FIXED POINT ANALYSIS OF THE CONSTANT MODULUS ALGORITHM

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ABSTRACT

The steady-state performance of adaptive equalizers can significantly vary when they are implemented in finite precision arithmetic, which makes it vital to analyze their performance in a quantized environment. In this paper we present a fixed point analysis for the steady-state mean square error (MSE) of a blind adaptive equalizer and the optimal value of the step-size that minimizes this MSE. Such expressions are useful for selecting the adequate wordlength of a blind equalizer to achieve a specific desired steady-state performance.

1. INTRODUCTION

In this paper, we derive expressions for the steady-state mean square error (MSE) of a blind adaptive equalizer and the optimal value of the step-size that minimizes this MSE. We focus on the constant modulus algorithm (CMA), which is among the most widely used algorithms for (fractionally-spaced) blind equalization [1, 2]. Its update equation is highly nonlinear, which makes it difficult to evaluate the steady-state MSE using conventional techniques that are used for analyzing the steady-state performance of adaptive filtering algorithms in general. A major feature of the approach proposed herein is that it bypasses the need for working directly with an update equation for the weight-error vector. This is achieved by exploiting an energy-preserving relation that in fact holds for a general class of adaptive algorithms (e.g., [3, 4, 5]). Throughout the paper, we use the channel-equalizer model used in [2, 6]. We focus on fractionally-spaced equalizer implementations due to their inherent advantages (see, e.g., [1, 2]).

A blind adaptive equalizer $\mathbf{w}$ is one that attempts to approximate a zero forcing equalizer $\mathbf{w}^o$ without knowledge of the channel impulse response $\mathbf{c}$ and without direct access to the transmitted sequence $\{s(\cdot)\}$ itself. A zero forcing equalizer leads to an overall channel-equalizer impulse response of the form

$$h_D = e^{j\theta}\,\text{col}[0,...,0,1,0,...,0]\ , \quad j=\sqrt{-1} \qquad (1)$$

for some constant phase shift $\theta \in [0, 2\pi]$, and where the unit entry is in some position D. Thus under such conditions, the output of the channel-equalizer system will be of the form $y(i) = s(i-D)e^{j\theta}$, for some $\{D, \theta\}$.

Approximating the zero forcing solution is achieved by seeking to minimize certain cost functions whose global minima generally occur at the location of zero forcing equalizers. The most popular adaptive blind algorithms are the so-called constant modulus algorithms [7]. They are derived as stochastic gradient methods for minimizing the cost function:

$$J_{CM}(\mathbf{w}) = \text{E}(|y(i)|^p - R_p|)^2$$

where $y(i) = \mathbf{u}_i\mathbf{w}$ is the equalizer output, $\mathbf{u}_i$ is an input row vector (regressor) to the equalizer, R_p is suitably chosen in order to guarantee that the global minima of $J_{CM}(\mathbf{w})$ occur at zero forcing solutions (see, e.g., [7]). In this paper we focus on the following stochastic gradient variant, known as CMA2-2, or simply the CMA. In this case, we select $p=2$,

$$R_2 = \frac{\text{E}\,|s(i)|^4}{\text{E}\,|s(i)|^2}$$

and the update equation for the weight estimates is given by

$$\mathbf{w}_{i+1} = \mathbf{w}_i + \mu\mathbf{u}_i^*\, f_e(i) \qquad (2)$$

where

$$f_e(i) = y(i)\ \left[R_2 - |y(i)|^2\right] \qquad (3)$$

with a step-size μ. The *row* vector $\mathbf{u}_i$ is the input data regressor to the adaptive equalizer and $y(i) = \mathbf{u}_i\mathbf{w}_i$ is the output of the adaptive equalizer. The symbol $*$ denotes complex conjugate transposition.

