RECURSIVE SOLUTIONS TO RATIONAL INTERPOLATION PROBLEMS *

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ABSTRACT

We describe a simple and straightforward recursive (and global) approach for the solution of rational interpolation problems. The derivation is based on a simple and well known matrix identity, namely the Schur (or Jacobi) reduction procedure, and exploits connections with structured matrices. We use the interpolation data to construct a convenient structure and then apply a recursive triangularization procedure. This leads to a transmission-line cascade of first-order J-lossless sections that makes evident the interpolation property. We also give state-space descriptions for each section and for the entire cascade.

1. INTRODUCTION

MANY signal processing, circuit theory and control application problems admit a formulation in terms of a rational interpolating function. The solution of such interpolation problems has been studied extensively in the literature. The recursions of Schur and Nevanlinna-Pick have been around for a long time (see, *e.g.* [1]). The corresponding tangential and matrix versions were later approached and solved by different authors and in different ways and contexts, including operator theory and the lifting of commutants [2, 3], approximation of Hankel operators [4], Krein spaces [5], reproducing kernel Hilbert spaces [6], system theory and state-space realizations [7], H^∞−control [8, 9], etc.. We describe here an alternative approach for the solution of rational interpolation problems. The derivation is simple, straightforward and is based on a simple and well known matrix identity, namely the Schur (or Jacobi) reduction procedure. This is a recursive procedure that performs the triangular factorization of an $n \times n$ matrix R. In the case of structured matrices (*i.e.* matrices that are solutions of certain Stein or Lyapunov equations), the Schur procedure reduces to an efficient ($O(n^2)$) recursive update of the so called generator matrix of R. Each step leads to a new generator with one more zero row than the previous one. We exploit this simple fact and construct a transmission-line cascade that makes evident the interpolation property. We may summarize the proposed scheme as follows: use the interpolation data to describe a convenient structured matrix, apply the recursive triangularization procedure to the corresponding generator matrix and construct the associated transmission-line cascade.

Let $H^\infty_{p\times q}$ denote the Hardy space of $p \times q$ rational matrix-valued functions that are analytic and bounded inside the open unit disc. A matrix valued function $S \in H^\infty_{p\times q}$ that is bounded by unity ($\|S\|_\infty < 1$) will be referred to as a function of Schur type. We use the notation $\mathcal{H}^k_A(z)$ to refer to the following block upper-triangular Toeplitz matrix

$$\begin{bmatrix} A(z) & \frac{1}{1!}A^{(1)}(z) & \frac{1}{2!}A^{(2)}(z) & \cdots & \frac{1}{(k-1)!}A^{(k-1)}(z) \\ & A(z) & \frac{1}{1!}A^{(1)}(z) & \cdots & \frac{1}{(k-2)!}A^{(k-2)}(z) \\ & & A(z) & & \\ & & & \ddots & \vdots \\ & O & & \ddots & \frac{1}{1!}A^{(1)}(z) \\ & & & & A(z) \end{bmatrix}$$

where A is a rational matrix function analytic at $z, k \geq 1$ is a positive integer and $A^{(i)}(z)$ denotes the i^{th} derivative of A at z. We also denote by $e_i = \begin{bmatrix} 0_{1\times i} & 1 & 0 \end{bmatrix}$ the i^{th} basis vector of the n−dimensional space of complex numbers $\mathbf{C}^{1\times n}$. We now state a general Hermite-Fejér interpolation problem that includes many of the classical problems as special cases. We consider m points $\{f_i\}_{i=0}^{m-1}$(not necessarily distinct) inside the open unit disc and we associate with each f_i a positive integer $r_i \geq 1$ and two row vectors $\mathbf{u}_i$ and $\mathbf{v}_i$, partitioned as follows

$$\mathbf{u}_i = \begin{bmatrix} u_1^{(i)} & \dots & u_{r_i}^{(i)} \end{bmatrix}, \quad \mathbf{v}_i = \begin{bmatrix} v_1^{(i)} & \dots & v_{r_i}^{(i)} \end{bmatrix}$$

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where $u_j^{(i)}$ and $v_j^{(i)}, (j = 1, \ldots, r_i)$ are $1 \times p$ and $1 \times q$ row vectors respectively.

