Regularized Robust Estimators for Time Varying Uncertain Discrete-Time Systems

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Abstract—**This paper addresses the issue of robust filtering for time varying uncertain discrete time systems. The proposed robust filters are based on a regularized least-squares formulation and guarantee minimum state error variances. Simulation results indicate their superior performance over other robust filter designs.**

keywords: regularization, least-squares, robust filter, regularization parameter, parametric uncertainty.

I. Introduction

The Kalman filter is the optimal linear least-mean-squares estimator for systems that are described by linear state-space Markovian models [1]. However, when the model is not accurately known, the performance of the filter can deteriorate appreciably. This filter sensitivity to modeling errors has led to several works in the literature on the development of robust state-space filters; robust in the sense that they attempt to limit, in certain ways, the effect of model uncertainties on the overall filter performance. Some of the well known approaches to state-space estimation in this regard are $\mathcal{H}_\infty$ filtering, mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filtering, set-valued estimation, guaranteed-cost designs and minimum variance filtering (see [2]-[11]). In [11], a robust filter design framework was proposed that performs regularization as opposed to de-regularization. The design in [11] involved choosing certain Ricatti variables so as to enforce a local optimality and robustness property. In this paper, we pursue the design of such regularized robust filters and consider two general classes of uncertain state-space models. We consider uncertain model descriptions that involve norm bounded uncertainties for the output matrices, and stochastic and polytopic uncertainties for the state matrices; both descriptions are common in applications. For each class, we shall design robust filters that bound the state error covariance matrix *globally*. The robustness criterion used is different from prior robust designs (e.g., $\mathcal{H}_\infty$, guaranteed-cost or set-valued estimation) in that, it is based on robust *regularization*. In this way, the resulting filters are well suited for online/real-time filtering applications involving both time-invariant and time-variant models. Simulation results are included to illustrate the superior performance of the proposed robust filters over other robust designs.

II. Least-Squares with Uncertainties

Let $J(x) = x^T Q x + R(x)$ denote a cost function with

$$R(x) = \Big((A+\delta A)x-(b+\delta b)\Big)^T W \Big((A+\delta A)x-(b+\delta b)\Big) \quad (1)$$

where δA denotes an $N \times n$ perturbation to A, δb denotes an $N \times 1$ perturbation to b, and $\{\delta A, \delta b\}$ are assumed to satisfy a model of the form

$$\begin{bmatrix} \delta A & \delta b \end{bmatrix} = H\Delta \begin{bmatrix} E_a & E_b \end{bmatrix} \quad (2)$$

where Δ is an arbitrary contraction, $\|\Delta\| \le 1$, and $\{H, E_a, E_b\}$ are known quantities of appropriate dimensions. Consider then the constrained two player game problem

$$\hat{x} = \arg\min_x \max_{\{\delta A, \delta b\}} J(x) \quad (3)$$

subject to (2). The following result is proven in [12].

Theorem 1: The problem (2)–(3) has a unique solution $\hat{x}$ that is given by

$$\hat{x} = \left[\widehat{Q} + A^T \widehat{W} A\right]^{-1} \left[A^T \widehat{W} b + \widehat{\beta} E_a^T E_b\right] \quad (4)$$

where

$$\widehat{Q} = Q + \widehat{\beta} E_a^T E_a \quad (5)$$
$$\widehat{W} = W + WH(\widehat{\beta} I - H^T W H)^\dagger H^T W \quad (6)$$

and the scalar $\widehat{\beta}$ is determined from the optimization

$$\widehat{\beta} = \arg\min_{\beta \ge \|H^T W H\|} G(\beta) \quad (7)$$

where the function $G(\beta)$ is defined as follows:

$$G(\beta) = x^T(\beta) Q x(\beta) + \beta \|E_a x(\beta) - E_b\|^2 + [Ax(\beta) - b]^T W(\beta) [Ax(\beta) - b] \quad (8)$$

with

$$W(\beta) = W + WH(\beta I - H^T W H)^\dagger H^T W$$
$$Q(\beta) = Q + \beta E_a^T E_a$$

and

$$x(\beta) = \Big[Q(\beta) + A^T W(\beta) A\Big]^{-1} \Big[A^T W(\beta) b + \beta E_a^T E_b\Big] \quad (9)$$

[The notation $X^\dagger$ denotes the pseudo-inverse of X.]

$\diamond$

It was shown in [12], [16] that the function $G(\beta)$ has a unique global minimum (and no local minima) over the interval $\beta \ge \|H^T W H\|$, which means that the determination of $\widehat{\beta}$ can be pursued by employing standard search procedures without worrying about convergence to undesired local minima. It was further argued in [11] that a good approximation for $\hat{\beta}$ is to choose it as $\hat{\beta} = (1+\alpha)\beta_l$ for some $\alpha > 0$ and where $\beta_l = \|H^T W H\|$.

III. The State Space Models

We now use Thm. 1 to design two robust filters. Each filter will be applicable to a particular class of model uncertainties. Thus consider an $n-$dimensional state-space model of the form:

$$x_{k+1} = F_k x_k + G_k w_k \quad (10)$$
$$y_k = (H_k + \Delta H_k) x_k + v_k, \quad k \ge 0 \quad (11)$$

where $\{w_k, v_k\}$ are uncorrelated white zero-mean random processes with variances

$$E w_k w_k^* = W_k, \qquad E v_k v_k^* = V_k$$

and x_0 is a zero-mean random variable that is uncorrelated with $\{w_k, v_k\}$ for all k. The uncertainties ΔH_k are modelled as

$$\Delta H_k = M_k \Delta_k E_k \quad (12)$$

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where M_k and E_k are known matrices, while Δ_k is an arbitrary contraction, $\Delta_k^T\Delta_k \leq I$.

