BELIEF CONTROL STRATEGIES FOR INTERACTIONS OVER WEAK GRAPHS

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ABSTRACT

In diffusion social learning over weakly-connected graphs, it has been shown that influential agents end up shaping the beliefs of non-influential agents. In this paper, we analyse this control mechanism more closely and reveal some critical properties. In particular, we characterize the set of beliefs that can be imposed on non-influential agents (i.e., the set of attainable beliefs) and how the graph topology of these latter agents helps resist manipulation but only to a certain degree. We also derive a design procedure that allows influential agents to drive the beliefs of non-influential agents to desirable attainable states. We illustrate the results with two examples.

Index Terms— Weakly-connected networks, social learning, diffusion strategy, belief control.

1. INTRODUCTION AND MOTIVATION

Several studies have examined the propagation of information over social networks and the influence of graph topology on this dynamics [1–12]. In recent work [11, 12], an intriguing phenomenon was revealed whereby it was shown that weakly-connected graphs enable certain agents to control the opinion of other agents to great degree, irrespective of the observations sensed by these latter agents. For example, agents can be made to believe that it is "raining" while they happen to be observing "sunny conditions". Weak graphs arise in many contexts, including in popular social platforms like Twitter and similar venues. In these graphs, the topology consists of multiple sub-networks where at least one sub-network (called a sending sub-network) feeds information in one direction to other network components without receiving back any (or being interested in any) information from them. For such networks, it was shown in [12] that, irrespective of the local observations sensed by the receiving networks, a sending sub-network can end up playing a domineering effect and influence the beliefs of the other groups in a significant manner. In particular, receiving agents can be made to arrive at incorrect inference decisions; they can also be made to disagree on their inferences among themselves.

The purpose of this article is three-fold. First, to show that the internal graph structure of receiving networks imposes a form of resistance to manipulation, but only to a certain degree. Second, to characterize the set of beliefs that can be imposed on receiving networks. And, third, given an attainable desirable response, to develop a control mechanism that allows sending networks to force the receiving networks to behave in that manner.

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1.1. Weakly-Connected Graphs

As described in [11, 12], a weakly-connected network consists of two types of sub-networks: S (sending) sub-networks and R (receiving) sub-networks. Each individual sub-network is a *connected* graph where any two agents are connected by a path. In addition, every sending sub-network is also *strongly*-connected, meaning that at least one of its agents has a self-loop. The flow of information between S and R sub-networks is asymmetric, as it only happens in one direction from S to R. Figure 1 shows one example of a weakly-connected network. The two top sub-networks are sending sub-networks and the two bottom sub-networks are receiving sub-networks. The weights on the connections from S to R networks are positive but can be small.

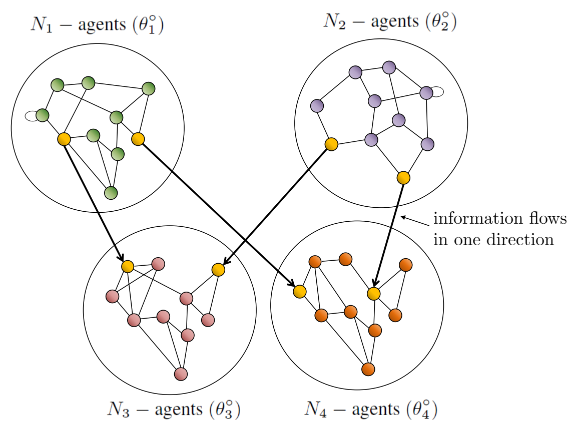

Fig. 1: An example of a weakly connected network. The two sub-networks on top are $S-$type, while the two sub-networks in the bottom are $R-$type.