Since this algorithm is based on instantaneous approximation of the true gradient vector of the cost function $J_{CM}(\mathbf{w})$, the equalizer output $y(i)$ need not converge to a zero forcing solution of the form $s(i-D)e^{j\theta}$ due to the presence of gradient noise. Therefore, the steady-state mean-square-error,

$$\text{MSE} = \lim_{i\to\infty} \text{E}\left|y(i) - s(i-D)e^{j\theta}\right|^2$$

is often used as a performance index of the adaptive equalization algorithm. Moreover, in finite precision implementations, quantization of the various equalizer quantities introduce errors that can cause the performance of the equalizer

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to vary significantly from the expected performance in the infinite precision case. In this paper we evaluate the MSE of a blind equalizer in a quantized environment, without directly using the weight error vector $\tilde{\mathbf{w}}_i = \mathbf{w}^o - \mathbf{w}_i$.

2. A QUANTIZED MATHEMATICAL MODEL

Figure 1 shows the quantized model used in the paper. Similar models have been used in the context of finite precision analyses of adaptive algorithms. In this figure, $Q[x]$ denotes the fixed point quantization of the value x, and the superscript q distinguishes quantized quantities from infinite precision quantities. Throughout the paper, rounding quantization is considered. It is also assumed that the saturation thresholds of the quantizers are properly chosen such that saturation errors are negligible. Thus, only rounding errors are considered. The variance σ^2 of the rounding error is related to the quantizer saturation threshold L according to

$$\sigma^2 = \frac{2^{-2B}L^2}{12}$$

where it is assumed that the quantizer uses B bits in addition to a sign bit. The values of B and L considered for quantization of the data ($\mathbf{u}_i$, $d(i)$, and $y(i)$) will be denoted by B_d and L_d and the ones considered for quantization of the equalizer coefficients will be denoted by B_c and L_c. The corresponding values of σ^2 will be denoted by σ_d^2 and σ_c^2, respectively. We can write

$$y^q(i) = \mathbf{u}_i^q \mathbf{w}_i^q + \gamma(i) \tag{4}$$

where $\gamma(i)$ is the quantization error that occurs in computing the term $\mathbf{u}_i^q \mathbf{w}_i^q$. The variance of $\gamma(i)$, σ_γ^2, depends on the procedure by which $y^q(i)$ is computed. If all N products involved in $\mathbf{u}_i^q \mathbf{w}_i^q$ are computed with high precision, summed, and the final result is quantized to B_d bits, then σ_γ^2 is approximately equal to σ_d^2. If each one of the N products is quantized to, say B_y bits, and the sum is then quantized to B_d bits with B_y being significantly greater than B_d, σ_γ^2 is equal to $\sigma_d^2 + N\sigma_y^2$. Moreover, the quantized error function of the CMA is given, from (3), by

$$\begin{aligned} f_e^q(i) &= Q\left[y^q(i)\ \left[R_2^q - Q\left[|y^q(i)|^2\right]\right]\right] \\ &= y^q(i)\ \left[R_2^q - |y^q(i)|^2 + e_1(i)\right] + e_2(i) \end{aligned} \tag{5}$$

where $R_2^q = Q[R_2]$, and $e_1(i)$ and $e_2(i)$ are two quantization errors of variances $\sigma_1^2 = \sigma_2^2 = \sigma_d^2$. Taking the above quantizations into effect, the CMA update equation becomes

$$\begin{aligned} \mathbf{w}_{i+1}^q &= \mathbf{w}_i^q + Q\left[\mu\ \mathbf{u}_i^{q*}\ f_e^q(i)\right] \\ &= \mathbf{w}_i^q + \mu\ \mathbf{u}_i^{q*}\ f_e^q(e^q(i)) - \mathbf{m}_i \end{aligned} \tag{6}$$

where $\mathbf{m}_i$ is a vector of multiplication quantization errors in the update term $\mu \mathbf{u}_i^{q*} f_e^q(e^q(i))$, each entry of which has variance σ_c^2. The weight error vector is now defined as