Tangential Hermite-Fejér: *Describe all Schur type functions* $S \in H^{\infty}_{p \times q}$ *that satisfy* $\mathbf{v}_i = \mathbf{u}_i \mathcal{H}_S^{r_i}(f_i)$ *for* $0 \leq i \leq m-1$.

This problem includes, among others, the following well known special cases

- **Scalar Carathéodory:** $m = 1, f_0 = 0, r_0 = n, p = q = 1$, $\mathbf{u}_0 = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}$ and $\mathbf{v}_0 = \begin{bmatrix} \alpha_0 & \alpha_1 & \ldots & \alpha_{n-1} \end{bmatrix}$. In this case, we are reduced to finding a scalar Schur function s such that $\frac{s^{(i)}(0)}{i!} = \alpha_i$ for $i = 0, 1, \ldots, n-1$.
- **Scalar Nevanlinna-Pick:** $m = n, f_i$ distinct, $r_i = 1$, $p = q = 1$, $\mathbf{u}_i = 1$ and $\mathbf{v}_i = \alpha_i$. In this case, we are reduced to finding a scalar Schur function s such that $s(f_i) = \alpha_i$.
- **Tangential NP:** $m = n, f_i$ distinct, $r_i = 1, \mathbf{u}_i = u_i$ and $\mathbf{v}_i = v_i$. In this case, we are reduced to finding a $p \times q$ Schur matrix function S such that $u_i S(f_i) = v_i$.

2. SOLVABILITY

The first step in our solution consists in constructing three matrices F, G and J directly from the interpolation data: F contains the information relative to the points f_i and the dimensions r_i, G contains the information relative to the direction vectors $\mathbf{u}_i$ and $\mathbf{v}_i$, and $J =$ diagonal$\{I_p, -I_q\}$ is a signature matrix, where I_p denotes a $p \times p$ identity matrix. The matrices F and G are constructed as follows: we associate with each f_i a Jordan block F_i of size $r_i \times r_i$

$$F_i = \begin{bmatrix} f_i & & & \\ 1 & f_i & & \\ & \ddots & \ddots & \\ & & 1 & f_i \end{bmatrix}$$

and two $r_i \times p$ and $r_i \times q$ matrices U_i and V_i respectively, which are composed of the row vectors associated with f_i

$$U_i = \begin{bmatrix} u_1^{(i)} \\ \vdots \\ u_{r_i}^{(i)} \end{bmatrix} \quad \text{and} \quad V_i = \begin{bmatrix} v_1^{(i)} \\ \vdots \\ v_{r_i}^{(i)} \end{bmatrix}$$

Then $F =$ diagonal$\{F_0, F_1, \ldots, F_{m-1}\}$ and

$$G = \begin{bmatrix} U_0 & V_0 \\ \vdots & \vdots \\ U_{m-1} & V_{m-1} \end{bmatrix}$$

Let $n = \sum_{i=0}^{m-1} r_i$ and $r = (p+q)$, then F and G are $n \times n$ and $n \times r$ matrices respectively. We shall denote the diagonal entries of F by $\{f_i\}_{i=0}^{n-1}$ (observe that F reduces to a diagonal matrix when $r_i = 1$ for all i). We shall show in the next section that by applying a simple recursive procedure to F and G we obtain a cascade structure that satisfies the interpolation conditions. Meanwhile, we associate with the interpolation problem the Lyapunov equation $R - FRF^* = GJG^*$, where F and G are as defined above (and the symbol $*$ stands for complex conjugation). Clearly, R is unique since F is a stable matrix ($|f_i| < 1, \forall i$). This construction allows us to give a simple proof for the existence of solutions [10].

Theorem 1 (Solvability Condition) *The tangential Hermite-Fejér problem is solvable if, and only if, R is positive definite.*

We say that R has a Toeplitz-like structure [11] with respect to (F, G, J) and G is called the generator matrix of R. We should stress at this point that we only know F, G and J whereas the matrix $R = [r_{ij}]_{i,j=0}^{n-1}$ is not known a priori. In fact, the recursive solution described in the next section does not need R explicitly. It only uses F, G and J.

