \mathbf{u}_i^q \tilde{\mathbf{w}}_i, \quad e_p(i) = \mathbf{u}_i^q\left(\tilde{\mathbf{w}}_{i+1} - \mathbf{m}_i\right).$$

Using (3) and (7), it is easy to see that the errors $\{e^q(i), e_a(i)\}$ are related via $e^q(i) = e_a(i) + v(i) + \xi(i)$. If we further subtract $\mathbf{w}^o$ from both sides of (9) and multiply by $\mathbf{u}_i^q$ from the left, we also find that the errors $\{e_p(i), e_a(i), e^q(i)\}$ are related via:

$$e_p(i) = e_a(i) - \frac{\mu}{\bar{\mu}(i)} \mathbf{F}^q(e^q(i)), \tag{11}$$

where we defined, for compactness, $\bar{\mu}(i) = 1/\|\mathbf{u}_i^q\|^2$. Substituting (11) into (9), we obtain the update relation

$$\tilde{\mathbf{w}}_{i+1} = \tilde{\mathbf{w}}_i - \bar{\mu}(i)\mathbf{u}_i^{q*}[e_a(i) - e_p(i)] + \mathbf{m}_i.$$

By evaluating the energies of both sides of this equation we obtain

$$\|\tilde{\mathbf{w}}_{i+1} - \mathbf{m}_i\|^2 + \bar{\mu}(i)|e_a(i)|^2 = \|\tilde{\mathbf{w}}_i\|^2 + \bar{\mu}(i)|e_p(i)|^2 \,. \tag{12}$$

This energy conservation relation, first established in [9, 10], holds for all adaptive algorithms whose recursions are of the form given by (2). *No approximations or assumptions are needed to establish* (12); it is an exact relation that shows how the energies of the weight error vectors at two successive time instants are related to the energies of the a-priori and a-posteriori estimation errors. The relation also has an interesting system-theoretic interpretation. It establishes that the mapping from $\left\{\tilde{\mathbf{w}}_i, \sqrt{\bar{\mu}(i)}e_p(i)\right\}$ to $\left\{\tilde{\mathbf{w}}_{i+1} - \mathbf{m}_i, \sqrt{\bar{\mu}(i)}e_a(i)\right\}$ is energy preserving (or lossless). Furthermore, combining (12) with (11), we see that both relations establish the existence of the feedback configuration shown in Fig. 2, where $\mathcal{T}$ denotes a lossless map and $q^{-1}$ denotes the unit delay operator. Here we can see that the multiplication quantization error acts as a disturbance input to the system. Such a disturbance mainly plays the same role as that of system nonstationarity [11, 12, 13].

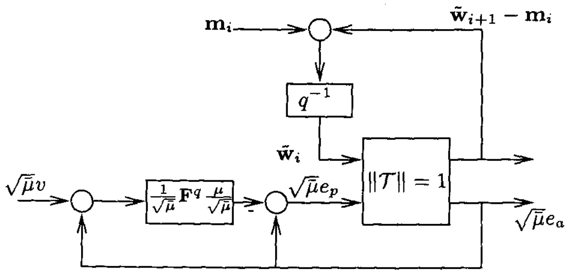


Figure 2: *Lossless mapping and a feedback loop.*

Relation (12) has several ramifications. It was used in [9, 10] to study the robustness and $l_2-$stability of adaptive filters and in [4, 11, 12, 13] to study the steady-state and tracking performances of various adaptive algorithms. Here we show its significance to finite precision analyses of adaptive algorithms.

First, we impose the following modeling assumption:

A.2 Quantization errors are zero-mean, mutually independent, and independent of all other signals.

This assumption is typical in the context of finite precision analysis of adaptive algorithms (see e.g., [5]–[8]), and it enables the derivation of closed-form expressions for the steady-state MSE. A more sophisticated nonlinear model for treating quantization errors, which takes into account the quantizer underflow effects, has been used in [14] for the LMS algorithm; though it does not lead to closed-form expressions.

