

OPTIMAL LINEAR FUSION FOR DISTRIBUTED SPECTRUM SENSING VIA SEMIDEFINITE PROGRAMMING

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ABSTRACT

As an enabling functionality of overlay cognitive radio networks, spectrum sensing needs to reliably detect licensed signal in the band of interest. To achieve reliable sensing, we propose a linear fusion scheme for distributed spectrum sensing to combine the sensing results from multiple spatially distributed cognitive radios. The optimal linear fusion design is formulated into a nonconvex optimization problem. We show that the optimal solution of such a nonconvex problem can be solved via semi-definite programming reformulation.

Index Terms— Spectrum sensing, distributed detection, cognitive radio, nonconvex optimization, and semi-definite programming.

1. INTRODUCTION

To improve the radio resource utilization, the Federal Communications Commission (FCC) is developing rules for unlicensed devices to dynamically access empty television bands or spectral holes. Cognitive radio (CR), as a promising technology to improve the spectral utilization, is defined as a radio system that continuously senses its spectral environment, dynamically identifies empty channels, and then operates in the empty channels.

Due to the channel impairments, i.e., shadowing and multipath fading, a single CR may not be able to reliably detect the licensed signal. To improve sensing reliability, it is of use to fuse the sensing results from multiple spatially distributed CRs to make a joint decision. A CR network equipped with such a data fusion capability has a better chance to detect the primary radios, hence mitigating the interference to the licensed transmissions.

In general, the data fusion methods fall into two categories, i.e., hard and soft decision combining. The optimal hard decision fusion scheme needs to determine the optimal thresholds at both the individual nodes and the fusion center.

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Since finding these optimal thresholds for correlated observations is an NP hard problem [1], one has to turn to suboptimal solutions. On the other hand, the system can make only one decision at the fusion center based on the soft decisions collected from individual nodes. It is well known that the optimal fusion scheme for soft decision combining is the likelihood ratio test (LRT). However, the LRT based detector is mathematically intractable due to its quadratic structure. In [2], it has been shown that a linear fusion scheme is comparable to the LRT approach with much better design flexibility. Alternatively, the linear structure allows quick adaptation to environmental changes [3]. The optimization of the linear fusion scheme can be formulated into a nonconvex problem, which can be solved using an iterative algorithm developed in [2].

In this paper, we present a faster algorithm to optimize the linear fusion scheme with applications in distributed spectrum sensing. We show that the formulated nonconvex optimization problem can be relaxed into a semi-definite program (SDP). SDP relaxation has recently become a popular convex approximation technique for various applications [4] [5]. However, we show that SDP relaxation can exactly solve our nonconvex optimization problem formulated for distributed spectrum sensing design.

2. SYSTEM MODEL

Consider a network of N spatially distributed CRs in the same area, each of which is sensing a channel of interest under the two hypotheses \mathcal{H}_0 and \mathcal{H}_1 . Specifically, \mathcal{H}_0 denotes the hypothesis that the primary signal is absent or far away, and \mathcal{H}_1 denotes the hypothesis that the primary signal is present in the vicinity. Each CR sends its real-valued observation u_i to the fusion center. Based on the received information $\mathbf{u} = [u_1, u_2, \dots, u_N]$, the fusion center makes a global decision $\mathcal{D}(u_1, u_2, \dots, u_N)$ on either \mathcal{H}_0 or \mathcal{H}_1 .

Suppose that the received vector \mathbf{u} at the fusion center follows an N -dimensional normal (Gaussian) distribution under each hypothesis, i.e.,

$$\mathbf{u} \sim \begin{cases} \mathcal{N}(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0), & \mathcal{H}_0 \\ \mathcal{N}(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1), & \mathcal{H}_1 \end{cases} \quad (1)$$

where $\boldsymbol{\mu}_0(\boldsymbol{\mu}_1)$ and $\boldsymbol{\Sigma}_0(\boldsymbol{\Sigma}_1)$ are the mean vector and covariance matrix of \mathbf{u} under $\mathcal{H}_0(\mathcal{H}_1)$. Refer to [2] for how such a model can be motivated in a practical distributed spectrum sensing system. Note that $\boldsymbol{\Sigma}_0 \succeq \mathbf{0}$ and $\boldsymbol{\Sigma}_1 \succeq \mathbf{0}$.