We shall consider two types of uncertainty descriptions for the state matrices F_k: one is in terms of polytopic uncertainties and the other is in terms of norm bounded stochastic uncertainties. In the first case, we assume that F_k lies inside a convex bounded polyhedral domain $\mathcal{K}_k$ described by m vertices as follows:

$$\mathcal{K}_k = \left\{ F_k = \sum_{i=1}^{i=m} \alpha_{i,k} F_{i,k}, \;\; \alpha_{i,k} \geq 0, \;\; \sum_{i=1}^{i=m} \alpha_{i,k} = 1 \right\} \tag{13}$$

Observe that $\mathcal{K}_k$ is allowed to vary with k. In the second case, we assume that F_k is described by

$$F_k = F_{k,c} + \Delta F_k, \quad \Delta F_k = N_k \bar{\Delta}_k J_k \tag{14}$$

for some known $F_{k,c}$ and where $\bar{\Delta}_k$ is a random matrix whose entries are zero mean and uncorrelated with each other, and such that

$$E \bar{\Delta}_k \bar{\Delta}_k^* \leq \rho_{\bar{\Delta}} I \tag{15}$$

for some known $\rho_{\bar{\Delta}}$.

IV. Robust State Space Filtering

When uncertainties are not present in the model (10)–(11), it is known that the optimal linear estimator for the state is the Kalman filter [18]. This filter admits a deterministic interpretation as the solution to a regularized least-squares problem as follows. Let [1]

$$\hat{x}_{k|k-1} \;\triangleq\; \text{an estimate of } x_k \text{ given } \{y_0, y_1, \ldots, y_{k-1}\}$$

$$\hat{x}_{k|k} \;\triangleq\; \text{an estimate of } x_k \text{ given } \{y_0, y_1, \ldots, y_{k-1}, y_k\}$$

Given the predicted estimate $\hat{x}_{k|k-1}$ and an observation y_k, the filtered estimate $\hat{x}_{k|k}$ that is computed by the Kalman filter is the solution of

$$\min_x \left[\|x - \hat{x}_{k|k-1}\|^2_{P_k^{-1}} \; + \; \|y_k - H_k x\|^2_{R_k^{-1}} \right] \tag{16}$$

where P_k and R_k are the state error covariance and the measurement noise covariance matrices, respectively. When uncertainties are present in $\{H_k, F_k\}$, we formulate a robust version of (16) by solving instead the min-max problem :

$$\min_x \max_{\delta H_k, \delta F_k} \left(\|x - \hat{x}_{k|k-1}\|^2_{P_k^{-1}} \; + \; \|y_k - (H_k + \delta H_k)x\|^2_{R_k^{-1}} \right) \tag{17}$$

This formulation was proposed in [11]. Compared with other robust designs, it has the advantage of performing regularization as opposed to de-regularization, a property that is useful for on-line/real-time operation since the resulting filter will not require existence conditions. In [11], the weighing matrices P_k in (17) were propagated through a Ricatti recursion that enforces a local optimality criterion. In our first filter below, we shall instead determine P_k so as to minimize the state error covariance matrix *globally*. We do so by showing how to reparametrize P_k and R_k in terms of a single parameter Q_k, over which the global minimization of the error covariance matrix reduces to a linear convex problem. In our second filter, we shall derive a more efficient procedure for updating P_k. The procedure does not require solving a linear convex problem at each iteration and has the same computational complexity as the Kalman filter.

[1] When uncertainties are not present, the qualification "estimate" refers to the linear-least-mean-squares estimate.

A. Polytopic Uncertainties

We consider first the case of polytopic uncertainties in F_k as in (13). Our objective is to design a robust linear estimator for the state variable x_k of the form

$$\hat{x}_{k|k} = F_{p,k}\hat{x}_{k|k-1} + K_{p,k} y_k, \quad k \geq 0 \tag{18}$$

$$\hat{x}_{k+1|k} = F_{k,c}\hat{x}_{k|k} \tag{19}$$

for some matrices $F_{p,k}$ and $K_{p,k}$ to be determined in order to minimize the worst case error variance of the state for all uncertainties, and where $F_{k,c}$ denotes the centroid of the polytope $\mathcal{K}_k$:

$$F_{k,c} = \frac{1}{m}\sum_{i=1}^{i=m} F_{i,k} \tag{20}$$

Assume first that the F_k are fixed; we will incorporate the uncertainties in F_k soon. With uncertainties in the output matrices H_k alone, problem (17) becomes

$$\min_x \max_{\delta H_k} \left(\|x - \hat{x}_{k|k-1}\|^2_{P_k^{-1}} \; + \; \|y_k - (H_k + \delta H_k)x\|^2_{R_k^{-1}} \right) \tag{21}$$

which can be written more compactly in the form (1)–(3) with the identifications:

$$\begin{aligned}
x &\longleftarrow \{x_k - \hat{x}_{k|k-1}\}, \quad b \longleftarrow y_k - H_k\hat{x}_{k|k-1} \\
\delta A &\longleftarrow M_k \Delta_k E_k \\
\delta b &\longleftarrow -M_k \Delta_k E_k \hat{x}_{k|k-1}, \quad Q \longleftarrow P_k^{-1} \\
W &\longleftarrow R_k^{-1}, \; H \longleftarrow M_k, \; E_a \longleftarrow E_k \\
E_b &\longleftarrow -E_k\hat{x}_{k|k-1}, \; \Delta \longleftarrow \Delta_k, \; A \longleftarrow H_k
\end{aligned}$$