Imposing the equality $\mathrm{E}\,\|\tilde{\mathbf{w}}_{i+1}\|^2 = \mathrm{E}\,\|\tilde{\mathbf{w}}_i\|^2$ in steady-state, and using (11) and A.2, it is straightforward to verify that the energy relation (12) leads to

$$\mathrm{E}\,\bar{\mu}(i)|e_a(i)|^2 = \mathrm{Tr}(\mathbf{M}) + \mathrm{E}\,\bar{\mu}(i)\left|e_a(i) - \frac{\mu}{\bar{\mu}(i)}\mathbf{F}^q(e^q(i))\right|^2,$$

where $\mathbf{M} = \mathrm{E}\,\mathbf{m}_i\mathbf{m}_i^*$. For iid multiplication errors, $\mathrm{Tr}(\mathbf{M}) = N\sigma_c^2$. Using (7), (8), and A.2, we obtain the following fundamental error variance relation in terms of $e_a(i)$ and $\bar{v}(i) \triangleq v(i) + \xi(i)$:

$$\mathrm{E}\,\bar{\mu}(i)|e_a(i)|^2 = \mathrm{Tr}(\mathbf{M}) + \mu^2\sigma_\eta^2\,\mathrm{Tr}(\mathbf{R}^q) + \mathrm{E}\,\bar{\mu}(i)\left|e_a(i) - \frac{\mu}{\bar{\mu}(i)}\mathbf{F}(e_a(i) + \bar{v}(i))\right|^2, \quad (13)$$

where $\mathrm{Tr}(\mathbf{R}^q) = \mathrm{E}\,\mathbf{u}_i^{q*}\mathbf{u}_i^q = \mathrm{Tr}(\mathbf{R}) + N\sigma_d^2$. This equation can now be solved for the steady-state excess mean-square-error (EMSE):

$$\zeta \triangleq \lim_{i\to\infty} \mathrm{E}\,|e_a(i)|^2 .$$

Observe from (4) that the desired MSE is given by MSE $= \sigma_v^2 + \zeta$, so that finding $\zeta$ is equivalent to finding the MSE. Moreover, for the infinite precision case, equation (13) is given by [4]:

$$\mathrm{E}\,\bar{\mu}(i)|e_a(i)|^2 = \mathrm{E}\,\bar{\mu}(i)\left|e_a(i) - \frac{\mu}{\bar{\mu}(i)}\mathbf{F}(e_a(i) + v(i))\right|^2. \quad (14)$$

Thus comparing (13) with (14), we can observe that two new terms exist on the RHS in the quantized case. Furthermore, using A.2, we can see that the LHS and the last term on the RHS of (13) are the same as those in (14) if we replace $v(i)$ by $\bar{v}(i)$ and $\|\mathbf{u}_i\|^2$ by $\|\mathbf{u}_i^q\|^2$. This is a significant observation in the context of finite precision analysis of adaptive algorithms, as it shows how to extend the results of the infinite precision case to those of the quantized case with minimal effort. In the literature, both cases have generally been studied separately.

Example. We now apply the above general procedure to the LMS algorithm. We solve both the infinite and finite precision energy equations (14) and (13), and show how to extract the results of the quantized case from those of the infinite precision case. Later we directly apply our general procedure to other algorithms. The reader will soon realize the convenience of working with such a procedure.

For LMS we have $\mathbf{F}(e(i)) = e(i) = e_a(i) + v(i)$. Substituting into (14) and using A.1, it follows immediately that

$$2\mu\zeta^{\mathsf{LMS}} = \mu^2\,\mathrm{E}\left(\|\mathbf{u}_i\|^2\,|e_a(i)|^2\right) + \mu^2\sigma_v^2\,\mathrm{Tr}(\mathbf{R}). \quad (15)$$

To solve for $\zeta^{\mathsf{LMS}}$ we consider two cases:

1. For sufficiently small $\mu$, we can assume that the term $\mu^2\,\mathrm{E}\,\|\mathbf{u}_i\|^2\,|e_a(i)|^2$ is negligible relative to the second term on the right-hand side of (15), so that

$$\zeta^{\mathsf{LMS}} = \frac{\mu}{2}\sigma_v^2\,\mathrm{Tr}(\mathbf{R}).$$

2. For larger values of $\mu$, equation (15) can be solved by imposing the following assumption [4]:

A.3 At steady state, $\mu^2\|\mathbf{u}_i\|^2$ is statistically independent of $|e_a(i)|^2$.

