STABILITY ANALYSIS OF AN ADAPTIVE STRUCTURE FOR SIGMA DELTA MODULATION

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ABSTRACT: In previous work [1], we have proposed an adaptive sigma delta modulator with improved dynamic range. The modulator adapts the step-size of the quantizer from estimates of the quantizer input. In this paper, we conduct a stability analysis of the new modulator.

1. INTRODUCTION

Adaptive Sigma Delta Modulation (ASDM) attempts to increase the dynamic range of sigma delta modulators while keeping the quantization noise as low as possible. ASDM achieves this objective by scaling either the input signal or the step-size of the quantizer through an estimation of the input signal strength. This estimation can be done from the input signal itself or from the modulator output as shown in Figure 1. Input scaling is shown in part a of the figure while step-size scaling is shown in part b. Using the input signal to perform the estimation is known as forward estimation while using the output signal is known as backward estimation. Adaptation could be done continuously or sporadicaly in time. Moreover, the value of the adaptation signal $d(n)$ could be continuous in amplitude or restricted to a specific range of values.

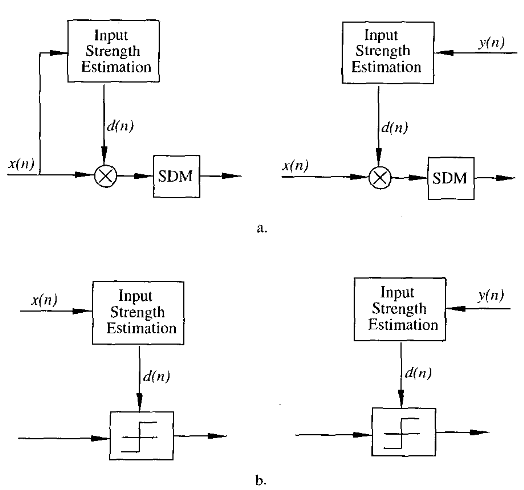

Figure 1: Adaptation schemes used in conventional ASDM's. a. Input scaling b. Quantizer step-size scaling.

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Several adaptation techniques have been investigated in the literature [2]-[6]. We developed an alternative scheme for adapting the quantization step-size in [1]. The scheme is based on estimating the amplitude of the quantizer input instead of the input signal itself. This estimate is then used to adapt the step size of the quantizer. In this paper, we perform a stability analysis of this proposed adaptive modulation structure.

2. NEW ASDM STRUCTURE

Figure 2 shows the basic structure of the proposed adaptive SDM with one bit quantizer from [1]. The modulation and demodulation blocks are shown in parts a and b, respectively. Higher-order ASDMs can be implemented by replacing the integrator by higher-order noise shaping filters. Furthermore, multi-loop and multi-stage adaptors can be adopted to improve the performance of the modulator.

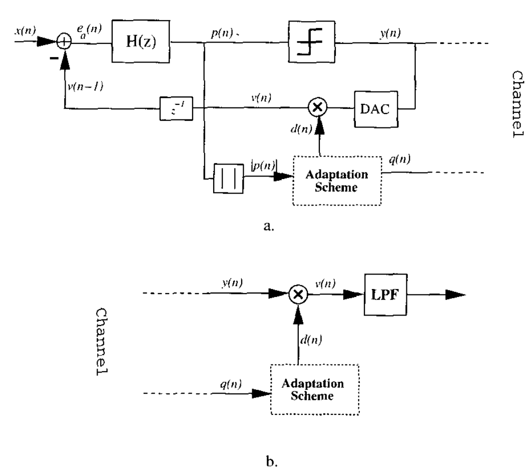

Figure 2: Block diagram of the proposed structure. a. Modulator b. Demodulator.

The error signal $e_a(n)$ is given by

$$e_a(n) = x(n) - v(n-1), \tag{1}$$

which is passed through the noise shaping filter $H(z)$. As a special case, if $H(z)$ is a simple integrator, then

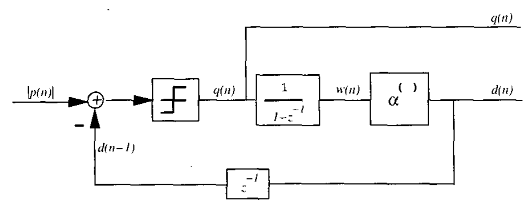

Figure 3: Adaptation scheme of the proposed modulator.

$$p(n) = p(n-1) + e_a(n) \;\text{ with }\; p(0) = 0. \qquad (2)$$

The filter output $p(n)$ is quantized using a one-bit quantizer to produce the signal $y(n)$. In other words,

$$y(n) = \text{sign}\,[p(n)]\,. \qquad (3)$$

The one-bit DAC is assumed to be ideal and thus has a unity transfer function.