We index strongly-connected sub-networks by $s = \{1, 2, \cdots, S\}$, and receiving sub-networks by $r = \{S+1, \ldots, S+R\}$. Each sub-network s has N_s agents, and the total number of agents in the S sub-networks is denoted by N_{gS}. Similarly, each sub-network r has N_r agents, and the total number of agents in the R sub-networks is denoted by N_{gR}. We let N denote the total number of agents across all sub-networks, i.e., $N = N_{gS} + N_{gR}$, and use $\mathcal{N} = \{1, 2, \cdots, N\}$ to refer to the indexes of all agents. We assign a pair of non-negative weights, $\{a_{k\ell}, a_{\ell k}\}$, to the edge connecting any two agents k and ℓ. The scalar $a_{\ell k}$ represents the weight with which agent k scales data arriving from agent ℓ and, similarly, for $a_{k\ell}$. We let $\mathcal{N}_k$ denote the neighborhood of agent k, which consists of all agents connected to k. Each agent k scales data arriving from its neighbors in a convex manner, i.e., the weights satisfy:

$$a_{\ell k} \geq 0, \quad \sum_{\ell \in \mathcal{N}_k} a_{\ell k} = 1, \quad a_{\ell k} = 0 \text{ if } \ell \notin \mathcal{N}_k \tag{1}$$

Using the same notation from [11, 12], we continue to assume that the agents are numbered such that the indexes of $\mathcal{N}$ represent first the agents from the S sub-networks, followed by those from the R sub-networks. In this way, if we collect the $\{a_{\ell k}\}$ into a large $N \times N$ combination matrix A, then A will have an upper block-triangular structure of the following form:

$$\overbrace{}^{\text{Subnetworks:}1,2,\ldots,S} \quad \overbrace{\phantom{A_{1,S+1} \; A_{1,S+2} \; \ldots \; A_{1,S+R}}}^{\text{Subnetworks:}S+1,S+2,\ldots,S+R}$$
$$\left[\begin{array}{cccc|cccc} A_1 & 0 & \ldots & 0 & A_{1,S+1} & A_{1,S+2} & \ldots & A_{1,S+R} \\ 0 & A_2 & \ldots & 0 & A_{2,S+1} & A_{2,S+2} & \ldots & A_{2,S+R} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & A_S & A_{S,S+1} & A_{S,S+2} & \ldots & A_{S,S+R} \\ \hline 0 & 0 & \ldots & 0 & A_{S+1} & A_{S+1,S+2} & \ldots & A_{S+1,S+R} \\ 0 & 0 & \ldots & 0 & 0 & A_{S+2} & \ldots & A_{S+2,S+R} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & A_{S+R} \end{array}\right] \tag{2}$$

The matrices $\{A_1, \cdots, A_S\}$ on the upper left corner are left-stochastic primitive matrices corresponding to the S strongly-connected sub-networks. Likewise, the matrices $\{A_{S+1}, \cdots, A_{S+R}\}$ in the lower right-most block correspond to the internal weights of the R sub-networks. We denote the block structure of A in (2) by:

$$A \;\triangleq\; \left[\begin{array}{c|c} T_{SS} & T_{SR} \\ \hline 0 & T_{RR} \end{array}\right] \tag{3}$$

2. DIFFUSION SOCIAL LEARNING

We assume that each sub-network has a true state value, denoted generically by θ°, which may differ from one sub-network to another. We denote by Θ the set of all possible states, by θ_s° the true state of sending sub-network s and by θ_r° the true state of receiving sub-network r, where both θ_s° and θ_r° are in Θ. At each time i, each agent k will possess a belief $\boldsymbol{\mu}_{k,i}(\theta)$, which represents a probability distribution over $\theta \in \Theta$. Agent k continuously updates its belief according to two information sources:

1. The first source consists of observational signals $\{\boldsymbol{\xi}_{k,i}\}$ streaming in locally at agent k. These signals are generated according to some known likelihood function parametrized by the true state of agent k. We denote the likelihood function by $L_k(.|\theta_r^\circ)$ if agent k belongs to receiving sub-network r or $L_k(.|\theta_s^\circ)$ if agent k belongs to sending sub-network s.
2. The second source consists of information received from the neighbors of agent k, denoted by $\mathcal{N}_k$. Agent k and its neighbors are connected by edges and they continuously communicate and share their opinions.