$$\tilde{\mathbf{w}}_i = \mathbf{w}^o - \mathbf{w}_i^q \tag{7}$$

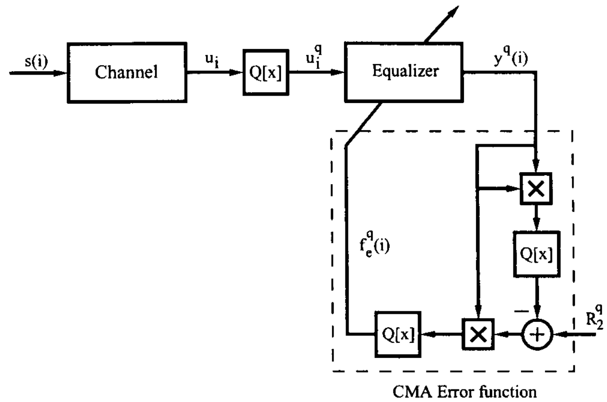

Figure 1: CMA *quantization model.*

3. QUANTIZED ENERGY RELATION

Based on the quantization model of the previous section, we now derive an energy preserving relation for quantized CMA. This energy relation will be used in the next section to derive a MSE expression for the quantized CMA.

Introduce the a-priori and a-posteriori estimation errors,

$$\begin{aligned} e_a(i) &= s(i-D)e^{j\theta} - y^q(i) = \mathbf{u}_i^q \mathbf{w}_i^o - \mathbf{u}_i^q \mathbf{w}_i = \mathbf{u}_i^q \tilde{\mathbf{w}}_i \\ e_p(i) &= \mathbf{u}_i^q\left(\tilde{\mathbf{w}}_{i+1} - \mathbf{m}_i\right) \end{aligned}$$

If we subtract $\mathbf{w}^o$ from both sides of (6) and multiply by $\mathbf{u}_i^q$ from the left, we find that the errors $\{e_p(i), e_a(i)\}$ are related via:

$$e_p(i) = e_a(i) - \frac{\mu}{\bar{\mu}(i)} f_e^q(i) \tag{8}$$

where we defined, for compactness, $\bar{\mu}(i) = 1/\|\mathbf{u}_i^q\|^2$. Substituting (8) into (6), we obtain the update relation

$$\tilde{\mathbf{w}}_{i+1} = \tilde{\mathbf{w}}_i - \bar{\mu}(i)\mathbf{u}_i^{q*}[e_a(i) - e_p(i)] + \mathbf{m}_i$$

By evaluating the energies of both sides of this equation we obtain

$$\|\tilde{\mathbf{w}}_{i+1} - \mathbf{m}_i\|^2 + \bar{\mu}(i)|e_a(i)|^2 = \|\tilde{\mathbf{w}}_i\|^2 + \bar{\mu}(i)|e_p(i)|^2 \tag{9}$$

To proceed, we impose the following modeling assumption:

<u>A.1</u> Quantization errors are zero-mean, mutually independent, and independent of all other signals.

This assumption is typical in the context of finite precision analysis of adaptive algorithms and it enables the derivation of closed-form expressions for the steady-state MSE. A more sophisticated nonlinear model for treating quantization errors, which takes into account the quantizer underflow effects, has been used in [8] for the LMS algorithm; though it does not lead to closed-form expressions.

Imposing the equality $\mathrm{E}\,\|\tilde{\mathbf{w}}_{i+1}\|^2 = \mathrm{E}\,\|\tilde{\mathbf{w}}_i\|^2$ in steady-state, and using (8) and A.1, it is straightforward to verify that the energy relation (9) leads to the following error variance relation, in terms of $e_a(i)$,

$$\mathrm{E}\left(\bar{\mu}(i)|e_a(i)|^2\right) = \mathrm{Tr}(\mathbf{M}) + \mathrm{E}\left(\bar{\mu}(i)\left|e_a(i) - \frac{\mu}{\bar{\mu}(i)}f_e^q(i)\right|^2\right) \quad (10)$$