At the fusion center, we propose a linear fusion rule as

$$T(\mathbf{u}) = \sum_{i=1}^N w_i u_i = \mathbf{w}^T \mathbf{u} \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\geq}} \gamma \quad (2)$$

where $\mathbf{w} = [w_1, w_2, \dots, w_N]^T$ are the weight coefficients. Since the linear combination of multiple Gaussian random variables is still Gaussian, it can be verified that

$$T(\mathbf{u}) \sim \begin{cases} \mathcal{N}(\mathbf{w}^T \boldsymbol{\mu}_0, \mathbf{w}^T \boldsymbol{\Sigma}_0 \mathbf{w}), & \mathcal{H}_0 \\ \mathcal{N}(\mathbf{w}^T \boldsymbol{\mu}_1, \mathbf{w}^T \boldsymbol{\Sigma}_1 \mathbf{w}), & \mathcal{H}_1 \end{cases} \quad (3)$$

The detection performance can be evaluated in terms of the probability of false alarm,

$$P_f = P(T(\mathbf{u}) \geq \gamma | \mathcal{H}_0) = Q\left(\frac{\gamma - \mathbf{w}^T \boldsymbol{\mu}_0}{\sqrt{\mathbf{w}^T \boldsymbol{\Sigma}_0 \mathbf{w}}}\right) \quad (4)$$

and the probability of detection

$$P_d = P(T(\mathbf{u}) \geq \gamma | \mathcal{H}_1) = Q\left(\frac{\gamma - \mathbf{w}^T \boldsymbol{\mu}_1}{\sqrt{\mathbf{w}^T \boldsymbol{\Sigma}_1 \mathbf{w}}}\right) \quad (5)$$

where $Q(\cdot)$ denotes the *complementary cumulative distribution* function, i.e., $Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{+\infty} e^{-\tau^2/2} d\tau$.

In CR networks, the probabilities of false alarm and detection have unique implications for the system performance. With the assumption that CRs always transmit in the channels in which they do not detect primary signals, $1 - P_d$ measures the probability that CRs cause interference to the primary transmissions (i.e., the probability of missed detection), and $1 - P_f$ measures opportunistic spectral utilization, i.e., the probability that the empty channel is available for CRs.

3. PROBLEM FORMULATION

Our objective is to find the optimal \mathbf{w} that minimizes the interference to the primary transmission while meeting some requirement on the opportunistic spectral utilization. Hence, the problem is formulated as one of maximizing P_d subject to some constraint on P_f , i.e.,

$$\begin{aligned} \max_{\mathbf{w}} \quad & P_d \\ \text{s.t.} \quad & P_f \leq \varepsilon. \end{aligned} \quad (6)$$

From (4), we can express the threshold γ as a function of \mathbf{w} and the required P_f (by setting $P_f = \varepsilon$):

$$\gamma = \mathbf{w}^T \boldsymbol{\mu}_0 + Q^{-1}(\varepsilon) \sqrt{\mathbf{w}^T \boldsymbol{\Sigma}_0 \mathbf{w}}. \quad (7)$$

Plugging (7) into (5) gives us an unconstrained optimization problem as

$$\max_{\mathbf{w}} Q\left(\frac{Q^{-1}(\varepsilon) \sqrt{\mathbf{w}^T \boldsymbol{\Sigma}_0 \mathbf{w}} - \mathbf{w}^T (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0)}{\sqrt{\mathbf{w}^T \boldsymbol{\Sigma}_1 \mathbf{w}}}\right). \quad (8)$$

Since $Q(\cdot)$ is a monotonically non-increasing function, (8) is equivalent to

$$\min_{\mathbf{w}} f(\mathbf{w}) = \frac{Q^{-1}(\varepsilon) \sqrt{\mathbf{w}^T \boldsymbol{\Sigma}_0 \mathbf{w}} - (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0)^T \mathbf{w}}{\sqrt{\mathbf{w}^T \boldsymbol{\Sigma}_1 \mathbf{w}}}. \quad (9)$$

Directly solving (9) is difficult since it is a nonconvex problem. To find the global optimal solution, we employ a divide-and-conquer strategy. By observing the problem structure in (9), we find that it can be reformulated into the following two subproblems. If $f(\mathbf{w}) \geq 0$, i.e., $P_d \leq 1/2$, (9) is equivalent to

$$\begin{aligned} \min_{\mathbf{z}} \quad & Q^{-1}(\varepsilon) \sqrt{\mathbf{z}^T \boldsymbol{\Sigma}_0 \mathbf{z}} - (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0)^T \mathbf{z} \\ \text{s.t.} \quad & \mathbf{z}^T \boldsymbol{\Sigma}_1 \mathbf{z} \geq 1 \end{aligned} \quad (10)$$

where $\mathbf{z} = \mathbf{w} / \sqrt{\mathbf{w}^T \boldsymbol{\Sigma}_1 \mathbf{w}}$. Otherwise, (9) is equivalent to

$$\begin{aligned} \max_{\mathbf{z}} \quad & -Q^{-1}(\varepsilon) \sqrt{\mathbf{z}^T \boldsymbol{\Sigma}_0 \mathbf{z}} + (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0)^T \mathbf{z} \\ \text{s.t.} \quad & \mathbf{z}^T \boldsymbol{\Sigma}_1 \mathbf{z} \leq 1. \end{aligned} \quad (11)$$