From Thm. 1, the solution $\hat{x}_{k|k}$ of (21) is given by

$$\begin{aligned}
\hat{x}_{k|k} = \hat{x}_{k|k-1} + &(P_k^{-1} + \hat{\beta}E_k^T E_k + H_k^T \hat{R}_k^{-1} H_k)^{-1} \\
&\{H_k^T \hat{R}_k^{-1}(y_k - H_k\hat{x}_{k|k-1}) - \hat{\beta}E_k^T E_k \hat{x}_{k|k-1}\}
\end{aligned} \tag{22}$$

where

$$\hat{R}_k^{-1} = (R_k - \hat{\beta}^{-1} M_k M_k^T)^{-1} \tag{23}$$

If we now introduce the matrix

$$Q_k \triangleq (P_k^{-1} + \hat{\beta}E_k^T E_k + H_k^T \hat{R}_k^{-1} H_k)^{-1} \tag{24}$$

then the expression for $\hat{x}_{k|k}$ becomes

$$\begin{aligned}
\hat{x}_{k|k} =& (I - \hat{\beta}Q_k E_k^T E_k - Q_k H_k^T \hat{R}_k^{-1} H_k)\hat{x}_{k|k-1} \\
&+ Q_k H_k^T \hat{R}_k^{-1} y_k
\end{aligned} \tag{25}$$

in terms of the parameter Q_k. Noting that w_k is a zero-mean white random process, we let the following be an estimate for x_{k+1} given the measurement y_k:

$$\hat{x}_{k+1|k} \triangleq F_{k,c}\hat{x}_{k|k} \tag{26}$$

We then get

$$\hat{x}_{k+1|k} = F_{p,k}\hat{x}_{k|k-1} + K_{p,k} y_k \tag{27}$$

where $F_{p,k}$ and $K_{p,k}$ are defined in terms of Q_k as

$$F_{p,k} = F_{k,c}(I - \hat{\beta}Q_k E_k^T E_k - Q_k H_k^T \hat{R}_k^{-1} H_k) \tag{28}$$

$$K_{p,k} = F_{k,c} Q_k H_k^T \hat{R}_k^{-1} \tag{29}$$

Denoting $\tilde{x}_k = x_k - \hat{x}_{k|k-1}$, we define the extended weight vector

$$\eta_k \triangleq \begin{pmatrix} x_k \\ \tilde{x}_k \end{pmatrix} \tag{30}$$

Then η_k satisfies

$$\eta_{k+1} = \bar{F}_k \eta_k + \bar{G}_k u_k \tag{31}$$

where

$$u_k = \begin{pmatrix} w_k \\ v_k \end{pmatrix}, \quad \bar{G} = \begin{pmatrix} G & 0 \\ G & -K_{p,k} \end{pmatrix} \tag{32}$$

$$\bar{F}_k = \begin{pmatrix} F_k & 0 \\ F_k - F_{p,k} - K_{p,k}H_k & F_{p,k} \end{pmatrix} \tag{33}$$

and the covariance matrix of η_k satisfies

$$\Pi_{k+1} = \bar{F}_k \Pi_k \bar{F}_k^T + \bar{G}_k S_k \bar{G}_k^T \tag{34}$$

where

$$S_k = \begin{pmatrix} W_k & 0 \\ 0 & V_k \end{pmatrix} \tag{35}$$

and Π_0 is the covariance matrix of η_0. Now observe that the expressions for $\{F_{p,k}, K_{p,k}\}$ are parametrized in terms of the single parameter Q_k. We shall then choose Q_k so as to minimize the covariance of η_k. In this way, the resulting filter will satisfy the robustness condition (21), in addition to minimizing the state error covariance. This is achieved as follows. First note that Q_k in (24) is to problem (21) as the matrix $\widehat{Q} + A^T \widehat{W} A$ in (4) is to problem (1)–(3). Therefore, Q_k must be positive definite so that the $\hat{x}_{k|k}$ is guaranteed to be the minimum of (21). Then we shall choose $Q_k > 0$ so as to minimize Π_{k+1} of (34). This can be obtained by solving

$$\min_{Q_k > 0} \quad \mathrm{Trace}(\Pi_{k+1}) \tag{36}$$

subject to the inequality

$$\Pi_{k+1} \geq \bar{F}_k \Pi_k \bar{F}_k^T + \bar{G}_k S_k \bar{G}_k^T \tag{37}$$

or, equivalently,

$$\begin{pmatrix} -\Pi_{k+1} & \bar{F}_k \Pi_k & \bar{G}_k S_k^{1/2} \\ \Pi_k \bar{F}_k^T & -\Pi_k & 0 \\ S_k^{T/2} \bar{G}_k^T & 0 & -I \end{pmatrix} \leq 0 \tag{38}$$

In order to incorporate the polytopic uncertainties in the F_k, as defined by the sets $\mathcal{K}_k$ in (13), we need to solve the above optimization problem with F_k taking values at the m vertices of the convex polytope $\mathcal{K}_k$, i.e., from the set $\{F_{1,k}, F_{2,k}, \ldots\ldots, F_{m,k}\}$. Since the inequality (38) is affine in F_k, the Q_k thus found will ensure minimum error covariance Π_k over all possible F_k in $\mathcal{K}_k$. Therefore, the time varying robust filter is given by (65)–(67), where Q_k is the positive definite solution of (51)–(38) with F_k taking values on the vertices of the convex polytope $\mathcal{K}_k$, and initializing $\Pi_0 = \mathrm{diag}\{P_o, \epsilon I\}$ for some positive definite P_o. Note that there always exists a solution to (51)–(38). This is because, at every time instant k, $Q_k = \epsilon I$ for $\epsilon > 0$ is a feasible solution. The filter is summarized in Table 1.

<u>Infinite horizon case :</u> In this paper, by the notion of stability in the infinite horizon case, we mean that the variables associated with the filter are bounded for all k. Assume $\|F_{k,c}\| < 1$ and choose Q_k to satisfy (51)–(38) as well as $\|\bar{F}_k\| < 1$. This additional constraint is easily represented in terms of a linear matrix inequality in the variable Q_k as

$$\begin{pmatrix} I & \bar{F}_k^T \\ \bar{F}_k & I \end{pmatrix} > 0 \tag{39}$$

This condition guarantees that Π_k will remain bounded for all k.