where $\mathbf{M} = \mathrm{E}(\mathbf{m}_i\mathbf{m}_i^*)$. For iid multiplication errors, $\mathrm{Tr}(\mathbf{M}) = N\sigma_c^2$. This equation can now be solved for the steady-state mean-square-error (MSE):

$$\zeta \triangleq \lim_{i\to\infty} \mathrm{E}\,|e_a(i)|^2$$

4. STEADY-STATE MSE OF THE QUANTIZED CMA

We now apply the above results to the CMA recursion (2). For mathematical tractability of the analysis, we impose the following two reasonable assumptions in *steady-state* $(i \to \infty)$ — for more motivation and explanation on these two assumptions, see [6, 9]:

A.2 The transmitted signal $s(i-D)$ and the estimation error $e_a(i)$ are independent in steady-state so that $\mathrm{E}(s^*(i-D)e_a(i)) = 0$, since $s(i-D)$ is assumed zero mean.

A.3 The scaled regressor energy $\mu^2\|\mathbf{u}_i\|^2$ is independent of $y^q(i)$ in steady-state.

We consider first the case of real-valued data $\{s(\cdot), y^q(\cdot), \mathbf{u}_i\}$. In this case, we can assume that the zero forcing response h_D that the adaptive equalizer attempts to achieve (*cf.* (1)) can be of either form $h_D = \pm[0,...,0,1,0,...,0]$. In the following, we continue with the choice $h_D = [0,...,0,1,0,...,0]$, which yields $e_a(i) = s(i-D) - y^q(i)$. A similar analysis holds for the case $h_D = [0,...,0,-1,0,...,0]$.

Substituting (5) into (10), we obtain

$$\mathrm{E}\left(\bar{\mu}(i)|e_a(i)|^2\right) = \mathrm{Tr}(\mathbf{M}) + \mathrm{E}\left(\bar{\mu}(i)\left|e_a(i) - \frac{\mu}{\bar{\mu}(i)}\left(y^q(i)\ \left[R_2^q - (y^q(i))^2 + e_1(i)\right] + e_2(i)\right)\right|^2\right) \quad (11)$$

We shall write more compactly

$$e_a \triangleq e_a(i), \quad \bar{\mu} \triangleq \bar{\mu}(i), \quad y \triangleq y^q(i),$$
$$\mathbf{u}^q \triangleq \mathbf{u}_i^q, \quad s \triangleq s(i-D), \quad e_1 \triangleq e_1(i), \quad e_2 \triangleq e_2(i)$$

for $i \to \infty$, so that (11) becomes, after expanding,

$$2\mu\,\mathrm{E}\left(e_a\ y\ \left[R_2^q - y^2 + e_1\right] + e_a\ e_2\right) = \mathrm{Tr}(\mathbf{M}) + \mu^2\,\mathrm{E}\left(\|\mathbf{u}^q\|^2\left(y\ \left[R_2^q - y^2 + e_1\right] + e_2\right)^2\right)$$

Using this equality we can now obtain an expression for the steady-state MSE, $\mathrm{E}(e_a^2)$. Replacing y by $s - e_a$, using assumptions A.1–A.3 and neglecting $2\mu\,\mathrm{E}(e_a^4)$, for sufficiently small μ and small e_a^2, it is straightforward to show that the steady-state MSE can be approximated by

$$\zeta^{\mathrm{CMA}}(real) \approx \frac{\mathrm{Tr}(\mathbf{M})/\mu + \mu\,\mathrm{E}(s^2 {R_2^q}^2 - 2R_2^q s^4 + s^2\sigma_1^2 + s^6 + \sigma_2^2)\,\mathrm{E}\,\|\mathbf{u}^q\|^2}{2\,\mathrm{E}(3s^2 - R_2^q)}$$