3. RECURSIVE SOLUTION

If R_1 denotes the Schur complement of r_{00} in R, then R_1 is also a Toeplitz-like matrix. To check this point, we let l_0 and g_0 denote the first column of R and the first row of G respectively. Then $l_0 = F l_0 f_0^* + G J g_0^*$. Moreover, if we define $d_0 = r_{00} = \frac{g_0 J g_0^*}{1 - f_0 f_0^*}$, then

$$R - l_0 d_0^{-1} l_0^* = \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & R_1 \end{bmatrix} \tag{1}$$

Let F_1 be the submatrix obtained after deleting the first row and column of F. Using (1) we can check easily that $R_1 - F_1 R_1 F_1^*$ can be factored as $G_1 J G_1^*$, where

$$\begin{bmatrix} 0_{1 \times r} \\ G_1 \end{bmatrix} = F l_0 h_0^* J + G J k_0^* J \tag{2}$$

and h_0 and k_0 are $r \times 1$ and $r \times r$ matrices respectively that satisfy the embedding relation

$$\begin{bmatrix} f_0 & g_0 \\ h_0 & k_0 \end{bmatrix} \begin{bmatrix} d_0 & \mathbf{0} \\ \mathbf{0} & J \end{bmatrix} \begin{bmatrix} f_0 & g_0 \\ h_0 & k_0 \end{bmatrix}^* = \begin{bmatrix} d_0 & \mathbf{0} \\ \mathbf{0} & J \end{bmatrix} \tag{3}$$

This shows that $R_1 = \left[r_{ij}^{(1)} \right]_{i,j=0}^{n-2}$ is indeed a Toeplitz-like matrix with respect to (F_1, G_1, J). This process may be repeated by defining the Schur complement R_2 of $r_{00}^{(1)}$ in R_1 and so on. In summary, if we let l_i and G_i denote the first column and the generator of the i^{th} Schur complement R_i respectively, then

$$\begin{bmatrix} l_i & 0_{1\times r} \\ & G_{i+1} \end{bmatrix} = \begin{bmatrix} F_i l_i & G_i \end{bmatrix} \begin{bmatrix} f_i^* & h_i^* J \\ Jg_i^* & Jk_i^* J \end{bmatrix} \quad (4)$$

where g_i is the first row of G_i, $\{f_i, g_i, h_i, k_i\}$ satisfy a relation similar to (3) with $d_i = r_{00}^{(i)}$, and F_i is the $(n-i)\times(n-i)$ submatrix obtained after deleting the first row and column of F_{i-1}. We now examine more closely the first-order discrete-time system that appears on the right-hand side of (4), viz.,

$$\begin{aligned} x_{k+1} &= x_k f_i^* + w_k J g_i^* \\ y_k &= x_k h_i^* J + w_k J k_i^* J \end{aligned}$$

where x_k is the state and w_k is a $1\times r$ row input vector at time k. Let $\Theta_i(z)$ denote the corresponding $r\times r$ transfer matrix

$$\Theta_i(z) = Jk_i^* J + Jg_i^* \left[z^{-1} - f_i^*\right]^{-1} h_i^* J \quad (5)$$

It follows from the embedding relation and from $|f_i| < 1$ that $\Theta_i(z)$ is a J-lossless matrix function, that is $\Theta_i(z)$ is analytic in $|z| < 1$ and

$$\begin{cases} \Theta_i(z) J \Theta_i^*(z) = J & \text{on } |z| = 1 \\ \Theta_i(z) J \Theta_i^*(z) < J & \text{in } |z| < 1 \end{cases}$$