<u>Assumed uncertain model</u>. Eqs. (10)–(13).

<u>Initial conditions</u>: $\hat{x}_0 = 0$, $\Pi_0 = \mathrm{diag}\{P_o, \epsilon I\}$.
<u>Step 1.</u> If $M_k = 0$, then set $\widehat{\beta}_k = 0$. Otherwise, set instead $\hat{\beta} = (1+\alpha)\beta_{l,k}$ where $\beta_{l,k} = \|M_k^T R_k^{-1} M_k\|$.

<u>Step 2.</u> Using Π_k, compute $\{Q_k, \Pi_{k+1}\}$ by solving

$$\min_{Q_k > 0} \quad \mathrm{Trace}(\Pi_{k+1})$$

subject to the inequality

$$\begin{pmatrix} -\Pi_{k+1} & \bar{F}_k \Pi_k & \bar{G}_k S_k^{1/2} \\ \Pi_k \bar{F}_k^T & -\Pi_k & 0 \\ S_k^{T/2} \bar{G}_k^T & 0 & -I \end{pmatrix} \leq 0$$

where $\{\bar{F}_k, \bar{G}_k, S_k\}$ are defined by (32),(33) and (35).
<u>Step 3.</u> Update $\hat{x}_k$ to $\hat{x}_{k+1}$ as

$$\hat{x}_{k+1} = F_{p,k}\hat{x}_k + K_{p,k} y_k$$

where

$$\begin{aligned} F_{p,k} &= F_{k,c}(I - \hat{\beta} Q_k E_k^T E_k - Q_k H_k^T \hat{R}_k^{-1} H_k) \\ K_{p,k} &= F_{k,c} Q_k H_k^T \hat{R}_k^{-1} \\ \hat{R}_k^{-1} &= (R_k - \hat{\beta}^{-1} M_k M_k^T)^{-1} \end{aligned}$$

with $F_{k,c}$ from (20).

Table 1: Regularized robust filter for polytopic uncertainties.

B. *Stochastic Uncertainties*

We now consider the case of norm bounded stochastic uncertainties in F_k as in (14). We derive two filters; one through convex optimization method and the other through a suboptimal ricatti equation method. We first look at a filter that is derived in terms of convex optimization.

B.1 A robust filter

Here again, our objective is to design a robust linear estimator for the state variable x_k of the form

$$\hat{x}_{k|k} = F_{p,k}\hat{x}_{k|k-1} + K_{p,k} y_k, \quad k \geq 0 \tag{40}$$

$$\hat{x}_{k+1|k} = F_{k,c}\hat{x}_{k|k} \tag{41}$$

for some matrices $F_{p,k}$ and $K_{p,k}$ to be determined in order to minimize the worst case error variance of the state for all uncertainties, and where $F_{k,c}$ denotes the nominal state matrix. Proceeding in the same manner as in the previous section from the robustness condition (21), we know that the expressions for $\{F_{p,k}, K_{p,k}\}$ will be parametrized in terms of the parameter Q_k. We shall then choose Q_k so as to minimize the covariance of η_k. Here η_k satisfies :

$$\eta_{k+1} = (\bar{F}_{k,c} + \bar{N}_k \bar{\Delta} \bar{J}_k)\eta_k + \bar{G}_k u_k \tag{42}$$

where

$$u_k = \begin{pmatrix} w_k \\ v_k \end{pmatrix} \quad (43)$$

$$\bar{F}_{k,c} = \begin{pmatrix} F_{k,c} & 0 \\ F_{k,c} - F_{p,k} - K_{p,k}H_k & F_{p,k} \end{pmatrix} \quad (44)$$

$$\bar{N}_k = \begin{pmatrix} N_k & 0 \\ N_k & 0 \end{pmatrix} \quad (45)$$

$$\bar{J}_k = \begin{pmatrix} J_k & 0 \\ 0 & 0 \end{pmatrix} \quad (46)$$

and the covariance matrix of η_k satisfies

$$\Pi_{k+1} = E\{(\bar{F}_{k,c}+\bar{N}_k\bar{\Delta}\bar{J}_k)\Pi_k(\bar{F}_{k,c}+\bar{N}_k\bar{\Delta}\bar{J}_k)^T\}+\bar{G}_kS_k\bar{G}_k^T \quad (47)$$

where E denotes the expectation operator and

$$S_k = \begin{pmatrix} W_k & 0 \\ 0 & V_k \end{pmatrix} \quad (48)$$

Let $\check{\Pi}_k$ satisfy

$$\check{\Pi}_{k+1} = \bar{F}_{k,c}\check{\Pi}_k\bar{F}_{k,c}^T + \bar{G}_kS_k\bar{G}_k^T + \rho_{\bar{\Delta}}\hat{\alpha}_k\bar{N}_k\bar{N}_k^T \quad (49)$$

and choose $\hat{\alpha}_k$ such that $\hat{\alpha}_k - \bar{J}_k\check{\Pi}_k\bar{J}_k^T > 0$. Then

$$\hat{\alpha}_kI - \bar{J}_k\Pi_k\bar{J}_k^T > 0 \quad (50)$$

and

$$\Pi_{k+1} \leq \bar{F}_{k,c}\Pi_k\bar{F}_{k,c}^T + \bar{G}_kS_k\bar{G}_k^T + \rho_{\bar{\Delta}}\hat{\alpha}_k\bar{N}_k\bar{N}_k^T$$