This result implies that the steady-state MSE is composed of two terms. The first term decreases with μ and increases with the multiplication error variance $\mathrm{Tr}(\mathbf{M})$. The second term increases with μ and the received signal variance, $\mathrm{E}\,\|\mathbf{u}^q\|^2$. Thus, unlike the infinite precision case (see, e.g., [6, 9]), the steady-state MSE is not a monotonically increasing function of μ. We can also see that in the noiseless case, and for non-constant modulus data $\{s(\cdot)\}$, there exists a finite optimal value of the step size, μ_o, that minimizes the above expression for the steady-state MSE, which is given by

$$\mu_o^{\mathrm{CMA}} = \sqrt{\mathrm{Tr}(\mathbf{M})/\left[\mathrm{E}\left(s^2 {R_2^q}^2 - 2R_2^q s^4 + s^2\sigma_1^2 + s^6 + \sigma_2^2\right)\mathrm{E}\,\|\mathbf{u}^q\|^2\right]}$$

where $\mathrm{E}\,\|\mathbf{u}^q\|^2 = \mathrm{E}\,\mathbf{u}_i^{q*}\mathbf{u}_i^q = \mathrm{E}\,\|\mathbf{u}_i\|^2 + N\sigma_d^2$. This expression shows that μ_o decreases with the signal variance, $E\|\mathbf{u}^q\|^2$, and increases with the multiplication error variance $\mathrm{Tr}(\mathbf{M})$. The corresponding minimum value of the steady-state MSE is then given by

$$\zeta_o^{\mathrm{CMA}}(real) = \frac{\sqrt{\mathrm{Tr}(\mathbf{M})\,\mathrm{E}\left(s^2 {R_2^q}^2 - 2R_2^q s^4 + s^2\sigma_1^2 + s^6 + \sigma_2^2\right)\mathrm{E}\,\|\mathbf{u}^q\|^2}}{\mathrm{E}(3s^2 - R_2^q)}$$

Here we may add that for complex-valued data, the steady-state MSE will have a different expression than that in the real-valued case. Following the same derivation, and assuming signal constellations that satisfy the circularity condition $\mathrm{E}\,s^2(i) = 0$, in addition to the condition $\mathrm{E}(2|s(i)|^2 - R_2) > 0$ (both of which hold for most constellations [7]), we can show that the steady-state MSE for complex-valued data, and for sufficiently small step-sizes, can be approximated by

$$\zeta^{\mathrm{CMA}}(complex) \approx \frac{\mathrm{Tr}(\mathbf{M})/\mu + \mu\,\mathrm{E}(|s|^2 {R_2^q}^2 - 2R_2^q|s|^4 + |s|^2\sigma_1^2 + |s|^6 + \sigma_2^2)\,\mathrm{E}\,\|\mathbf{u}^q\|^2}{2\,\mathrm{E}(2|s|^2 - R_2^q)}$$

In this case, the optimum value of the algorithm step size still has the same value as in the real-valued data case, while the minimum achievable steady-state MSE is given by

$$\zeta_o^{\mathrm{CMA}}(complex) = \frac{\sqrt{\mathrm{Tr}(\mathbf{M})\,\mathrm{E}\left(|s|^2 {R_2^q}^2 - 2R_2^q|s|^4 + |s|^2\sigma_1^2 + |s|^6 + \sigma_2^2\right)\mathrm{E}\,\|\mathbf{u}^q\|^2}}{\mathrm{E}(2|s|^2 - R_2^q)}$$

Finally, we may add that, for the infinite precision case $(\sigma_c^2 = \sigma_d^2 = 0)$, the expressions for the steady-state MSE reduce to the expressions obtained in [6].

