ORTHONORMAL REALIZATION OF FAST FIXED-ORDER RLS ADAPTIVE FILTERS

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ABSTRACT

The existing derivations of fast RLS adaptive filters are dependent on the shift structure in the input regression vectors. This structure arises when a tapped-delay line (FIR) filter is used as a modeling filter. In this paper, we show, unlike what original derivations may suggest, that fast fixed-order RLS adaptive algorithms are not limited to FIR filter structures. We show that fast recursions in both explicit and array forms exist for more general data structures, such as orthonormally-based models. One of the benefits of working with an orthonormal basis is that fewer parameters can be used to model long impulse responses.

1. INTRODUCTION

Fast RLS adaptive filtering algorithms represent an attractive way to compute the least squares solution of growing length data efficiently, in $\mathcal{O}(M)$ computations per sample, where M is the filter order. The low complexity that is achieved by these algorithms is a direct consequence of the shift structure that is characteristic of regression vectors in FIR adaptive implementations. Recently, the authors showed that the input data structure that arises from more general networks, such as Laguerre filters, can be exploited to derive fast order-recursive [1] and fixed-order filters [2, 3] as well.

In this paper, we show that fast fixed-order RLS adaptive algorithms can also be derived for general orthonormal bases (see e.g., [4, 5]) in both explicit and array forms.

2. THE EXTENDED FAST TRANSVERSAL FILTER

Given a column vector $y_N \in \mathbb{C}^{N+1}$ and a data matrix $H_N \in \mathbb{C}^{(N+1)\times M}$, the exponentially-weighted least squares problem seeks the column vector $w \in \mathbb{C}^M$ that solves

$$\min_w \left[\lambda^{N+1}\|\Pi^{-1/2}w\|^2 + \|W_N^{1/2}(y_N - H_N w)\|^2\right]. \quad (1)$$

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The matrix Π is a positive-definite regularization matrix, and $W = (\lambda^N \oplus \lambda^{N-1} \oplus \cdots \oplus 1)$. The symbol $*$ denotes complex conjugate transposition. The individual entries of y_N will be denoted by $\{d(i)\}$, and the individual rows of the matrix H_N will be denoted by $\{u_i\}$. The RLS algorithm computes the optimal solution of problem (1) recursively as follows:

$$\begin{aligned} w_{N+1} &= w_N + g_{N+1}[d(N+1) - u_{N+1}w_N] && (2)\\ g_{N+1} &= \lambda^{-1}P_N u_{N+1}^* \gamma(N+1) && (3)\\ \gamma^{-1}(N+1) &= 1 + \lambda^{-1}u_{N+1}P_N u_{N+1}^* && (4)\\ P_{N+1} &= \lambda^{-1}P_N - g_{N+1}\gamma^{-1}(N+1)g_{N+1}^* && (5) \end{aligned}$$

with $w_{-1} = 0$ and $P_{-1} = \Pi$. When the regression vectors possess shift structure, it is well known that these recursions can be replaced by more efficient ones. Now, consider the generalized orthonormal network of Fig. 1 with transfer function (the case of equal poles, which corresponds to a Laguerre network, is treated in [2]):

$$G(z) = \sum_{i=0}^{M-1} w_i \frac{\sqrt{1-|a_i|^2}}{1-a_i z^{-1}} \prod_{k=0}^{i-1} \frac{z^{-1}-a_k^*}{1-a_k z^{-1}}, \quad |a_k| < 1, \quad (6)$$

with coefficients $\{w_k\}$.

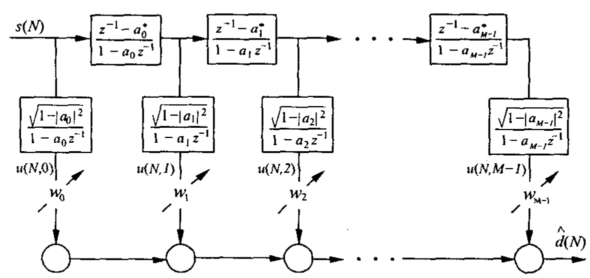

Fig. 1. *Transversal orthonormal structure for adaptive filtering.*

The input to the orthonormal network at time N is denoted by $s(N)$. Using (6) we can relate two successive regression vectors u_N and u_{N+1} as