This suggests that we can choose Q_k by constructing a sequence of matrices $\hat{\Pi}_k$ so as to solve:

$$\min_{Q_k > 0} \quad \text{Trace}(\hat{\Pi}_{k+1}) \quad (51)$$

subject to the inequality

$$\hat{\Pi}_{k+1} \geq \bar{F}_k\hat{\Pi}_k\bar{F}_k^T + \bar{G}_kS_k\bar{G}_k^T + \rho_{\bar{\Delta}}\hat{\alpha}_k\bar{N}_k\bar{N}_k^T \quad (52)$$

or, equivalently,

$$\begin{pmatrix} -\hat{\Pi}_{k+1} + \rho_{\bar{\Delta}}\hat{\alpha}_k\bar{N}_k\bar{N}_k^T & \bar{F}_k\hat{\Pi}_k & \bar{G}_kS_k^{1/2} \\ \hat{\Pi}_k\bar{F}_k^T & -\hat{\Pi}_k & 0 \\ S_k^{T/2}\bar{G}_k^T & 0 & -I \end{pmatrix} \leq 0 \quad (53)$$

Note that $\Pi_k \leq \hat{\Pi}_k$ for every k. The filter hence derived is summarized in Table 2.

Infinite horizon case : We now describe some sufficient conditions for infinite horizon stability (i.e., for the matrices $\{\hat{\Pi}_k, \check{\Pi}_k\}$ to remain bounded). Assume that $\|F_{k,c}\| < 1$ and choose Q_k to satisfy (51)–(53) as well as $\|\bar{F}_{k,c}\| < 1$. This additional constraint is easily represented in terms of a linear matrix inequality in the variable Q_k as

$$\begin{pmatrix} I & \bar{F}_{k,c}^T \\ \bar{F}_{k,c} & I \end{pmatrix} > 0 \quad (54)$$

Let a fixed $\hat{\alpha}$ be chosen in the following way. Consider the recursion

$$\check{\Pi}_{k+1} = \bar{F}_{k,c}\check{\Pi}_k\bar{F}_{k,c}^T + \bar{G}_kS\bar{G}_k^T + \rho_{\bar{\Delta}}\hat{\alpha}_k\bar{N}_k\bar{N}_k^T$$

Then

$$\|\check{\Pi}_{k+1}\| \leq \|\bar{F}_{k,c}\|^2\|\check{\Pi}_k\| + \|\bar{G}_kS\bar{G}_k^T\| + \rho_{\bar{\Delta}}\hat{\alpha}_k\|\bar{N}_k\bar{N}_k^T\|$$

Let $\beta = \|\bar{F}_{k,c}\|^2 < 1$, $\|\bar{N}_k\bar{N}_k^T\| < \gamma$ and since Q_k lies inside a bounded set $\mathcal{C}$, assume $\|\bar{G}_kS_k\bar{G}_k^T\| < \zeta$. Then

$$\|\check{\Pi}_{k+1}\| \leq \beta^k\|\check{\Pi}_0\| + \sum_{i=0}^{i=k-1}\beta^{k-1-i}\zeta + \sum_{i=0}^{i=k-1}\beta^{k-1-i}\rho_{\bar{\Delta}}\hat{\alpha}_k\gamma$$

or

$$\|\check{\Pi}_{k+1}\| \leq \|\check{\Pi}_0\| + \frac{1}{1-\beta}(\zeta + \rho_{\bar{\Delta}}\hat{\alpha}_k\gamma)$$

We then have

$$\begin{aligned} \|\bar{J}_{k+1}\check{\Pi}_{k+1}\bar{J}_{k+1}^T\| &\leq \|\check{\Pi}_{k+1}\|\|\bar{J}_{k+1}\|^2 \\ &\leq \chi(\|\check{\Pi}_0\| + \frac{1}{1-\beta}(\zeta + \rho_{\bar{\Delta}}\hat{\alpha}_k\gamma)) \end{aligned}$$

where $\|\bar{J}_{k+1}\|^2 \leq \chi$. If $\hat{\alpha}_k = \hat{\alpha}$ can be chosen for each k such that

$$\hat{\alpha} > \chi\left\{\|\check{\Pi}_0\| + \frac{1}{1-\beta}(\zeta + \rho_{\bar{\Delta}}\hat{\alpha}\gamma)\right\} \quad (55)$$

we see that $\|\bar{J}_{k+1}\check{\Pi}_{k+1}\bar{J}_{k+1}^T\|$ is bounded for all k. A fixed $\hat{\alpha}$ chosen according to condition (55) along with (54) guarantee that $\hat{\Pi}_k$ and $\check{\Pi}_k$ will remain bounded for all k.

Assumed uncertain model. Eqs. (14).

Initial conditions: $\hat{x}_0 = 0$, $\hat{\Pi}_0 = \check{\Pi}_0 = \text{diag}\{P_o, \epsilon I\}$.

Step 1. If $M_k = 0$, then set $\widehat{\beta}_k = 0$. Otherwise, set instead $\hat{\beta} = (1+\alpha)\beta_{l,k}$ where $\beta_{l,k} = \|M_k^TR_k^{-1}M_k\|$. Choose $\hat{\alpha}_k$ such that $\hat{\alpha}_k - \bar{J}_k\check{\Pi}_k\bar{J}_k^T > 0$ for $\check{\Pi}_k$ satisfying

$$\check{\Pi}_{k+1} = \bar{F}_{k,c}\check{\Pi}_k\bar{F}_{k,c}^T + \bar{G}_kS_k\bar{G}_k^T + \rho_{\bar{\Delta}}\hat{\alpha}_k\bar{N}_k\bar{N}_k^T$$

Step 2. Using $\hat{\Pi}_k$ and $\hat{\alpha}_k$ compute, $\{Q_k, \hat{\Pi}_{k+1}\}$ by solving

$$\min_{Q_k > 0} \quad \text{Trace}(\hat{\Pi}_{k+1})$$

subject to the inequality

$$\begin{pmatrix} -\hat{\Pi}_{k+1} + \rho_{\bar{\Delta}}\hat{\alpha}_k\bar{N}_k\bar{N}_k^T & \bar{F}_k\hat{\Pi}_k & \bar{G}_kS_k^{1/2} \\ \hat{\Pi}_k\bar{F}_k^T & -\hat{\Pi}_k & 0 \\ S_k^{T/2}\bar{G}_k^T & 0 & -I \end{pmatrix} \leq 0$$

where $\{\bar{F}_k, \bar{G}_k, S_k\}$ are defined by (32),(33) and (35).