Since the sign of $Q^{-1}(\varepsilon)$ is undetermined, the problems (10) and (11) are still nonconvex in general. In the next section, we show how to solve (10) via SDP relaxation. Once (10) is solved, (11) can be solved using a similar approach.

4. SDP RELAXATION

The proposed SDP relaxation approach is based on a judicious reformulation of the original problem in (11). To see this, we introduce a new variable

$$\alpha = Q^{-1}(\varepsilon) \sqrt{\mathbf{z}^T \boldsymbol{\Sigma}_0 \mathbf{z}} - (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0)^T \mathbf{z}. \quad (12)$$

Thus, (10) can be transformed into

$$\begin{aligned} \min_{\mathbf{z}} \quad & \alpha^2 \\ \text{s.t.} \quad & \alpha = Q^{-1}(\varepsilon) \sqrt{\mathbf{z}^T \boldsymbol{\Sigma}_0 \mathbf{z}} - (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0)^T \mathbf{z} \\ & \mathbf{z}^T \boldsymbol{\Sigma}_1 \mathbf{z} \geq 1 \\ & \alpha \geq 0 \end{aligned} \quad (13)$$

where we utilize the fact that minimizing α^2 is equivalent to minimizing α since α is nonnegative. Furthermore, it can be shown that (13) is equivalent to

$$\begin{aligned} \min_{\mathbf{z}} \quad & \alpha^2 \\ \text{s.t.} \quad & \left[\alpha + (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0)^T \mathbf{z} \right]^2 = Q^{-2}(\varepsilon) \mathbf{z}^T \boldsymbol{\Sigma}_0 \mathbf{z} \\ & \mathbf{z}^T \boldsymbol{\Sigma}_1 \mathbf{z} \geq 1 \\ & \alpha \geq 0 \end{aligned} \quad (14)$$

which is a nonconvex quadratic program with two quadratic constraints. By introducing a new variable $\mathbf{x} = [\mathbf{z}^T \alpha]^T$, we can write problem (14) as

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{x}^T \mathbf{F} \mathbf{x} \\ \text{s.t.} \quad & \mathbf{x}^T \mathbf{G}_1 \mathbf{x} = 0 \\ & \mathbf{x}^T \mathbf{H} \mathbf{x} \geq 1 \end{aligned} \quad (15)$$

where

$$\mathbf{F} = \begin{bmatrix} \mathbf{0}^{N \times N} & \mathbf{0}^{N \times 1} \\ \mathbf{0}^{1 \times N} & 1 \end{bmatrix}, \quad (16)$$

$$\mathbf{G}_1 = \begin{bmatrix} (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0)(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0)^T - Q^{-2}(\varepsilon)\boldsymbol{\Sigma}_0 & \boldsymbol{\mu}_1 - \boldsymbol{\mu}_0 \\ (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0)^T & 1 \end{bmatrix}, \quad (17)$$

and

$$\mathbf{H} = \begin{bmatrix} \boldsymbol{\Sigma}_1 & \mathbf{0}^{N \times 1} \\ \mathbf{0}^{1 \times N} & 0 \end{bmatrix}, \quad (18)$$

with \mathbf{F} , \mathbf{G}_1 , and $\mathbf{H} \in \mathbf{S}^{N+1}$.

Since $\mathbf{F} \succeq \mathbf{0}$ and $\mathbf{H} \succeq \mathbf{0}$, an optimal solution of (15) must satisfy

$$\mathbf{x}^T \mathbf{H} \mathbf{x} = 1. \quad (19)$$

Thus, (15) is equivalent to

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{x}^T \mathbf{F} \mathbf{x} \\ \text{s.t.} \quad & \mathbf{x}^T \mathbf{G}_1 \mathbf{x} = 0, \quad \mathbf{x}^T \mathbf{H} \mathbf{x} = 1. \end{aligned} \quad (20)$$