$$\begin{aligned} u_{M+1,N} &= [u(N+1,0) \quad u_N] = [u_{N+1} \quad u(N,M-1)]\Phi \\ &= \bar{u}_{M+1,N+1}\Phi, && (7) \end{aligned}$$

where Φ is the $(M+1)\times(M+1)$ matrix (e.g., for $M=5$)

$$\Phi = \begin{bmatrix} 1 & a_0^* & 0 & 0 & 0 & 0 \\ 0 & b_1 b_0 & a_1^* & 0 & 0 & 0 \\ 0 & -a_1 b_2 b_0 & b_1 b_2 & a_2^* & 0 & 0 \\ 0 & a_1 a_2 b_3 b_0 & -a_2 b_3 b_1 & b_3 b_2 & a_3^* & 0 \\ 0 & -a_1 a_2 a_3 b_0/b_4 & a_2 a_3 b_1/b_4 & -a_3 b_2/b_4 & b_3/b_4 & 0 \\ 0 & \frac{a_1 a_2 a_3 a_4 b_0}{b_4} & \frac{-a_2 a_3 a_4 b_1}{b_4} & \frac{-a_3 a_4 b_2}{b_4} & \frac{-a_4 b_3}{b_4} & 1 \end{bmatrix} \tag{8}$$

where we defined $b_i \triangleq \sqrt{1-|a_i|^2}$. Note that the regression vectors are not shifted versions of each other. Still, we shall show that fast RLS algorithms are possible.

2.1. Forward Estimation Problem

Consider the input data matrix $H_{M,N}$ and define the coefficient matrix (note that we are now indicating explicitly the column dimension of H_N, since we will be dealing with order-recursive relations):

$$P_{M,N}^{-1} = (\lambda^{N+1}\Pi_M^{-1} + H_{M,N}^* W_N H_{M,N}).$$

Now suppose that one more column is appended to $H_{M,N}$ from the left, i.e.,

$$H_{M+1,N} = [\; x_{0,N} \quad H_{M,N} \;] \tag{9}$$

and let

$$P_{M+1,N}^{-1} = (\lambda^{N+1}\Pi_{M+1}^{-1} + H_{M+1,N}^* W_N H_{M+1,N})$$

where $\Pi_{M+1}^{-1} = (\mu \oplus \Pi_M^{-1})$. Then it is easy to verify that

$$P_{M+1,N} = \begin{bmatrix} 0 & 0 \\ 0 & P_{M,N} \end{bmatrix} + \frac{1}{\zeta_M^f(N)} \begin{bmatrix} 1 \\ -w_{M,N}^f \end{bmatrix} \begin{bmatrix} 1 & -w_{M,N}^{f*} \end{bmatrix} \tag{10}$$

where $w_{M,N}^f$ is the solution to the least-squares problem

$$\min_{w_M^f} \left[\mu\lambda^{N+1} \|\Pi_M^{-1/2} w_M^f\|^2 + \|W_N^{1/2}(x_{0,N} - H_{M,N} w_M^f)\|^2 \right]$$

whose minimum cost we denote by $\xi_M^f(N)$. It holds that $\zeta_M^f(N) = \mu\lambda^{N+1} + \xi_M^f(N)$. Now, the following equations constitute the update of the quantities of this problem:

$$w_{M,N}^f = w_{M,N-1}^f + k_{M,N} f_M(N) \tag{11}$$

$$k_{M+1,N} = \begin{bmatrix} 0 \\ k_{M,N} \end{bmatrix} + \frac{\alpha_M^*(N)}{\lambda\zeta_M^f(N-1)} \begin{bmatrix} 1 \\ -w_{M,N-1}^f \end{bmatrix} \tag{12}$$

$$\zeta_M^f(N) = \lambda\zeta_M^f(N-1) + \alpha_M^*(N) f_M(N) \tag{13}$$

$$\gamma_{M+1}(N) = \gamma_M(N) \frac{\lambda\zeta_M^f(N-1)}{\zeta_M^f(N)} \tag{14}$$

where $k_{M,N} \triangleq g_{M,N}\gamma_M^{-1}(N)$ is the normalized gain vector, $\alpha_M(N)$ and $f_M(N)$ are the *a priori* and *a posteriori* forward prediction errors, related via $f_M(N) = \alpha_M(N)\gamma_M(N)$. Note that no information on the data structure is needed in order to derive these equations (see [1]).