Using these two pieces of information, each agent k then updates its belief according to the following diffusion social learning rule [1]:

$$\begin{cases} \boldsymbol{\psi}_{k,i}(\theta) = \dfrac{\boldsymbol{\mu}_{k,i-1}(\theta) L_k(\boldsymbol{\xi}_{k,i}|\theta)}{\sum_{\theta' \in \Theta} \boldsymbol{\mu}_{k,i-1}(\theta') L_k(\boldsymbol{\xi}_{k,i}|\theta')} \\ \boldsymbol{\mu}_{k,i}(\theta) = \displaystyle\sum_{\ell \in \mathcal{N}_k} a_{\ell k}\, \boldsymbol{\psi}_{\ell,i}(\theta) \end{cases} \tag{4}$$

A consensus-based strategy can also be employed, as was done in [2, 13], although the latter reference focuses instead on the problem of pure averaging and not on social learning and requires the existence of certain anchor nodes. Here, we assume all agents are homogeneous and focus on the diffusion strategy due to its enhanced performance, as observed in [1] and as further explained in the treatments [14, 15]. Other models for social learning can be found in [3, 4, 6, 9, 10]. In the first step of (4), agent k updates its belief, $\boldsymbol{\mu}_{k,i-1}(\theta)$, based on its observed private signal $\boldsymbol{\xi}_{k,i}$ by means of the Bayesian rule and obtains an intermediate belief $\boldsymbol{\psi}_{k,i}(\theta)$. In the second step, agent k learns from its social neighbors.

When agents of sending sub-networks follow this model, they can learn their own true states. It was shown in [1] that

$$\lim_{i\to\infty} \boldsymbol{\mu}_{k,i}(\theta_s^\circ) \overset{a.s.}{=} 1 \tag{5}$$

for any agent k that belongs to sending sub-network s. On the other hand, agents of receiving sub-networks will not be able to find their true states. Instead, their beliefs will converge to a fixed distribution defined over the true states of the *sending* sub-networks as follows. First, we collect in the column vectors

$$\boldsymbol{\mu}_{\mathcal{S},i}(\theta) \;\triangleq\; \text{col}\left\{\boldsymbol{\mu}_{1,i}(\theta), \boldsymbol{\mu}_{2,i}(\theta), \ldots, \boldsymbol{\mu}_{N_{gS},i}(\theta)\right\} \tag{6}$$

$$\boldsymbol{\mu}_{\mathcal{R},i}(\theta) \;\triangleq\; \text{col}\left\{\boldsymbol{\mu}_{N_{gS}+1,i}(\theta), \boldsymbol{\mu}_{N_{gS}+2,i}(\theta), \ldots, \boldsymbol{\mu}_{N,i}(\theta)\right\} \tag{7}$$

all beliefs from all agents in the S and R sub-networks respectively. Note that these belief vectors are evaluated at a specific $\theta \in \Theta$. Then, under some technical assumptions, it was shown in [12, 16] that

$$\lim_{i\to\infty} \boldsymbol{\mu}_{\mathcal{R},i}(\theta) = W^\mathsf{T} \left(\lim_{i\to\infty} \boldsymbol{\mu}_{\mathcal{S},i}(\theta)\right) \tag{8}$$

where W is the $N_{gS} \times N_{gR}$ matrix given by:

$$W \quad \triangleq \quad T_{SR}(I - T_{RR})^{-1} \tag{9}$$

and I denotes the identity matrix of size N_{gR}. The matrix W has non-negative entries and the sum of the entries in each of its columns is equal to one [11]. We can expand (8) to reveal the influence of the sending networks more explicitly as follows.