Applying SDP relaxation [6] and eliminating the hidden constraint $\mathbf{X} = \mathbf{x}\mathbf{x}^T \in \mathbf{S}^{N+1}$, we can obtain a standard SDP problem as follows:

$$\begin{aligned} \min_{\mathbf{X} \in \mathbf{S}^{N+1}} \quad & \text{tr}(\mathbf{F}\mathbf{X}) \\ \text{s.t.} \quad & \text{tr}(\mathbf{G}_1\mathbf{X}) = 0, \quad \text{tr}(\mathbf{H}\mathbf{X}) = 1 \\ & \mathbf{X} \succeq \mathbf{0} \end{aligned} \quad (21)$$

which has linear equality constraints and a matrix nonnegativity constraint on the unknown variable \mathbf{X} . Recall that $\text{tr}(\mathbf{F}\mathbf{X}) = \sum_{i=1}^{N+1} \sum_{j=1}^{N+1} F_{ij} X_{ji}$ is in the form of a general real-valued linear function on \mathbf{S}^{N+1} , which shows that SDP is a generalized framework of linear programming over matrices. As a result, (21) is a relaxation of (15) since we have removed the rank-one constraint.

5. RANK-ONE SOLUTION

In this section, we will show that there exists at least one rank-one solution for (21) such that its optimal objective value is the same as that of (15), and thus this rank-one solution is the optimal solution of (15). The development of the rank-one solution for our particular problem in (21) needs a special *rank-one decomposition* technique proposed in [7]:

¹ \mathbf{S}^n denotes the set of $n \times n$ symmetric matrices.

² \succeq denotes the matrix inequality, i.e., $\mathbf{A} \succeq \mathbf{B}$ means that $\mathbf{A} - \mathbf{B}$ is positive semi-definite.

Lemma 1 Let $\mathbf{X} \in \mathbf{S}^n, \mathbf{X} \succeq \mathbf{0}$ be a matrix with rank r . Given $\mathbf{G} \in \mathbf{S}^n$, \mathbf{X} can be decomposed into

$$\mathbf{X} = \sum_{i=1}^r \mathbf{x}_i \mathbf{x}_i^T \quad (22)$$

where the decomposed vectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_r$ satisfy

$$\mathbf{x}_i^T \mathbf{G} \mathbf{x}_i = \frac{\text{tr}(\mathbf{G}\mathbf{X})}{r}, \quad i = 1, 2, \dots, r. \quad (23)$$

□

This decomposition can be proven by construction, and its pseudo code is given as follows:

Rank-One Decomposition Procedure

Input: $\mathbf{X} \succeq \mathbf{0}$, and $\mathbf{G} \in \mathbf{S}^n$.

Step 1: Apply any decomposition that yields $\mathbf{X} = \sum_{i=1}^r \mathbf{x}_i \mathbf{x}_i^T$; e.g., eigendecomposition.

Step 2: If $\mathbf{x}_i^T \mathbf{G} \mathbf{x}_i = \text{tr}(\mathbf{G}\mathbf{X})/r$ for all i then output $\mathbf{x}_1, \dots, \mathbf{x}_r$ and return; otherwise find i, j such that $\mathbf{x}_i^T \mathbf{G} \mathbf{x}_i > \text{tr}(\mathbf{G}\mathbf{X})/r$ and $\mathbf{x}_j^T \mathbf{G} \mathbf{x}_j < \text{tr}(\mathbf{G}\mathbf{X})/r$.

Step 3: Determine β such that $(\mathbf{x}_i + \beta \mathbf{x}_j)^T \mathbf{G} (\mathbf{x}_i + \beta \mathbf{x}_j) = (1 + \beta^2) \text{tr}(\mathbf{G}\mathbf{X})/r$.

Step 4: $\mathbf{x}_i := (\mathbf{x}_i + \beta \mathbf{x}_j) / \sqrt{1 + \beta^2}$, $\mathbf{x}_j := (-\beta \mathbf{x}_i + \mathbf{x}_j) / \sqrt{1 + \beta^2}$.

Step 5: Repeat Step 2.

The above decomposition technique has been used in [7] to solve nonconvex quadratic programs with two quadratic constraints. However, the method in [7] is not directly applicable to solving the problem (21). Thus, we give the following theorem to find a rank-one solution for (21).