Step 3. Update $\hat{x}_k$ to $\hat{x}_{k+1}$ as

$$\hat{x}_{k+1} = F_{p,k}\hat{x}_k + K_{p,k}y_k$$

where

$$\begin{aligned} F_{p,k} &= F_{k,c}(I - \hat{\beta}Q_kE_k^TE_k - Q_kH_k^T\hat{R}_k^{-1}H_k) \\ K_{p,k} &= F_{k,c}Q_kH_k^T\hat{R}_k^{-1} \\ \hat{R}_k^{-1} &= (R_k - \hat{\beta}^{-1}M_kM_k^T)^{-1} \end{aligned}$$

Table 2: A regularized robust filter for stochastic uncertainties.

B.2 Another Robust Filter

Consider again equations (47)–(51). We will now show how to generate a *new* sequence of matrices $\tilde{\Pi}_k$ and $\hat{\Pi}_k$ such that $\Pi_k \leq \tilde{\Pi}_k \leq \hat{\Pi}_k$. This construction will enable us to avoid the solution of the optimization problem (51)-(53) at each iteration thus reducing the computational complexity. At every iteration we would find a suboptimal Q_k that minimizes the bound $\tilde{\Pi}_k$ of the error covariance matrix Π_k. At time instant $k+1$, assuming we have $\Pi_k \le \tilde{\Pi}_k \le \hat{\Pi}_k$, then the state error covariance is bounded by the $(2,2)$ block element of the matrix $\tilde{\Pi}_{k+1}$ defined by

$$\tilde{\Pi}_{k+1} = \bar{F}_{k,c}\hat{\Pi}_k\bar{F}_{k,c}^T + \bar{G}_k S_k \bar{G}_k^T + \rho_{\bar{\Delta}}\hat{\alpha}_k \bar{N}_k \bar{N}_k^T$$

The sequence of matrices $\hat{\Pi}_k$ are restricted to have a structure of the form:

$$\hat{\Pi}_k = \begin{pmatrix} Y_k & Y_k - Z_k \\ Y_k - Z_k & Y_k - Z_k \end{pmatrix} \quad (56)$$

and are generated as will be explained in the sequel. Moreover, with at time instant k, with $\hat{\Pi}_k$ given as in (56) we have that

$$\tilde{\Pi}_{k+1} = \begin{pmatrix} \tilde{Y}_{k+1} & \tilde{X}_{k+1} \\ \tilde{X}_{k+1}^T & \tilde{Y}_{k+1} - \tilde{Z}_{k+1} \end{pmatrix} \quad (57)$$

where

$$\begin{aligned}
\tilde{Y}_{k+1} &= F_{k,c}Y_kF_{k,c}^T + \rho_{\bar{\Delta}}\hat{\alpha}_k N_k N_k^T + G_k W_k G_k^T \\
\tilde{Z}_{k+1} &= F_{k,c}Z_kF_{k,c}^T - F_{k,c}Q_kH_k^TV_k^{-1}H_kQ_kF_{k,c}^T \\
&+F_{k,c}(Y_k - Z_k)H_k^TV_k^{-1}H_kQ_kF_{k,c}^T \\
&+F_{k,c}Q_kH_k^TV_k^{-1}H_k(Y_k - Z_k)F_{k,c}^T \\
&-\beta^2F_{k,c}Q_kE_k^TE_kZ_kE_k^TE_kQ_kF_{k,c} \\
&-F_{k,c}Q_kH_k^TV_k^{-1}H_k(Y_k - Z_k)H_k^TV_k^{-1}H_kQ_kF_{k,c}^T \\
\tilde{X}_{k+1} &= \beta F_{k,c}Y_kE_k^TE_kQ_kF_{k,c}^T + F_{k,c}(Y_k - Z_k)F_{p,k}^T \\
&+\rho_{\bar{\Delta}}\hat{\alpha}_k N_k N_k^T + G_k W_k G_k^T
\end{aligned}$$

Also, the $(2,2)$ block element of $\tilde{\Pi}_{k+1}$ is given by

$$\tilde{\Pi}_{k+1}^{2,2} = C_{1,k} + C_{2,k} \quad (58)$$

where

$$\begin{aligned}
C_{1,k} &= F_{k,c}(Y_k - Z_k)F_{k,c}^T + \rho_{\bar{\Delta}}\hat{\alpha}_k N_k N_k^T \\
&+G_kW_kG_k^T + F_{k,c}Q_kH_k^TV_k^{-1}H_kQ_kF_{k,c}^T \\
&-F_{k,c}(Y_k - Z_k)H_k^TV_k^{-1}H_kQ_kF_{k,c}^T \\
&-F_{k,c}Q_kH_k^TV_k^{-1}H_k(Y_k - Z_k)F_{k,c}^T \\
&+F_{k,c}Q_kH_k^TV_k^{-1}H_k(Y_k - Z_k)H_k^TV_k^{-1}H_kQ_kF_{k,c}^T \\
C_{2,k} &= \beta^2F_{k,c}Q_kE_k^TE_kZ_kE_kE_k^TQ_kF_{k,c}
\end{aligned}$$