2.2. Backward Estimation Problem

Similarly to the forward estimation problem, assume that one more column is appended to $H_{M,N}$ from the right, i.e.,

$$\bar{H}_{M+1,N} = [\; H_{M,N} \quad x_{M,N} \;] \tag{15}$$

and define the corresponding coefficient matrix as

$$\bar{P}_{M+1,N}^{-1} = (\lambda^{N+1}\bar{\Pi}_{M+1}^{-1} + \bar{H}_{M+1,N}^* W_N \bar{H}_{M+1,N})$$

where

$$\bar{\Pi}_{M+1}^{-1} = \begin{bmatrix} \Pi_M^{-1} & c \\ c^* & \delta \end{bmatrix} \tag{16}$$

for some constant vector c and scalar δ to be specified. Inverting both sides, we obtain:

$$\bar{P}_{M+1,N} = \begin{bmatrix} P_{M,N} & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{\zeta_M^b(N)} \begin{bmatrix} -q_N \\ 1 \end{bmatrix} \begin{bmatrix} -q_N^* & 1 \end{bmatrix} \tag{17}$$

This equation has two main differences with respect to the definition of the variables $w_{M,N}^f$ and $\zeta_M^f(N)$, for the forward prediction problem. The vector q_N is the sum of two quantities,

$$q_N = w_{M,N}^b + t_N \tag{18}$$

where

$$t_N = \lambda^{N+1} P_{M,N} c\,. \tag{19}$$

The first term of (18) is the solution to the least-squares problem:

$$\min_{w_M^b} \left[\lambda^{N+1} \|\Pi_M^{-1/2} w_M^b\|^2 + \|W_N^{1/2}(x_{M,N} - H_{M,N} w_M^b)\|^2 \right]$$

where $\xi_M^b(N)$ is the corresponding minimum cost. Substituting Eq. (5) into (19), we obtain a recursive relation for t_N (which is analogous to the time-update for $w_{M,N}^b$), and it further implies the following time-update for q_N:

$$q_N = q_{N-1} + \eta_M(N) k_{M,N}$$

where $\eta_M(N) = \varepsilon_M(N)\gamma_M(N)$, and

$$\varepsilon_M(N) = \beta_M(N) - u_{M,N} t_{N-1}\,.$$

In addition, the quantity $\zeta_M^b(N)$ is defined by

$$\zeta_M^b(N) \triangleq \xi_M^b(N) + \lambda^{N+1}(\delta - c^* t_N - c^* w_{M,N}^b - w_{M,N}^{b*} c)\,.$$

Although the update of these terms may look complicated, using the time-update for $w_{M,N}^b$ and t_N, we obtain after some manipulations

$$\zeta_M^b(N) = \lambda\zeta_M^b(N-1) + \varepsilon_M^*(N)\eta_M(N)\,. \tag{20}$$

Also, multiplying (17) from the right by $\bar{u}_{M+1,N+1}^*$, we obtain, similar to the forward estimation problem,

$$\begin{bmatrix} k_{M,N} \\ 0 \end{bmatrix} = \bar{k}_{M+1,N} - \nu_M(N) \begin{bmatrix} -w_{M,N-1}^b \\ 1 \end{bmatrix} \tag{21}$$

where $\nu_M(N) = \varepsilon_M(N)/\lambda\zeta_M^b(N-1)$. The quantity $\nu_M(N)$ is referred to as the *rescue variable* and can be directly obtained as the last entry of $\bar{k}_{M+1,N}$ (to be computed further ahead).

Proceeding similarly to the derivation of (14), we obtain

$$\gamma_M(N) = \bar{\gamma}_{M+1}(N)[1 - \bar{\gamma}_{M+1}(N)\varepsilon_M(N)\nu_M(N)]^{-1} .$$

Note that the variables $\varepsilon_M(N)$ and $\eta_M(N)$ play roles similar to the a priori and a posteriori backward prediction problems. However, although all the quantities related to the backward prediction problems satisfy identical recursive equations, here they have different interpretations.