Theorem 1 Let $\bar{\mathbf{X}}$ be a solution to the SDP relaxation problem (21) and suppose $r = \text{rank}(\bar{\mathbf{X}})$. A rank-one solution to (21) can be obtained from $\bar{\mathbf{X}}$ by performing the following steps:

- i) Apply the decomposition according to Lemma 1 on $\bar{\mathbf{X}}$ with respect to \mathbf{G}_1 to obtain vectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_r$.
- ii) For any l satisfying $\mathbf{x}_l^T \mathbf{H} \mathbf{x}_l > 0$, $1 \leq l \leq r$, let

$$\bar{\mathbf{x}}_l = \frac{\mathbf{x}_l}{\sqrt{\mathbf{x}_l^T \mathbf{H} \mathbf{x}_l}}.$$

We have $\bar{\mathbf{x}}_l \bar{\mathbf{x}}_l^T$ as the optimal rank-one solution for (21). □

The above result can be proven by showing that the rank-one matrix $\bar{\mathbf{x}}_l \bar{\mathbf{x}}_l^T$ essentially satisfies the Karush-Kuhn-Tucker (KKT) [6] optimality conditions of (21). The proof is omitted due to the space limit, but will be provided in the future publication [8].

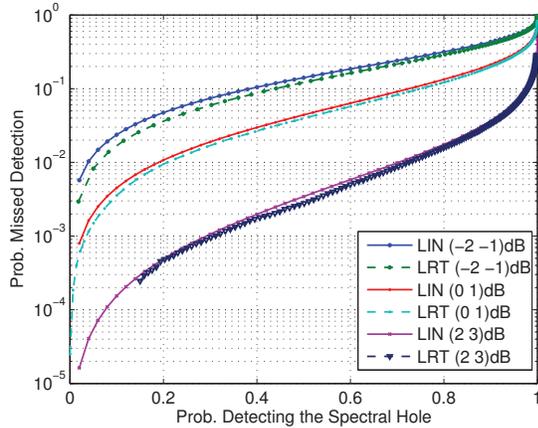


Fig. 1. $(1 - P_d)$ versus $(1 - P_f)$, with $N = 2$ and $\gamma = 2$ at various SNR levels. The SNR levels of the two CRs are given in the parentheses.

6. NUMERICAL EXAMPLES

In this section, we evaluate the detection performance of the optimal linear fusion scheme. By assuming that \mathcal{H}_0 and \mathcal{H}_1 are equally probable, we can define the signal-to-noise ratio (SNR) of each CR as

$$\text{SNR}_i = \frac{2(\mu_{1,i} - \mu_{0,i})^2}{\sigma_{1,ii} + \sigma_{0,ii}} \quad (24)$$

where $\sigma_{j,ii}$ is the (i, i) th entry of Σ_j ($j = 0, 1$). Let $\mathbf{1}$ denote an all-one vector. We choose $\Sigma_0 = \text{diag}(\mathbf{1})$ and $\Sigma_1 = \kappa \Sigma_0$, where κ implies the difference between Σ_0 and Σ_1 .

In Figure 1, we illustrate the probability of missed detection (i.e., $1 - P_d$) versus the probability of detecting the spectral hole (i.e., $1 - P_f$) over various SNR levels. The curve of the optimal linear detector is denoted by LIN and the LRT detector serves as a performance benchmark. It can be observed that the optimal linear detector approaches the LRT performance limit.

In Figure 2, we show how the difference between Σ_0 and Σ_1 affects the detection performance. It can be observed that the optimal linear detector approximates the LRT detector well if the difference between Σ_0 and Σ_1 is small (e.g., $\kappa \approx 1$). In the special case where $\kappa = 1$, the LRT detector degenerates into a linear detector. On the other hand, the linear detector might be far away from the optimum if the difference between the two covariance matrices is large.

7. CONCLUSION

In this paper, we have proposed a linear fusion strategy for distributed spectrum sensing in CR networks via SDP by building on the results of [2]. We have shown that the optimization of such a linear fusion scheme can be efficiently and exactly

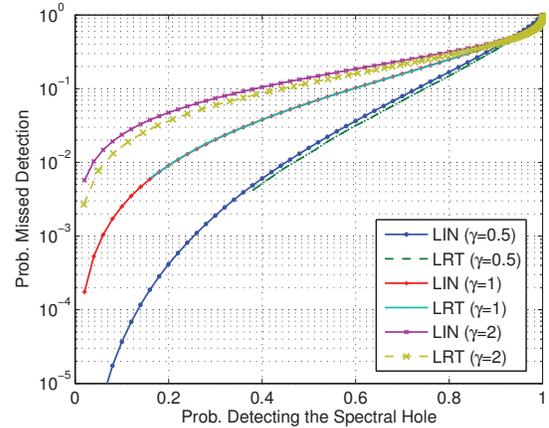


Fig. 2. $(1 - P_d)$ versus $(1 - P_f)$, with $N = 2$ and various γ values.

solved via careful SDP reformulation. The resultant linear detector can achieve performance comparable to that of the optimal LRT approach. The results provide a simple and effective design approach for distributed spectrum sensing in CR networks to achieve improved sensing reliability.

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