The cascade $\Theta(z) = \Theta_0(z)\Theta_1(z)\ldots\Theta_{n-1}(z)$ has an intrinsic interpolation property, which follows directly from the generator recursion (4). Using the embedding relation (3) we readily conclude that

$$\begin{aligned} g_0\Theta_0(f_0) &= g_0 J k_0^* J + g_0 J g_0^* \frac{f_0}{1 - f_0 f_0^*} h_0^* J \\ &= g_0 J k_0^* J + f_0 d_0 h_0^* J = \mathbf{0} \end{aligned}$$

(This will also follow from the derivation in the next section). Similarly, using (4) and (5) we conclude that $g_1 = g_0\Theta_0^{(1)}(f_0) + e_1 G\Theta_0(f_0)$ and hence $g_0\Theta^{(1)}(f_0) + e_1 G\Theta(f_0) = \mathbf{0}$. This argument can be easily extended and leads to the following result

Theorem 2 (Interpolation) *Consider the i^{th} Jordan block F_i and let s_i be the total size of the previous Jordan blocks ($s_i = \sum_{j=0}^{i-1} r_j,\ s_0 = 0$), then*

$$\begin{bmatrix} e_{s_i}G & e_{s_i+1}G & \ldots & e_{s_i+r_i-1}G \end{bmatrix} \mathcal{H}_{\Theta}^{r_i}(f_i) = \mathbf{0} \quad (6)$$

Notice that the row vector on the left hand-side of (6) is composed of the r_i row vectors in $\begin{bmatrix} U_i & V_i \end{bmatrix}$. If we partition $\Theta(z)$ accordingly with J

$$\Theta(z) = \begin{bmatrix} \Theta_{11}(z) & \Theta_{12}(z) \\ \Theta_{21}(z) & \Theta_{22}(z) \end{bmatrix}$$

then it follows from the last theorem that the Schur matrix function $S = -\Theta_{12}\Theta_{22}^{-1}$ satisfies the required interpolation conditions. Moreover,

Lemma (Solutions) *All solutions S to the tangential Hermite-Fejér problem are given through a linear fractional transformation of a Schur type matrix function $K \in H_{p\times q}^{\infty}$*

$$S(z) = -\left[\Theta_{11}(z)K(z) + \Theta_{12}(z)\right]\left[\Theta_{21}(z)K(z) + \Theta_{22}(z)\right]^{-1}$$

4. FURTHER SIMPLIFICATIONS

Using the embedding relation (or the J-losslessness of $\Theta_i(z)$) we verify easily that h_i and k_i are given by

$$h_i = \Theta_i^{-1}\left\{\frac{1}{d_i}\frac{\tau_i - f_i}{1 - \tau_i f_i^*} J g_i^*\right\},\ k_i = \Theta_i^{-1}\left\{I_r - \frac{1}{d_i}\frac{J g_i^* g_i}{1 - \tau_i f_i^*}\right\}$$

where Θ_i is an arbitrary J-unitary matrix ($\Theta_i J \Theta_i^* = J$) and $|\tau_i| = 1$. This allows us to express $\Theta_i(z)$ in terms of f_i and g_i only

$$\Theta_i(z) = \left\{I_r + [B_i(z) - 1]\frac{J g_i^* g_i}{g_i J g_i^*}\right\}\Theta_i \quad (7)$$

where $B_i(z)$ is a Blaschke factor

$$B_i(z) = \frac{z - f_i}{1 - z f_i^*}\frac{1 - \tau_i f_i^*}{\tau_i - f_i}$$

Notice that $g_i\Theta_i(f_i) = \mathbf{0}$ readily follows from (7). We now remark that if G_i is a generator of R_i then $G_i U_i$ is also a generator for any J-unitary matrix U_i. Hence we can always assume that the leftmost element of g_i is nonzero (by choosing a J-unitary permutation U_i, for instance). So assume we fix $\tau_i = 1$ and choose Θ_i (using elementary rotations or as described below) such that $g_i\Theta_i$ is reduced to the form $g_i\Theta_i = \begin{bmatrix} \delta_i & 0 & \ldots & 0 \end{bmatrix}$, where δ_i is a scalar ($= \sqrt{d_i(1 - |f_i|^2)}$). In this case, $\Theta_i(z)$ and the generator recursion are further simplified to