Let w_k^T denote the row in W^T that corresponds to receiving agent k and partition it into sub-vectors as follows[1]:

$$w_k^\mathsf{T} = \left[\begin{array}{c|c|c|c} w_{k,N_1}^\mathsf{T} & w_{k,N_2}^\mathsf{T} & \ldots & w_{k,N_S}^\mathsf{T} \end{array}\right] \tag{10}$$

where the $\{N_1, N_2, \ldots, N_S\}$ are the number of agents in each sub-network $s \in \{1, 2, \ldots, S\}$. Moreover, let

$$e_{\theta,\theta_s^\circ} \;\triangleq\; \begin{cases} \mathbb{1}_{N_s}, & \text{if} \quad \theta = \theta_s^\circ \\ \mathbf{0}_{N_s}, & \text{otherwise} \end{cases} \tag{11}$$

where $\mathbb{1}_{N_s}$ denotes a column vector of length N_s whose elements are all one. Similarly, $\mathbf{0}_{Ns}$ denotes a column vector of length N_s whose elements are all zero. Then, according to (8), we have

$$\lim_{i\to\infty} \boldsymbol{\mu}_{k,i}(\theta) = \sum_{s=1}^{S} w_{k,N_s}^\mathsf{T} e_{\theta,\theta_s^\circ} \tag{12}$$

Result (12) means that the belief of receiving agent k will converge to a distribution defined over the true states of the *sending* sub-networks: $\Theta^\bullet = \{\theta_1^\circ, \theta_2^\circ, \ldots, \theta_S^\circ\}$.

[1] The index of the row in W^T that corresponds to agent k is $k - N_{gS}$.

3. BELIEF CONTROL MECHANISM

Results (8) and (12) suggest that it should be possible for influential agents to control the steady-state beliefs of receiving agents. Expression (8), in particular, shows how the limiting distributions of the sending sub-networks determine the limiting distributions of the receiving sub-networks through the matrix $W = T_{SR}(I - T_{RR})^{-1}$. Two questions arise at this stage: (a) can receiving agents be driven to arbitrary beliefs or does the network structure limit the scope of control by the influential agents? and (b) even if there is a limit to what influential agents can accomplish, how can they ensure that receiving agents will end up with particular beliefs?

To answer these questions, we start by referring to (12). Let $q_k(\theta)$ denote the desired final distribution for receiving agent k. We would like to examine first how the entries of T_{SR} should be designed to force the receiving agent to converge to this specific $q_k(\theta)$. We would also like to examine whether it is possible to force agent k to converge to *any* $q_k(\theta)$.

3.1. Characterizing Attainable Beliefs

As is already evident from (12), the belief $q_k(\theta)$ needs to be a probability distribution defined over the true states of all *sending* sub-networks, $\Theta^\bullet = \{\theta_1^\circ, \theta_2^\circ, \ldots, \theta_S^\circ\}$. We assume, without loss of generality, that the true states of the sending sub-networks are distinct, so that $|\Theta^\bullet| = S$. If two or more sending sub-networks have the same true state, we can blend them together and treat them as corresponding to one sending sub-network; although this enlarged component is not strongly-connected, it nevertheless consists of strongly-connected elements and the same arguments will apply.

We collect the desirable limiting beliefs for all receiving agents into the vector:

$$q_{\mathcal{R}}(\theta) \;\triangleq\; \text{col}\left\{q_{N_{gS}+1,i}(\theta), q_{N_{gS}+2,i}(\theta), \ldots, q_{N,i}(\theta)\right\} \tag{13}$$

which has length N_{gR}. Then, from (8), we must have:

$$q_{\mathcal{R}}^{\mathsf{T}}(\theta) = \left(\lim_{i\to\infty} \boldsymbol{\mu}_{\mathcal{S},i}(\theta)\right)^{\mathsf{T}} W \tag{14}$$