The adapter generates a scaling signal $d(n)$, which is an approximation of the amplitude of the quantizer input signal $p(n)$. The encoded signal $v(n)$ is then given by

$$v(n) = y(n)d(n). \qquad (4)$$

Notice that if $d(n) = |p(n)|$ then we shall have

$$V(z)/X(z) = 1. \qquad (5)$$

The adaptation block used in this study is shown in Figure 3, which is similar in structure to a delta modulator with an additional exponent term α. The purpose of this additional term is to increase the tracking capability of the adapter.

The adaptation signal $d(n)$ is constructed as follows:

$$d(n) = \alpha^{q(n)} d(n-1), \qquad (6)$$

where the binary sequence $q(n)$ is generated from

$$q(n) = \begin{cases} +1, & \text{if } |p(n)| > d(n-1), \\ -1, & \text{otherwise.} \end{cases} \qquad (7)$$

In other words,

$$q(n) = \text{sign}[|p(n)| - d(n-1)]. \qquad (8)$$

The two binary sequences $y(n)$ and $q(n)$ are carried out to the demodulation part as shown in Figure 2b. There, the signal $v(n)$ is reconstructed using equations (4) and (6). Finally, the reconstructed signal is filtered using a low-pass filter as usually done in conventional sigma delta modulators.

3. STABILITY ANALYSIS OF THE MODULATOR

In this section the stability of the new modulator is analyzed. The analysis is restricted to the case where $H(z)$ is a simple integrator as in (2).

3.1. Equivalent Structure of the Modulator

Consider the modulator shown in Figure 2 with the adapter shown in Figure 3. Taking the logarithm of both sides of equation (6) we get

$$\log_\alpha(d(n)) = \log_\alpha(d(n-1)) + q(n). \qquad (9)$$

Using the fact that the logarithm is an increasing function, we can write

$$\begin{aligned} q(n) &\triangleq \text{sign}\,[|p(n)| - d(n-1)] \\ &= \text{sign}\,[\log_\alpha(|p(n)|) - \log_\alpha(d(n-1))] \end{aligned} \qquad (10)$$

Now let

$$x_d(n) \triangleq \log_\alpha(|p(n)|), \qquad (11)$$

and

$$y_d(n) \triangleq \log_\alpha(d(n)). \qquad (12)$$

From equations (9)-(12) we get

$$y_d(n) = y_d(n-1) + \text{sign}\,[x_d(n) - y_d(n-1)]\,. \qquad (13)$$

This dynamic equation characterizes a delta modulator as illustrated in Figure 4 part a. Its linearized version is shown in part b.

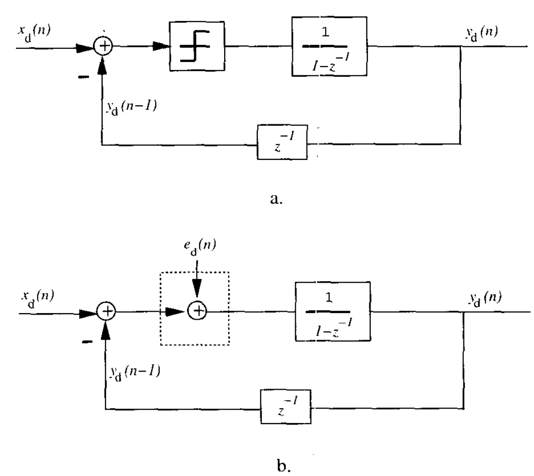

Figure 4: A Delta Modulator. a. Typical b. Linearized

Therefore, we can redraw the adapter of Figures 2 and 3 in an equivalent form utilizing equations (11)-(13), as shown in Figure 5. The adaptation block together with the quantizer of the modulator now look like a log-PCM [7], except that the PCM block is replaced by a delta modulator.