2.3. Exploiting Data Structure

We still need to evaluate $\bar{k}_{M,N}$. For this purpose, we need to identify the variable that is affected by the input data structure. Thus, consider any invertible matrix Φ such as in (8). From Eq. (7), it follows that

$$\bar{H}_{M+1,N+1} = \begin{bmatrix} 0 \\ H_{M+1,N} \end{bmatrix} \Phi^{-1}$$

where $H_{M+1,N}$ and $\bar{H}_{M+1,N+1}$ are the corresponding augmented input data matrices. We then get

$$\begin{aligned} \bar{P}_{M+1,N+1} &= (\lambda^{N+2}\bar{\Pi}_{M+1}^{-1} + \bar{H}_{M+1,N+1}^{*}W_{N+1}\bar{H}_{M+1,N+1})^{-1} \\ &= (\lambda^{N+2}\bar{\Pi}_{M+1}^{-1} + \Phi^{-*}H_{M+1,N}^{*}W_N H_{M+1,N}\Phi^{-1})^{-1} \end{aligned}$$

Note that if we could choose

$$\bar{\Pi}_{M+1}^{-1} = \lambda^{-1}\Phi^{-*}\Pi_{M+1}^{-1}\Phi^{-1} \qquad (22)$$

we obtain a simpler relation between $\{\bar{P}_{M+1,N+1}, P_{M+1,N}\}$:

$$\boxed{\bar{P}_{M+1,N+1} = \Phi P_{M+1,N}\Phi^{*}} \qquad (23)$$

In order for this relation to hold, we need to show how to choose Π_M, c, and δ in order to satisfy (22). Substituting (16) into (22), we get

$$\begin{bmatrix} \Pi_M^{-1} & c \\ c^{*} & \delta \end{bmatrix} = \lambda^{-1}\Phi^{-*}\begin{bmatrix} \mu & 0 \\ 0 & \Pi_M^{-1} \end{bmatrix}\Phi^{-1} . \qquad (24)$$

Now, the matrix Φ^{-*} can be defined block-wise as

$$\Phi^{-*} = \begin{bmatrix} \bar{v} & \bar{T} \\ 0 & m \end{bmatrix}$$

where

$$m = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix} \text{ and}$$

$$\bar{v} = \begin{bmatrix} \frac{a_1a_2a_3a_4b_0}{b_4} & \frac{-a_2a_3a_4b_1}{b_4} & \frac{-a_3a_4b_2}{b_4} & \frac{-a_4b_3}{b_4} & 1 \end{bmatrix}^T .$$

Initialization

μ is a small positive number; Π is the solution to (25);
c is given by (26).
$\zeta_M^f(-1) = \mu/\lambda$
$\zeta_M^b(0) = \lambda^{-1}[\Pi^{-1}]_{M-1,M-1} - c^{*}\Pi c$
$w_{M,0} = w_{M,-1}^f = 0$
$q_0 = \Pi^{-1}c$

For $N > 0$, repeat:

$u(N) = a_0u(N-1) + \sqrt{1-|a_0|^2}s(N)$

$\alpha_M(N-1) = u(N) - u_{M,N-1}w_{M,N-2}^f$
$f_M(N-1) = \gamma_M(N-1)\alpha_M(N-1)$
$k_{M+1,N-1} = \begin{bmatrix} 0 \\ k_{M,N-1} \end{bmatrix} + \frac{\alpha_M^{*}(N-1)}{\lambda\zeta_M^f(N-2)}\begin{bmatrix} 1 \\ -w_{M,N-2}^f \end{bmatrix}$
$\zeta_M^f(N-1) = \lambda\zeta_M^f(N-2) + \alpha_M^{*}(N-1)f_M(N-1)$
$w_{M,N-1}^f = w_{M,N-2}^f + k_{M,N-1}f_M(N-1)$
$\bar{\gamma}_{M+1}(N) = \gamma_M(N-1)\frac{\lambda\zeta_M^f(N-2)}{\zeta_M^f(N-1)}$
$\bar{k}_{M+1,N} = \Phi k_{M+1,N-1}$
$\nu_M(N) =$ (last entry of $\bar{k}_{M+1,N}$)
$k_{M,N} = \bar{k}_{1:M,N} + \nu_M(N)q_{N-1}$
$\varepsilon_M(N) = \lambda\zeta_M^b(N-1)\nu_M^{*}(N)$
$\gamma_M(N) = \frac{\bar{\gamma}_{M+1}(N)}{1-\bar{\gamma}_{M+1}(N)\varepsilon_M(N)\nu_M(N)}$
$\eta_M(N) = \gamma_M(N)\varepsilon_M(N)$
$\zeta_M^b(N) = \lambda\zeta_M^b(N-1) + \varepsilon_M^{*}(N)\eta_M(N)$
$q_N = q_{N-1} + k_{M,N}\eta_M(N)$