Using A.3, and (15), we directly obtain

$$\zeta^{\mathsf{LMS}} = \frac{\mu\sigma_v^2\,\mathrm{Tr}(\mathbf{R})}{2 - \mu\,\mathrm{Tr}(\mathbf{R})}.$$

Now for the quantized case, substituting in (13) and using A.1 and A.2, we directly obtain

$$2\mu\zeta^{\mathsf{LMS}} = \mu^2\,\mathrm{E}\left(\|\mathbf{u}_i^q\|^2\,|e_a(i)|^2\right) + \mu^2\sigma_{\bar{v}}^2\,\mathrm{Tr}(\mathbf{R}^q).$$

For small enough values of $\mu$, we have

$$\zeta^{\mathsf{LMS}} = \frac{1}{2}\left(\mu^{-1}\,\mathrm{Tr}(\mathbf{M}) + \mu\sigma_{\bar{v}}^2\,\mathrm{Tr}(\mathbf{R}^q)\right),$$

where $\sigma_{\bar{v}}^2 = \sigma_v^2 + \sigma_\xi^2$. For larger values of $\mu$, using A.2 and A.3, we obtain

$$\zeta^{\mathsf{LMS}} = \frac{\mu^{-1}\,\mathrm{Tr}(\mathbf{M}) + \mu\sigma_{\bar{v}}^2\,\mathrm{Tr}(\mathbf{R}^q)}{2 - \mu\,\mathrm{Tr}(\mathbf{R}^q)}.$$

This example shows that the finite precision results could be easily obtained from the infinite precision results.

Table 2: *Infinite and finite precision expressions for the EMSE:*

| ALGORITHM | Infinite Precision | Finite Precision |
|---|---|---|
| LMS (small $\mu$) | $\frac{\mu}{2}\sigma_v^2\,\mathrm{Tr}(\mathbf{R})$ | $\frac{1}{2}\,\mathrm{Tr}(\mathbf{M})/\mu + \frac{\mu}{2}\sigma_{\bar{v}}^2\,\mathrm{Tr}(\mathbf{R}^q)$ |
| LMS (large $\mu$) | $\frac{\mu\sigma_v^2\,\mathrm{Tr}(\mathbf{R})}{2-\mu\,\mathrm{Tr}(\mathbf{R})}$ | $\frac{\mathrm{Tr}(\mathbf{M})/\mu+\mu\sigma_{\bar{v}}^2\,\mathrm{Tr}(\mathbf{R}^q)}{2-\mu\,\mathrm{Tr}(\mathbf{R}^q)}$ |
| LMF (small $\mu$) | $\frac{\mu}{2}\left(\frac{\xi_v^6}{3\sigma_v^2}\right)\mathrm{Tr}(\mathbf{R})$ | $\frac{1}{2}\,\mathrm{Tr}(\mathbf{M})/\mu + \frac{\mu}{2}\left(\frac{\xi_{\bar{v}}^6+\sigma_d^2}{3\sigma_{\bar{v}}^2}\right)\mathrm{Tr}(\mathbf{R}^q)$ |
| LMF (large $\mu$) | $\frac{\mu\xi_v^6\,\mathrm{Tr}(\mathbf{R})}{6\sigma_v^2-9\mu\xi_v^4\,\mathrm{Tr}(\mathbf{R})}$ | $\frac{\mathrm{Tr}(\mathbf{M})/\mu+\mu\left(\xi_{\bar{v}}^6+\sigma_d^2\right)\mathrm{Tr}(\mathbf{R}^q)}{6\sigma_{\bar{v}}^2-9\mu\xi_{\bar{v}}^4\,\mathrm{Tr}(\mathbf{R}^q)}$ |
| LMMN (small $\mu$) | $\frac{\mu}{2}\left(\frac{\delta^2\sigma_v^2+2\delta\bar{\delta}\xi_v^4+\bar{\delta}^2\xi_v^6}{\delta+3\bar{\delta}\sigma_v^2}\right)\mathrm{Tr}(\mathbf{R})$ | $\frac{1}{2}\,\mathrm{Tr}(\mathbf{M})/\mu + \frac{\mu}{2}\left(\frac{\delta^2\sigma_{\bar{v}}^2+2\delta\bar{\delta}\xi_{\bar{v}}^4+\bar{\delta}^2\xi_{\bar{v}}^6+2\sigma_d^2}{\delta+3\bar{\delta}\sigma_{\bar{v}}^2}\right)\mathrm{Tr}(\mathbf{R}^q)$ |
| LMMN (large $\mu$) | $\frac{\mu(\delta^2\sigma_v^2+2\delta\bar{\delta}\xi_v^4+\bar{\delta}^2\xi_v^6)\,\mathrm{Tr}(\mathbf{R})}{2(\delta+3\bar{\delta}\sigma_v^2)-\mu(\delta^2+6\delta\bar{\delta}\sigma_v^2+9\bar{\delta}\xi_v^4)\,\mathrm{Tr}(\mathbf{R})}$ | $\frac{\mathrm{Tr}(\mathbf{M})/\mu+\mu(\delta^2\sigma_{\bar{v}}^2+2\delta\bar{\delta}\xi_{\bar{v}}^4+\bar{\delta}^2\xi_{\bar{v}}^6+2\sigma_d^2)\,\mathrm{Tr}(\mathbf{R}^q)}{2(\delta+3\bar{\delta}\sigma_{\bar{v}}^2)-\mu(\delta^2+6\delta\bar{\delta}\sigma_{\bar{v}}^2+9\bar{\delta}\xi_{\bar{v}}^4)\,\mathrm{Tr}(\mathbf{R}^q)}$ |
| SA | $\frac{A}{2}\left(A+\sqrt{A^2+4\sigma_v^2}\right)$ | $\frac{B}{2}\left(B+\sqrt{B^2+4\sigma_v^2}\right)$ |