Evaluating this expression at the successive states $\{\theta_1^\circ, \theta_2^\circ, \ldots, \theta_S^\circ\}$, we get

$$\underbrace{\begin{bmatrix} q_{\mathcal{R}}^{\mathsf{T}}(\theta_1^\circ) \\ q_{\mathcal{R}}^{\mathsf{T}}(\theta_2^\circ) \\ \vdots \\ q_{\mathcal{R}}^{\mathsf{T}}(\theta_S^\circ) \end{bmatrix}}_{\triangleq\, Q} = \underbrace{\begin{bmatrix} \mathbb{1}_{N_1}^{\mathsf{T}} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \mathbb{1}_{N_2}^{\mathsf{T}} & \ldots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & \mathbb{1}_{N_S}^{\mathsf{T}} \end{bmatrix}}_{\triangleq\, E} W \tag{15}$$

where Q is the $S \times N_{gR}$ matrix that collects the desired beliefs for all receiving agents. Using (9), we rewrite (15) more compactly as

$$ET_{SR} = Q\,(I - T_{RR}) \tag{16}$$

Therefore, given Q and T_{RR}, the design problem becomes one of finding a matrix T_{SR} that satisfies (16) subject to

$$\mathbb{1}^{\mathsf{T}} T_{SR} + \mathbb{1}^{\mathsf{T}} T_{RR} = \mathbb{1}^{\mathsf{T}} \tag{17}$$

$$T_{SR} \succcurlyeq 0 \tag{18}$$

$$T_{SR}(j,k) = 0, \text{ if } j \text{ does not feed } k \tag{19}$$

The first condition (17) is because the entries on each column of A defined in (3) add up to one. The second condition (18) ensures that each element of T_{SR} is a non-negative combination weight (the operator $\succcurlyeq$ is for element-wise comparison). The third condition (19) takes into account the network structure, i.e., if receiving agent k is not connected to sending agent j, the corresponding entry in T_{SR} should be zero.

It turns out that the solution to the control problem (16)–(19) can be characterized analytically, along with conditions for when a solution exists. We provide the answer here but omit the derivations due to space limitations. Let $t_{SR,k}$, $t_{RR,k}$ and q_k, respectively, denote the columns of T_{SR}, T_{RR} and Q that correspond to receiving agent k. Then,

$$v_k \;\triangleq\; q_k - Q t_{RR,k} \tag{20}$$

denotes the column that corresponds to agent k in the right hand-side of relation (16). If agent k is not connected to any agent of sub-network s, then the corresponding entry in v_k should be zero, i.e.,

$$v_k(s) = q_k(\theta_s^\circ) - \sum_{\ell > N_{gS}} a_{\ell k} q_\ell(\theta_s^\circ) = 0 \tag{21}$$

Otherwise, no solution exists for the problem. This means that to force receiving agent k to adopt a specific belief at θ_s°, this agent must be connected to at least one agent from sending sub-network s, or the cumulative influence from the agent's neighbours must match the desired limiting belief. In this case, we set the corresponding entries of $t_{SR,k}$ to zero.

If agent k is connected to some agents of sub-network s, then the corresponding entry in v_k should be non-negative, i.e.,

$$v_k(s) = q_k(\theta_s^\circ) - \sum_{\ell > N_{gS}} a_{\ell k} q_\ell(\theta_s^\circ) \geq 0 \tag{22}$$

Let $t_{SR,k}^{s}$ denote the block of $t_{SR,k}$ that includes the weights with which agent k scales data arriving from all agents of sending sub-network s. Some of these entries should be set to zero if they correspond to the agents not connected to agent k. We remove these zero elements from $t_{SR,k}^{s}$ and label the modified block as $t_{SR,k}^{s'}$. Then, if condition (22) is satisfied, then the solution takes the following form:

$$t_{SR,k}^{s'} = \frac{v_k(s)}{N_s^k} \mathbb{1}_{N_s^k} + \left(I_{N_s^k} - \frac{1}{N_s^k} \mathbb{1}_{N_s^k} \mathbb{1}_{N_s^k}^{\mathsf{T}}\right) y_{N_s^k} \tag{23}$$

where N_s^k is the number of agents of sending sub-network s that are connected to agent k and $y_{N_s^k}$ is an arbitrary vector that must be chosen so that $t_{SR,k}^{s'} \succeq 0$.