$\epsilon_M(N) = d(N) - u_{M,N}w_{M,N-1}$
$e_M(N) = \gamma_M(N)\epsilon_M(N)$
$w_{M,N} = w_{M,N-1} + k_{M,N}e_M(N)$

Table 1: *The extended fast transversal filter for orthonormal bases.*

Expanding (24), we find that

$$\lambda\Pi_M^{-1} - \bar{T}\Pi_M^{-1}\bar{T}^{*} = \mu\bar{v}\bar{v}^{*} . \qquad (25)$$

Hence, if $|a_k| < \sqrt{\lambda}$, this Lyapunov equation admits a unique positive definite solution Π_M. This is because all the eigenvalues of $\bar{T}$ are either a_k^{*} or 0, and the pair $(\lambda^{-1/2}\bar{T}, \bar{v})$ is controllable. From (24), we then obtain

$$\begin{aligned} c &= \lambda^{-1}\bar{T}\Pi_M^{-1}m^{*} \\ \delta &= \lambda^{-1}m\Pi_M^{-1}m^{*} = \lambda^{-1}[\Pi^{-1}]_{M-1,M-1} . \end{aligned} \qquad (26)$$

From (23), we can now obtain similar relations between $\{g_{M+1,N}, \bar{g}_{M+1,N+1}\}$ and $\{\gamma_{M+1}(N), \bar{\gamma}_{M+1}(N+1)\}$, and it is straightforward to show that

$$\bar{\gamma}_{M+1}(N+1) = \gamma_{M+1}(N)$$

and

$$\boxed{\bar{k}_{M+1,N+1} = \Phi k_{M+1,N}} \qquad (27)$$

This relation shows that the time update of the gain vector $k_{M,N}$, which is necessary to update the optimal solution

$$w_{M,N+1} = w_{M,N} + k_{M,N+1} e_M(N+1)$$

can be efficiently performed in three main steps: (1) Order update $k_{M,N} \rightarrow k_{M+1,N}$; (2) Time-update $k_{M+1,N} \rightarrow \bar{k}_{M+1,N+1}$; (3) Order downdate $\bar{k}_{M+1,N+1} \rightarrow k_{M,N+1}$ [i.e, Eq. (12), (27) and (21)]. Table 1 shows the resulting generalized FTF algorithm.

Note that when $a_k = 0$, we have $\Phi = I$ and therefore $\bar{k}_{M+1,N+1} = k_{M+1,N}$, in which case the recursions collapse to the FTF algorithm [6]. Equation (27) is the only recursion that uses the fact that the input data has structure. For the orthonormal basis considered here, this multiplication is essentially a convolution, and can be performed with $\mathcal{O}(M)$ operations. The cost of the usual FIR FTF algorithm is known to be $\mathcal{O}(7M)$ operations [6]. The overall cost for our extended filter is $\mathcal{O}(8M)$ operations.