Following the same procedure, we can extend the infinite precision results given in [4] to obtain the corresponding quantized results for the algorithms given in Tab. 1. The expressions are shown in Tab. 2. In this table, $\bar{\delta} = 1-\delta$, $\xi_v^4 = \mathrm{E}\,|v(i)|^4$, $\xi_v^6 = \mathrm{E}\,|v(i)|^6$, $\xi_{\bar{v}}^4 = \mathrm{E}\,|\bar{v}(i)|^4$, $\xi_{\bar{v}}^6 = \mathrm{E}\,|\bar{v}(i)|^6$, $A = \sqrt{\frac{\pi}{8}}\mu\,\mathrm{Tr}(\mathbf{R})$, and $B = \sqrt{\frac{\pi}{8}}\left(\mu^{-1}\,\mathrm{Tr}(\mathbf{M}) + \mu\,\mathrm{Tr}(\mathbf{R}^q)\right)$. For the case of the SA we used the assumption that the quantization of the estimation error does not introduce any errors in its sign [8].

Here we may add that the results obtained for the LMF and LMMN algorithms are new as no finite precision analysis exists for them in the literature. The results for the SA are the same as those obtained in [8] by using the independence assumptions. Here we have shown that the same result holds without the independence assumptions. Moreover, by differentiating the EMSE expressions in Tab. 2 with respect to $\mu$, we obtain several new expressions for the optimum step-sizes that achieve the lowest EMSE. Due to space limitations, we do not list these expressions here.

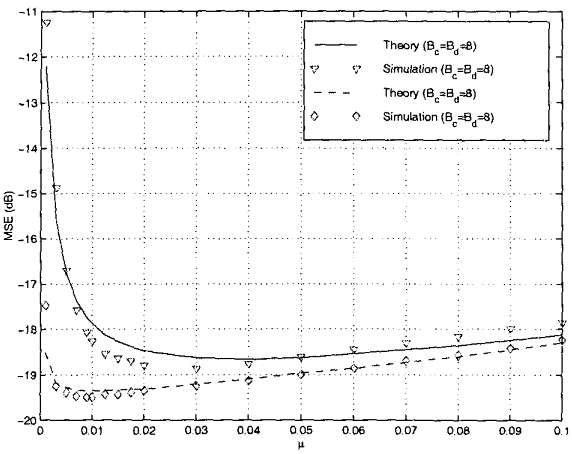


Figure 3: *Theory and simulation* MSE *for the* LMMN *vs.* $\mu$.

## 4. SIMULATION RESULTS

Figure 3 compares the simulation and theoretical results of the steady state MSE of the LMMN algorithm, with $\delta$=0.5, for a large range of $\mu$ and two values of the wordlength. In the simulations, the unknown system weight vector $\mathbf{w}^o$ is of length 10 and the elements of the input vector, $\mathbf{u}_i$, are white Gaussian of unit variance. The plant noise is chosen to be a linear combination of normally and uniformly distributed independent random variables of variances $\sigma_n^2 = 10^{-6}$ and $\sigma_u^2 = 10^{-2}/12$, respectively. Each simulation result is the steady state statistical average of 50 runs, with up to 20,000 iterations in each run. We can see from the figure that the theoretical and experimental MSE are in good match.

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