In deriving the above expressions, without loss of generality, we have chosen the weighing matrices R_k such that $\hat{R}_k = V_k$. It is usually hard to find a positive-definite Q_k that minimizes $\tilde{\Pi}_{k+1}^{2,2}$. Hence, we will find a suboptimal solution as follows. We shall bound $C_{2,k}$ by λI for some $\lambda > 0$. The choice of λI will become clear in the sequel. Then we will find a lower bound for $C_{1,k} + \lambda I$, which occurs at the lower bound of $C_{1,k}$. After some considerable algebra, we can show that for

$$Q_{k,opt} = (Y_k - Z_k) - (Y_k - Z_k)H_k^T\bar{R}_{e,k}^{-1}H_k(Y_k - Z_k)$$

and $\bar{R}_{e,k} = V_k + H_k(Y_k - Z_k)H_k^T$, we have

$$\frac{\partial C_{1,k}}{Q_k} = 0 \quad \text{and} \quad \frac{\partial^2 C_{1,k}}{Q_k} > 0 \quad (59)$$

That is, $Q_{k,opt}$ minimizes $C_{1,k}$. It can be seen that $Q_{k,opt}$ is positive definite and hence guarantees a unique solution to the problem (21). Now note that $\tilde{\Pi}_{k+1}^{2,2}$ is quadratic in the variable Q_k and hence its value, for any arbitrary Q_k, is proportional to $\|Q_k - Q_{k,opt}\|_F$, where $\|.\|_F$ denotes the Frobenius norm of the argument. Also note that the set of all Q_k that guarantee $C_{2,k} < \lambda I$ is a convex bounded set $\mathcal{C}$ about the origin in the normed vector space of all matrices of dimension $n \times n$. If $Q_{k,opt}$ lies inside $\mathcal{C}$, we choose Q_k as $Q_{k,opt}$. Otherwise, Q_k is chosen as a matrix that is closest to $Q_{k,opt}$ in the Frobenius norm and simultaneously lying inside the set $\mathcal{C}$. Now we will derive an upper bound for $\tilde{\Pi}_{k+1}$ in the form (which is compatible with the form we started with in (56)):

$$\hat{\Pi}_{k+1} = \begin{pmatrix} Y_{k+1} & Y_{k+1} - Z_{k+1} \\ Y_{k+1} - Z_{k+1} & Y_{k+1} - \hat{Z}_{k+1} \end{pmatrix} \quad (60)$$

for some matrices Y_{k+1} and Z_{k+1}. Choose ψ_k as the maximum singular value of $I + B$ where

$$\begin{aligned}
B &= F_{k,c}Q_kH_k^TV_k^{-1}H_k(Y_k - Z_k)H_k^TV_k^{-1}H_kQ_kF_{k,c}^T \\
&+\beta^2F_{k,c}Q_kE_k^TE_kZ_kE_k^TE_kQ_kF_{k,c} \\
&+F_{k,c}Q_kH_k^TV_k^{-1}H_kQ_kF_{k,c}^T \\
&-F_{k,c}Q_kH_k^TV_k^{-1}H_k(Y_k - Z_k)F_{k,c}^T \\
&-\beta F_{k,c}Z_kQ_kE_k^TE_kF_{k,c}^T
\end{aligned}$$

Assumed uncertain model. Eqs. (10)–(13) and (14)–(15).
Initial conditions: $\hat{x}_0 = 0$, $Y_0 = I$, $Z_0 = \mu I$,

$$\Pi_0 = \begin{pmatrix} Y_0 & Y_0 - Z_0 \\ Y_0 - Z_0 & Y_0 - Z_0 \end{pmatrix}$$

Step 1a. Using $\{V_k, H_k, Y_k, Z_k\}$ compute $\{R_{e,k}, Q_k\}$:

$$\begin{aligned}
R_{e,k} &= V_k + H_k(Y_k - Z_k)H_k^T \\
Q_{k,opt} &= (Y_k - Z_k) - (Y_k - Z_k)H_k^T\bar{R}_{e,k}^{-1}H_k(Y_k - Z_k)
\end{aligned}$$

Step 1b. If $M_k = 0$, then set $\widehat{\beta}_k = 0$. Otherwise, set $\widehat{\beta}_k = (1+\alpha)\beta_{l,k}$, $\alpha > 0$. Determine the largest $Q_k = \xi Q_{k,opt}$ for some positive ξ such that $C_{2,k} < \lambda I$ for some small $\lambda > 0$.

Step 2. Compute the parameters:

$$\begin{aligned}
F_{p,k} &= F_{k,c}(I - Q_k\beta E_k^TE_k - Q_kH_k^T\hat{R}_k^{-1}H_k) \\
K_{p,k} &= F_{k,c}Q_kH_k^T\hat{R}_k^{-1} \\
B &= F_{k,c}Q_kH_k^TV_k^{-1}H_k(Y_k - Z_k)H_k^TV_k^{-1}H_kQ_kF_{k,c}^T \\
&+\beta^2F_{k,c}Q_kE_k^TE_kZ_kE_k^TE_kQ_kF_{k,c} \\
&+F_{k,c}Q_kH_k^TV_k^{-1}H_kQ_kF_{k,c}^T \\
&-F_{k,c}Q_kH_k^TV_k^{-1}H_k(Y_k - Z_k)F_{k,c}^T \\
&-\beta F_{k,c}Z_kQ_kE_k^TE_kF_{k,c}^T \\
\psi_k &= \|I + B\| \\
\hat{\alpha}_k &= \|\bar{J}_k\check{\Pi}_k\bar{J}_k^T\|
\end{aligned}$$