3. THE EXTENDED FAST ARRAY ALGORITHM

Using the expressions for $\{P_{M+1,N}, \bar{P}_{M+1,N+1}\})$ in (10) and (17) (and ignoring the order index M), the FTF algorithm can be further motivated in a different manner by noticing that its recursions perform at each iteration the low rank factorization $\nabla_{\{P_N,\Phi\}} =$

$$\begin{bmatrix} P_N & 0 \\ 0 & 0 \end{bmatrix} - \Phi \begin{bmatrix} 0 & 0 \\ 0 & P_{N-1} \end{bmatrix} \Phi^* = L_N J L_N^* \qquad (28)$$

where J is an $r \times r$ signature matrix and

$$L_N = \begin{bmatrix} \begin{matrix} -q_N \\ 1 \end{matrix} & \Phi \begin{bmatrix} 1 \\ -w^f_{M,N-1} \end{bmatrix} \end{bmatrix} \begin{bmatrix} \frac{1}{\zeta_M^{b/2}(N)} & \\ & \frac{1}{\zeta_M^{f/2}(N-1)} \end{bmatrix}$$

More generally, it can be shown (see, e.g. [7, 3]) that by forcing the initial difference to have low rank, and a certain inertia, we end up forcing all the successive differences to have a similar property. This fact is the basis for the existence of a fast recursion that does not necessarily propagate the difference $\nabla_{\{P_{M,N},\Phi\}}$ explicitly. That is, we need to find a matrix $P_0 = \Pi$ such that the difference $\nabla_{\{\Pi,\Phi\}}$ has low rank. Expanding this difference, we obtain

$$\nabla_{\{P_0,\Phi\}} = \begin{bmatrix} \lambda^{-1}\Pi - T\Pi T^* & -T\Pi v^* \\ -v\Pi T^* & -v\Pi v^* \end{bmatrix} \qquad (29)$$

where $[T]_{ij} = [\bar{T}]_{M-1-j,M-1-i}$ and $v = \bar{v}^T$. The matrix Π that results from solving (25), under the condition $|a_k| < \sqrt{\lambda}$, leads to a rank 2 difference with $J = (1 \oplus -1)$.

Alternatively, we can find another Π that leads to a rank 3 difference and requires instead the condition $|\sqrt{\lambda} a_k| < 1$. Thus consider the matrix difference $\lambda^{-1}\Pi - T\Pi T^*$ in Eq. (29). We proceed to find a positive definite matrix Π such that this difference has a rank 1 matrix factorization of the form

$$\lambda^{-1}\Pi - T\Pi T^* = hh^* . \qquad (30)$$

Again, from the properties of Lyapunov equations, we know that this equation admits a unique Hermitian solution, since all the eigenvalues of T are equal to a_k^* or 0. Moreover, since $|\sqrt{\lambda} a_k| < 1$, any vector h such that the pair $(\lambda^{1/2}T, h)$ is controllable, will result in a positive-definite solution Π.

Hence, we can choose a vector h (and consequently, Π) such that the difference $\lambda^{-1}\Pi - T\Pi T^*$ has rank one (and inertia 1). It then follows that the rank of $\nabla_{\{P_0,\Phi\}}$ in (29) will be 3 and $J = (1 \oplus -1 \oplus 1)$. For either choice of Π, the resulting fast array algorithm can be summarized as follows.

(Fast Array Algorithm) *Consider input regression vectors arising from the orthonormal structure of Figure 1. The solution to the minimization problem (1) can be recursively computed as follows. Start with $w_{-1} = 0$, $\gamma^{-1/2}(0) = 1$, $k_0 = 0$, L_0 and J from the factorization (28) at time 0, and repeat for each $N \geq 0$,*

$$\begin{bmatrix} \gamma^{-1/2}(N) & \frac{1}{\sqrt{\lambda}}[u_{N+1} \;\; u(N, M-1)]L_N \\ \Phi \begin{bmatrix} 0 \\ g_N \gamma^{-1/2}(N) \end{bmatrix} & \frac{1}{\sqrt{\lambda}} L_N \end{bmatrix} \Theta_N = \begin{bmatrix} \gamma^{-1/2}(N+1) & 0 \\ \begin{bmatrix} g_{N+1}\gamma^{-1/2}(N+1) \\ 0 \end{bmatrix} & L_{N+1} \end{bmatrix}$$

where Θ_N is a $(1 \oplus J)$-unitary matrix that produces the zero entries in the above post-array, and L_N is $(M+1) \times r$. The product with the matrix Φ defined in Eq. (8) can be implemented fast by convolution. Moreover,

$$w_{N+1} = w_N + g_{N+1}[d(N+1) - u_{N+1} w_N].$$

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