Theorem 1 *(***Attainable beliefs and control mechanism***) Let C denote an $S \times N_{gR}$ binary matrix, with as many rows as the number of sending sub-networks and as many columns as the number of receiving agents. The $(s,r)-$th entry of C is one if receiving agent r is connected to sending sub-network s; otherwise, it is zero. Define the difference matrix*

$$V \;\triangleq\; Q(I - T_{RR}) \tag{24}$$

Then, a given belief matrix Q is attainable if, and only if, the entries of V will be zero wherever the entries of C are zero, and the entries *of V will be non-negative wherever the entries of C are one. That is, the matrices V and C must have the same structure with the unit entries of C translated into non-negative entries in V. In that case, the control mechanism can be implemented by selecting the columns of the combination matrix T_{SR} according to (23).*

The belief of a particular receiving agent k can be controlled to q_k, (the $k-$th column of Q), if the corresponding columns of V and C have the same structure, as described above. ■

We next illustrate the results with some examples.

3.2. Case Involving Broadband Influence

Consider the network shown in Fig. 2 (*Left*). It consists of $N = 8$ agents, two sending sub-networks and one receiving sub-network, with the following combination matrix:

$$A = \left[\begin{array}{ccccc|ccc} 0.2 & 0.2 & 0.8 & 0 & 0 & 0 & 0 & \times \\ 0.5 & 0.4 & 0.1 & 0 & 0 & \times & 0 & 0 \\ 0.3 & 0.4 & 0.1 & 0 & 0 & \times & \times & 0 \\ 0 & 0 & 0 & 0.4 & 0.3 & \times & 0 & \times \\ 0 & 0 & 0 & 0.6 & 0.7 & 0 & \times & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0.2 & 0.3 & 0.2 \\ 0 & 0 & 0 & 0 & 0 & 0.1 & 0.2 & 0.3 \\ 0 & 0 & 0 & 0 & 0 & 0.1 & 0.2 & 0.1 \end{array}\right] \quad (25)$$

We assume that there are 3 possible states $\Theta = \{\theta_1^\circ, \theta_2^\circ, \theta_3^\circ\}$, where θ_1° is the true event for the first sending sub-network, θ_2° is the true event for the second sending sub-network, and θ_3° is the true event for the receiving sub-network.

Let us design T_{SR} so that all receiving agents' beliefs converge to the same belief, say,

$$Q = \begin{bmatrix} 0.2 & 0.2 & 0.2 \\ 0.8 & 0.8 & 0.8 \end{bmatrix} \quad (26)$$

Computing v_k defined in (20) for each receiving agent k, we obtain:

$$v_6 = \begin{bmatrix} 0.12 \\ 0.48 \end{bmatrix}, v_7 = \begin{bmatrix} 0.06 \\ 0.24 \end{bmatrix}, v_8 = \begin{bmatrix} 0.08 \\ 0.24 \end{bmatrix} \quad (27)$$

Note that in this example, all receiving agents are connected to all sending sub-networks. Therefore the only requirement is to have all the entries of v_k non-negative, which is the case. Therefore, from (23), one possible choice for T_{SR} is the following:

$$T_{SR} = \begin{bmatrix} 0 & 0 & 0.08 \\ 0.06 & 0 & 0 \\ 0.06 & 0.06 & 0 \\ 0.48 & 0 & 0.32 \\ 0 & 0.24 & 0 \end{bmatrix} \quad (28)$$

To verify that the beliefs of the receiving agents converge in this case to the desired belief (26), we compute the matrix W^T from (9):

$$\begin{array}{c} 0.2 \\ \leftarrow \\ 0.2 \\ \leftarrow \\ 0.2 \\ \leftarrow \end{array} \left[\begin{array}{ccc|cc} 0.0169 & 0.0839 & 0.0992 & 0.7390 & 0.0610 \\ 0.0322 & 0.0394 & 0.1284 & 0.4441 & 0.3559 \\ 0.1034 & 0.0318 & 0.0648 & 0.6678 & 0.1322 \end{array}\right] \begin{array}{c} 0.8 \\ \rightarrow \\ 0.8 \\ \rightarrow \\ 0.8 \\ \rightarrow \end{array} \quad (29)$$

Then, by (12), we compute the limiting beliefs at θ_1° and θ_2°, by summing the elements of the corresponding blocks and we obtain the required limiting beliefs (26).