Step 3. Now update $\{Y_k, Z_k, \Pi_k, \hat{x}_k\}$ to $\{Y_{k+1}, Z_{k+1}, \Pi_{k+1}, \hat{x}_{k+1}\}$ as follows:

$$\begin{aligned}
Y_{k+1} &= F_{k,c}Y_kF_{k,c}^T + \rho_{\bar{\Delta}}\hat{\alpha}_kN_kN_k^T + G_kW_kG_k^T + \psi_k^2 I \\
Z_{k+1} &= F_{k,c}Z_kF_{k,c}^T - F_{k,c}Q_kH_k^TV_k^{-1}H_kQ_kF_{k,c}^T \\
&+F_{k,c}(Y_k - Z_k)H_k^TV_k^{-1}H_kQ_kF_{k,c}^T \\
&+F_{k,c}Q_kH_k^TV_k^{-1}H_k(Y_k - Z_k)F_{k,c}^T \\
&-F_{k,c}Q_kH_k^TV_k^{-1}H_k(Y_k - Z_k)H_k^TV_k^{-1}H_kQ_kF_{k,c}^T \\
&+\psi_k^2 I - I \\
\check{\Pi}_{k+1} &= \bar{F}_{k,c}\check{\Pi}_k\bar{F}_{k,c}^T + \bar{G}_kS_k\bar{G}_k^T + \rho_{\bar{\Delta}}\hat{\alpha}_k\bar{N}_k\bar{N}_k^T \\
\hat{x}_{k+1} &= F_{p,k}\hat{x}_k + K_{p,k}y_k
\end{aligned}$$

Table 3: A second regularized robust filter for stochastic uncertainties.

TABLE IV

ERROR VARIANCE WITH POLYTOPIC UNCERTAINTIES IN F_k.

Filters	error variance
Proposed filter	22.8dB
Regularized robust filter of [11]	26.9dB
Guaranteed-cost filter [2]	30dB
Set-valued filter [3]	34.47dB
Kalman filter with nominal model	31.18dB

TABLE V

ERROR VARIANCE WITH STOCHASTIC UNCERTAINTIES IN F_k.

Filters	error variance
Proposed filter from table 2	21.85dB
Proposed filter from table 3	21.20dB
Regularized robust filter of [11]	22.68dB
Guaranteed-cost filter [2]	25.3dB
Set-valued filter [3]	25.9dB
Kalman filter with nominal model	39.5dB

Now with

$$Y_{k+1} = \psi_k^2 I + \tilde{Y}_{k+1} \quad (61)$$
$$Z_{k+1} = \tilde{Z}_{k+1} + \psi_k^2 I - I \quad (62)$$

$\hat{\Pi}_{k+1}$ is an upper bound of $\tilde{\Pi}_{k+1}$. This is because

$$\hat{\Pi}_{k+1} - \tilde{\Pi}_{k+1} = \begin{pmatrix} \psi_k^2 I & I+B \\ (I+B)^T & I \end{pmatrix} > 0 \quad (63)$$

We will now in fact choose $\hat{\alpha}_k$ such that $\hat{\alpha}_k - \bar{J}_k T_k \bar{J}_k^T > 0$ for T_k satisfying

$$T_k = \bar{F}_{k-1,c} T_{k-1} \bar{F}_{k-1,c}^T + \bar{G}_{k-1} S_{k-1} \bar{G}_{k-1}^T + \rho_{\bar{\Delta}} \hat{\alpha}_{k-1} \bar{N}_{k-1} \bar{N}_{k-1}^T \quad (64)$$

Note that $\hat{\alpha}_k$ chosen as explained above implies that $\hat{\alpha}_k$ satisfies (50). We now state the filter :

$$\hat{x}_{k+1|k} = F_{p,k} \hat{x}_{k|k-1} + K_{p,k} y_k \quad (65)$$

where $F_{p,k}$ and $K_{p,k}$ are defined in terms of Q_k as

$$F_{p,k} = F_{k,c}(I - \hat{\beta} Q_k E_k^T E_k - Q_k H_k^T \hat{R}_k^{-1} H_k) \quad (66)$$
$$K_{p,k} = F_{k,c} Q_k H_k^T \hat{R}_k^{-1} \quad (67)$$

and Q_k is determined in terms of Y_k and Z_k at every k as explained before. The designed filter is shown in Table 3.

V. SIMULATIONS

To illustrate the operation of the filter developed for deterministic uncertainties, we choose an implementation of order 2 with $E_k = [.12 \quad .12]$, and $M_k = 1$ for all k. The uncertain state matrices F_k are assumed to lie inside the convex polytope

$$F_k = \begin{pmatrix} .9802 & .0196 + \delta \\ 0 & .5802 + \delta \end{pmatrix} \quad (68)$$

with $|\delta| \leq 0.4982$ The uncertainties in the output matrices H_k are determined by $M_k = 1$, $E_k = [.4 \quad .4]$ and $G_k = [-6 \quad 1]$. Table 2 shows the average squared state-error values (averaged over 50 experiments) for the Kalman filter, the proposed filter, the set-valued estimation filter [3], the guaranteed cost filter [2] and the filter of [11]. To illustrate the filter developed for stochastic uncertainties, we choose an implementation of order 2 with $E_k = [.8 \quad .8]$, $M_k = 1$ for all k. The uncertain state matrices F_k are assumed to be

$$F_k = \begin{pmatrix} .9802 + \bar{\Delta} & .4196 + \bar{\Delta} \\ \bar{\Delta} & .8802 + \bar{\Delta} \end{pmatrix} \quad (69)$$

with $|\bar{\Delta}| \leq 0.4982$. Table 3 shows the average squared state-error values in this case.

VI. CONCLUSION

In this paper we developed two regularized robust filters for state-space estimation. The design procedure is through the solution of a regularized weighted recursive least squares problem and it enforces minimum state error variance. The proposed filters outperform earlier robust designs and are suitable for on-line/real-time filtering applications since they do not require existence conditions.

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