$$\Theta_i(z) = \Theta_i \begin{bmatrix} B_i(z) & 0 \\ 0 & I_{r-1} \end{bmatrix}$$

$$\begin{bmatrix} 0_{1\times r} \\ G_{i+1} \end{bmatrix} = G_i\Theta_i \begin{bmatrix} 0 & 0 \\ 0 & I_{r-1} \end{bmatrix} + \Phi_i G_i \Theta_i \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad (8)$$

where $\Phi_i = \frac{1 - f_i^*}{1 - f_i}(F_i - f_i I_{n-i})(I_{n-i} - f_i^* F_i)^{-1}$. The generator recursion (8) has the following simple array interpretation

- Multiply G_i by Θ_i and keep the last $r-1$ columns;
- Multiply the first column of $G_i\Theta_i$ by Φ_i;
- These two steps result in G_{i+1}.

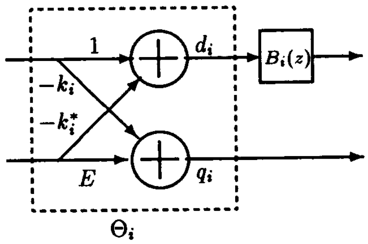

Figure 1: **Internal structure of $\Theta_i(z)$.**

Let $J = 1 \bigoplus -E$ and partition $g_i = [\ g_{i0} \quad y_i\ ]$, where g_{i0} is a scalar and y_i is a $1 \times (r-1)$ row vector. If we define the row vector $k_i = g_{i0}^{-1} y_i E$, then a possible choice for Θ_i would be [11]

$$\Theta_i = \begin{bmatrix} 1 & -k_i \\ -k_i^* & E \end{bmatrix} \begin{bmatrix} d_i & 0 \\ 0 & q_i \end{bmatrix}$$

where $d_i = (1 - k_i E k_i^*)^{-1/2}$ is a scalar and q_i is an $(r-1) \times (r-1)$ matrix that satisfies $q_i E q_i^* = (E - k_i^* k_i)^{-1}$. Figure 1 depicts a section $\Theta_i(z)$. The first column of G_i goes through the top line and the last $(r-1)$ columns propagate through the bottom line. Figure 2 shows a scattering interpretation where $\Sigma(z)$ is the scattering matrix associated with $\Theta(z)$. The solution S is the transfer matrix from the top left $(1 \times p)$ input to the bottom left $(1 \times q)$ output, with a Schur type load $(-K)$ at the right end. Using the state-space description of the first order sections $\Theta_i(z)$ we can verify that the cascade $\Theta(z)$ is given by

$$\Theta(z) = \left\{ I - (1 - z\tau) J G^* (I - zF^*)^{-1} R^{-1} (I - \tau F)^{-1} G \right\} \Theta$$

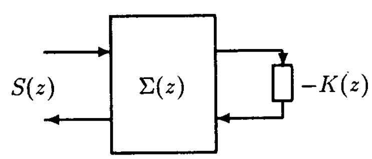

Figure 2: **Scattering interpretation.**

where Θ is an arbitrary J-unitary matrix and $|\tau| = 1$. Expressions similar to those of $\Theta(z)$ are used in [7] to solve interpolation problems. Note however, that it involves the inverse of R explicitly, whereas our recursive solution avoids this computation and uses only the matrices F and G, which are constructed directly from the interpolation data. The previous discussion leads to the following procedure: use the interpolation data to form F and G, apply the generator recursion (8) and use $\{f_i, g_i\}$ to construct the sections $\Theta_i(z)$ as in figure 1. It is worth mentioning that this procedure includes the recursions of Schur and Nevanlinna-Pick as special cases [10].

5. CONCLUSION

We described a simple alternative approach for the solution of rational interpolation problems, which is based on a recursive construction of a cascade or transmission-line structure. We also derived a state-space realization for the cascade and remarked that our solution does not require the computation of R^{-1}. The overall procedure requires $O(rn^2)$ operations (additions and multiplications). We remark that the derivation presented here can be easily extended to the solution of time-varying interpolation problems [12].

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