There are three advantages of using the log-DM over the log-PCM in our case. The first advantage is that log-PCM usually requires a multi-bit DAC after the PCM block, to reconstruct its analog input, introducing a source of nonlinearity in the overall modulator. Clearly log-DM does not suffer from this problem since the quantizer used inside the DM is single-bit and thus has a linear behavior.

Furthermore, it is found through simulation that the use of log-PCM introduces large tones at the modulator output,

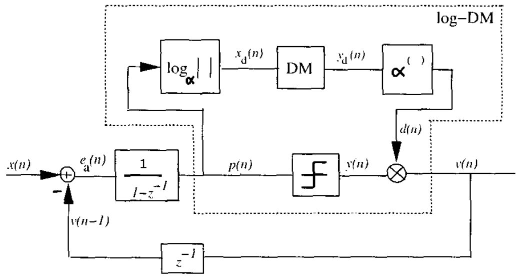

Figure 5: Equivalent form of the ASDM.

especially when the number of PCM-levels is small. These tones are usually undesirable when dealing with speech signals. On the other hand, the log-DM does not introduce large tones to the modulator since the output of the DM is inherently analog.

Finally, log-PCM offers an SNR performance that is ideally independent of the input signal strength. However, this feature is not practical since it requires a PCM with infinite dynamic range [7]. The log-DM offers practically unlimited dynamic range provided that it is given enough tracking time.

The delta modulator can be linearized by replacing its quantizer by an additive quantization noise $e_d(n)$ as shown in Figure 4b. The noise $e_d(n)$ is assumed to be uniformly distributed in an interval $[-\Delta, \Delta]$ (usually $\Delta = 1$ for single bit DM). The transfer function of the linearized DM can now be written as

$$Y_d(z) = \frac{\frac{1}{1-z^{-1}}}{1 + \frac{z^{-1}}{1-z^{-1}}}(X_d(z) + E_d(z)), \qquad (14)$$

which simplifies to

$$Y_d(z) = X_d(z) + E_d(z). \qquad (15)$$

In the time domain, we can write

$$y_d(n) = x_d(n) + e_d(n). \qquad (16)$$

From (12) we have

$$d(n) = \alpha^{y_d(n)}, \qquad (17)$$

so that

$$d(n) = \alpha^{x_d(n)+e_d(n)}. \qquad (18)$$

Substituting the expression for $x_d(n)$ from equation (11), we get

$$d(n) = \alpha^{log_\alpha(|p(n)|)+e_d(n)} = |p(n)|\alpha^{e_d(n)}. \qquad (19)$$

Substituting back into (4), we have

$$v(n) = p(n)\alpha^{e_d(n)}. \qquad (20)$$

Finally, if we denote

$$K(n) \triangleq \alpha^{e_d(n)}, \qquad (21)$$

then we arrive at the expression

$$\boxed{v(n) = p(n)K(n)} \qquad (22)$$

This result shows that we can approximate the adapter and quantizer in the main loop of Figure 2a by a time varying gain $K(n)$. Figure 6 shows the resulting equivalent structure of our ASDM. Since the distribution of the random error signal $e_d(n)$ is known, the distribution of the variable gain $K(n)$ can be readily defined. Our further analysis is based on the following basic assumptions:

1. All random processes are stationary.
2. The variable gain $K(n)$ is independent of everything else.

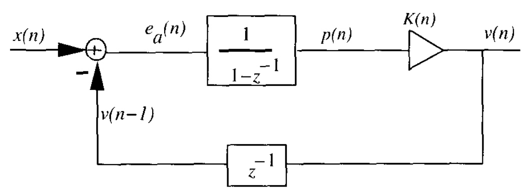

Figure 6: The ASDM as a linear time variant (LTV) system.

3.2. Mean Analysis

In this section, we show that the input signal $x(n)$ and the output signal $v(n)$ have the same mean. As a result, the error signal $e_a(n)$ has zero mean. To show this, refer back to equations (1), (2) and (22). Then,

$$E\{p(n)\} = E\{\Big(1 - K(n-1)\Big)p(n-1) + x(n)\}. \qquad (23)$$

Based on the independence and stationarity assumptions, we can write

$$E_p = (1 - E_K)E_p + E_x. \qquad (24)$$

Solving for E_p we get

$$E_p = \frac{1}{E_K}E_x. \qquad (25)$$

We also know from equation (22) that

$$E\{v(n)\} = E\{K(n)p(n)\}. \qquad (26)$$

Therefore, $E_v = E_K E_p$. Substituting the expression for E_p we get $E_v = E_x$, and consequently,

$$E_{ea} = E_x - E_v = 0. \qquad (27)$$

Thus, we conclude that for any arbitrary stationary and bounded input, the expected value of the error $e_a(n)$ is zero.