5. SIMULATION RESULTS

We now provide some simulation results that compare the experimental performance with the one predicted by the derived expressions. The channel considered in this simulation is given by $\mathbf{c} = [0.1, 0.3, 1, -0.1, 0.5, 0.2]$. A 4-tap FIR filter is used as a $\frac{T}{2}$-fractionally spaced quantized equalizer, with $B_c = B_d = 8$, and 9. In this simulation, the transmitted signal was 6-PAM constellated, $s(i) \in \{1, 0.6, 0.2, -0.2, -0.6, -1\}$ with $\mathrm{E}\, s^6 = 0.3489$, $\mathrm{E}\, s^4 = 0.3771$, $\mathrm{E}\, s^2 = 0.4667$, and $R_2 = 0.808$. The value of $\|\mathbf{u}_i\|^2$ is the norm of the received signal vector. The value of $\mathrm{E}\,\|\mathbf{u}_i\|^2$ was computed as the average over 10,000 realizations of $\|\mathbf{u}_i\|^2$. The value of experimental MSE was obtained as the average over 100 repeated runs.

Figures 2 and 3 are plots of the experimental MSE and the theoretical MSE versus the step-size μ for $B_c = B_d =$ 8 and 9 bits, respectively. It can be seen from the figure that the theoretical results reasonably match the experimental results. We can also see that, for $B_c = B_d =$ 8 bits, the experimental MSE reaches a minimum value of -30.13 dB, which corresponds to an optimal value of μ equal to 1.5×10^{-2}, while our theory predicted a minimum achievable MSE of -30.38 dB at $\mu_o = 0.94 \times 10^{-2}$. For $B_c = B_d =$ 9 bits, the experimental MSE reaches a minimum value of -32.11 dB, which corresponds to an optimal value of μ equal to 10^{-2}. On the other hand, our theory predicted a minimum achievable MSE of -33.38 dB at $\mu_o = 0.47 \times 10^{-2}$.

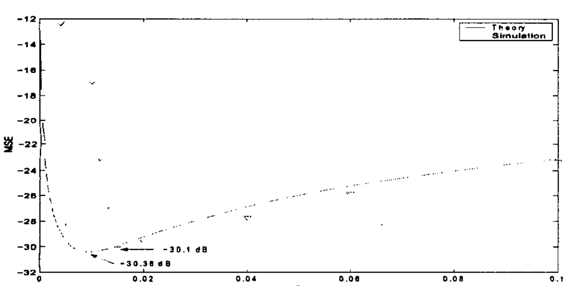

Figure 2: *Theoretical and simulation* MSE *of* CMA *for* $B_c = B_d = 8$.

Here we note that the experimental value results validate that the steady-state MSE is not a monotonically increasing function of μ, as predicted by our analytical results. Furthermore, the experimental values of the minimum achievable MSE match reasonably well the analytical values. Thus, the derived results for the minimum MSE can be reliable in predicting the best steady-state performance, which the CMA can achieve for a given wordlength. However, the experimental values for the optimum step size are lower than the corresponding predicted analytical values. This is due to quantizer underflow effects that were not taken into consideration in our quantization model. Thus, a more conservative (larger) design value for μ_o should be taken into consideration to account for this effect.

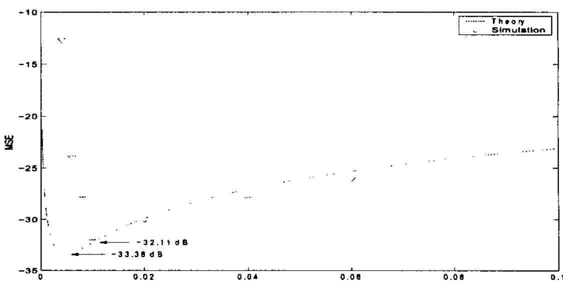

Figure 3: *Theoretical and simulation* MSE *of* CMA *for* $B_c = B_d = 9$.

6. CONCLUSIONS

In this paper, a finite precision analysis of a blind equalizer using the CMA algorithm was presented. Expressions for the steady-state mean-square-error were derived. It was found that, unlike the infinite precision case, the MSE is not a monotonically increasing function of the step size. The value of the optimum step size that minimizes MSE and the corresponding value of the MSE were derived. Simulation results show reasonable match with the analytical results. However, taking the effects of quantizer underflow into consideration remains an open issue for future work.

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