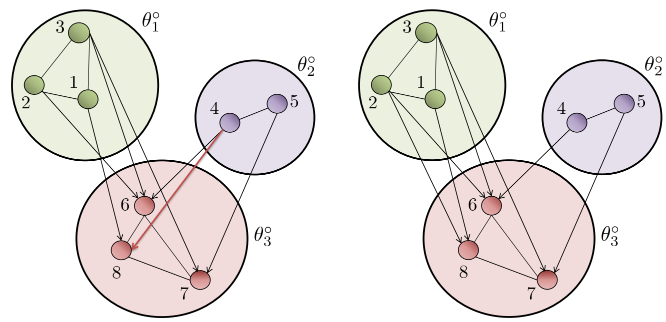

Fig. 2: (*Left*) Broadband influence. (*Right*) Narrowband influence.

3.3. Case Involving Narrowband Influence

Consider now the network shown in Fig. 2 (*Right*) with combination matrix:

$$A = \left[\begin{array}{ccccc|ccc} 0.2 & 0.2 & 0.8 & 0 & 0 & 0 & 0 & \times \\ 0.5 & 0.4 & 0.1 & 0 & 0 & \times & 0 & \times \\ 0.3 & 0.4 & 0.1 & 0 & 0 & \times & \times & 0 \\ 0 & 0 & 0 & 0.4 & 0.3 & \times & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0.7 & 0 & \times & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0.2 & 0.3 & 0.1 \\ 0 & 0 & 0 & 0 & 0 & 0.1 & 0.2 & 0.6 \\ 0 & 0 & 0 & 0 & 0 & 0.1 & 0.2 & 0 \end{array}\right] \quad (30)$$

The main difference between both networks in Fig. 2 is that now agent 8 is not connected to sending sub-network 2. Let us consider the case where we want to design T_{SR} so that the desired limiting beliefs are as follows (i.e., a situation where discord among the agents is being promoted):

$$Q = \begin{bmatrix} 0.8 & 0.7 & 0.8 \\ 0.2 & 0.3 & 0.2 \end{bmatrix} \quad (31)$$

Computing v_k for each receiving agent k, we obtain:

$$v_6 = \begin{bmatrix} 0.49 \\ 0.11 \end{bmatrix}, v_7 = \begin{bmatrix} 0.16 \\ 0.14 \end{bmatrix}, v_8 = \begin{bmatrix} 0.3 \\ 0 \end{bmatrix} \quad (32)$$

Note that in this example, agent 8 is not connected to the second sending sub-network, but the controlling scheme can still work because condition (21) is satisfied. Therefore, one possible choice for T_{SR} is the following:

$$T_{SR} = \begin{bmatrix} 0 & 0 & 0.3/2 \\ 0.49/2 & 0 & 0.3/2 \\ 0.49/2 & 0.16 & 0 \\ 0.11 & 0 & 0 \\ 0 & 0.14 & 0 \end{bmatrix} \quad (33)$$

To verify that the beliefs of the agents converge in this case to the desired belief, let us compute W^T from (9):

$$\begin{array}{c} 0.8 \\ \leftarrow \\ 0.7 \\ \leftarrow \\ 0.8 \\ \leftarrow \end{array} \left[\begin{array}{ccc|cc} 0.0309 & 0.3737 & 0.3954 & 0.1539 & 0.0461 \\ 0.0586 & 0.2200 & 0.4214 & 0.0724 & 0.2276 \\ 0.1883 & 0.3193 & 0.2924 & 0.0588 & 0.1412 \end{array}\right] \begin{array}{c} 0.2 \\ \rightarrow \\ 0.3 \\ \rightarrow \\ 0.2 \\ \rightarrow \end{array} \quad (34)$$

Then, by (12), we compute the limiting beliefs at θ_1° and θ_2°, by summing the elements of the corresponding blocks and we obtain the desired beliefs as in (31).

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