3.3. BIBO Stability of the Modulator

Consider the modulator structure shown in Figure 6. The signal $p(n)$ is given by

$$p(n) = p(n-1) + x(n) - v(n-1). \quad (28)$$

From equation (22) we have

$$p(n) = p(n-1) + x(n) - K(n-1)p(n-1). \quad (29)$$

Thus, the dynamic equation for the signal $p(n)$ can be expressed as

$$p(n) = \Big(1 - K(n-1)\Big)p(n-1) + x(n). \quad (30)$$

Since $p(0) = 0$, the forced response of $p(n)$ is

$$p(n) = \sum_{i=1}^{n} \prod_{j=i}^{n} (1 - K(j-1))\, x(i). \quad (31)$$

When the input signal $x(n)$ is bounded, i.e.,

$$|x(n)| \leq \Lambda < \infty, \quad (32)$$

for some Λ, then we get

$$|p(n)| \leq \Lambda \sum_{i=1}^{n} \prod_{j=i}^{n} |1 - K(j-1)|. \quad (33)$$

Lemma *The signal* $p(n)$ *will be bounded if*

$$|1 - K(n)| \leq L < 1, \quad (34)$$

for some L *and for all* n. *In this case, the modulator output will be bounded by*

$$|v(n)| \leq \alpha^{\Delta} \frac{\Lambda}{1-L}. \quad (35)$$

Proof: If equation (34) is satisfied, i.e., if the quantity $1 - K(n)$ is uniformly bounded by 1, then from equation (33) we conclude that

$$|p(n)| \leq \Lambda \sum_{i=1}^{n} L^{n-i} = \Lambda \sum_{j=0}^{n-1} L^{j}, \quad (36)$$

so that

$$|p(n)| \leq \frac{\Lambda}{1-L}. \quad (37)$$

Also, from equation (22), we conclude that

$$|v(n)| \leq \max(K(n)) \frac{\Lambda}{1-L}. \quad (38)$$

Referring to equation (21), the maximum of $K(n)$ is given by

$$\max(K(n)) = \alpha^{\Delta}. \quad (39)$$

Substituting back into (38) we get (35) ◇

To complete the argument, we need to show when condition (34) is satisfied. In other words, we need to determine the range of values for the exponent term α such that the quantity $|1 - K(n)|$ is uniformly bounded by one.

Corollary *Assume that* $e_d(n)$ *is a uniformly distributed random variable between* $[-\Delta, \Delta]$. *If* α *is chosen such that*

$$2^{-\frac{1}{\Delta}} < \alpha < 2^{\frac{1}{\Delta}}, \quad (40)$$

then a bound L *can be found that satisfies condition (34).*

Proof: Let $L = 1 - \epsilon$ where ϵ is a sufficiently small positive number. Then we can write

$$-1 + \epsilon \leq 1 - K(n) \leq 1 - \epsilon. \quad (41)$$

Therefore,

$$\epsilon \leq K(n) \leq 2 - \epsilon. \quad (42)$$

Since $K(n) = \alpha^{e_d(n)}$ and $e_d(n)$ is a uniform random variable between $[-\Delta, \Delta]$, then we can write

$$-\epsilon \leq [\alpha^{-\Delta}, \alpha^{\Delta}] \leq 2 - \epsilon. \quad (43)$$

In other words, the closed interval $[\alpha^{-\Delta}, \alpha^{\Delta}]$ lies entirely inside the interval $[-\epsilon, 2-\epsilon]$. Solving for α we get

$$(2-\epsilon)^{-\frac{1}{\Delta}} \leq \alpha \leq (2-\epsilon)^{\frac{1}{\Delta}}. \quad (44)$$

Since this inequality is true for any small $\epsilon > 0$ then

$$2^{-\frac{1}{\Delta}} < \alpha < 2^{\frac{1}{\Delta}} \quad \diamond \quad (45)$$

4. CONCLUSION

In this work we studied the stability of the ASDM structure in [1]. In particular, a range of values for the modulation exponent term to guarantee stability has been derived.

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