Foraging Behavior of Fish Schools via Diffusion Adaptation

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***Abstract*—Fish organize themselves into schools as a way to defend against predators and improve foraging efficiency. In this work we develop a model for food foraging and explain how a school of fish can move as a group if every fish were to employ a distributed strategy, known as diffusion adaptation. The algorithm assumes the fish sense the general direction of food and can also infer the general direction of motion of their neighbors. The result indicates that a simple diffusion algorithm can account for the foraging behavior. The study also reveals that some form of communication among the fish is crucial to achieve schooling.**

***Index Terms*—Distributed signal processing, self-organization, diffusion, adaptation, fish schools, adaptive networks.**

I. INTRODUCTION

Self-organization is a remarkable property of nature and it has been observed in several physical and biological systems. Examples include fish joining together in schools, chemicals forming spirals, and sand grains assembling into rippling dunes [1]. In self-organizing systems, a global pattern emerges from interactions among the individual components of the system. In this work we focus on the schooling behavior of fish while searching for food.

Fish form schools and move together in remarkable harmony. Biologically, there are advantages provided by the schooling behavior such as defense against predators and foraging efficiency [2]. Many models have been proposed in the literature to explain the schooling behavior of fish [3]-[8]. Most of the models assume that a fish adjusts its swimming direction along the average direction of its neighboring fish. While these models can help explain the group behavior, they nevertheless neglect the influence of local information processing by the fish and how coordinated processing can help direct the fish school towards a particular food source. In other words, we are not only interested in explaining the pattern formation of fish moving together; we are also interested in explaining how the fish school moves towards a food source (i.e., towards a specific destination). To do so, we model the fish group as a dynamic network and employ diffusion strategies to explain the foraging behavior in the presence of a food source.

Our analysis employs distributed estimation algorithms over cognitive, adaptive networks [10]-[12]. Distributed estimation algorithms are based on the principle that the network nodes obtain their estimates by communicating only with their neighbors. One class of distributed estimation algorithms is known as diffusion algorithms, whereby nodes perform an adaptation step using the available measurements, followed by a diffusion step which requires combining the estimates from the neighboring nodes [9][12]. We show that by employing the diffusion algorithm, the fish are able to move as a group in the direction of a food source.

The diffusion algorithm consists of two steps: adaptation and diffusion. Let $\psi_{k,i}$ and $\omega_{k,i}$ be adaptation and diffusion results for the kth fish at time i. The algorithm is generally described as follows:

$$\begin{cases} \psi_{k,i} = f_a\left(\omega_{k,i-1}, p_{k,i}\right) & \text{(Adaptation)} \\ \omega_{k,i} = f_d\left(\psi_{l,i}, l \in \mathcal{N}_{k,i}\right) & \text{(Diffusion)} \end{cases} \tag{1}$$

In adaptation, $\psi_{k,i}$ is updated by combining the previous diffusion result, $\omega_{k,i-1}$, with a current local estimation of the desired parameter vector, $p_{k,i}$, via some adaptation function f_a. In diffusion, $\omega_{k,i}$ is updated by fusing adaptation results from the neighboring nodes including itself, $\mathcal{N}_{k,i}$, via some diffusion function f_d. Note that, in general, the neighbors of the kth fish vary with time.

The organization of the paper is as follows. In Section II, we describe the foraging algorithm in detail. Simulation results are presented in Section III. Finally, conclusions are made in Section IV.

II. ALGORITHM

A. Food Detection Model

The signal intensity from a food source is assumed to be inversely proportional to a certain order of the distance between a fish and the food. Let d be the distance between the fish and the food source. Then the intensity is modeled as $Gd^{-\alpha}$, where G is a constant and α is a path loss exponent. To take uncertain fluctuations into account, the intensity is modeled as an exponential distribution, say,

$$y(n) = \begin{cases} v_0(n), & \text{if food does not exist} \\ v_1(n), & \text{if food exists} \end{cases} \tag{2}$$

where $v_0(n)$ and $v_1(n)$ are both exponential random variables with means σ^2 and $Gd^{-\alpha} + \sigma^2$, respectively.

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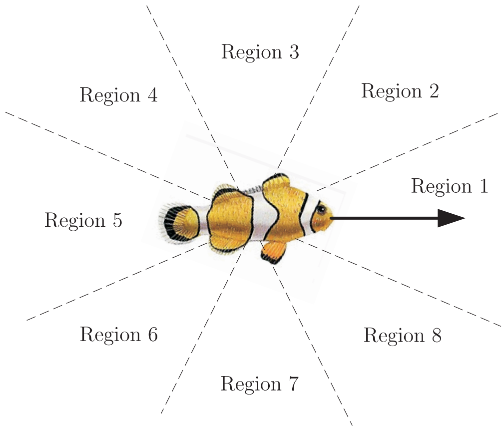

Fig. 1. Phase sectorization of a fish with $Z = 8$ (dash lines). The arrow represents the swimming direction of the fish.

Each fish detects the existence of food in the ambient area by conducting a hypothesis test. The detector is given by

$$T = \sum_{n=1}^{N} y(n) \underset{H_0}{\overset{H_1}{\gtrless}} \tau \tag{3}$$

where N and τ denote the number of samples and the threshold of the detector, respectively. To choose the threshold, we have to specify an optimal criterion. Due to lack of *a priori* probability and cost functions, we adopt the Neyman-Pearson criterion. Under the null hypothesis, $y(n)$ has exponential distribution with mean σ^2, and thus $2y(n)/\sigma^2$ becomes a χ^2 random variable with 2 degrees of freedom. Therefore, $2T/\sigma^2$ is χ^2-distributed with $2N$ degrees of freedom. The probability of false alarm can be represented by the right-tail probability of a χ^2 random variable, i.e.,

$$P_{FA} = \Pr\left(\frac{2T}{\sigma^2} \geq \frac{2\tau}{\sigma^2}\middle| H_0\right) = Q_{2N}\left(\frac{2\tau}{\sigma^2}\right) \tag{4}$$

where $Q_n(x)$ is the probability of a χ^2 random variable with n degrees of freedom being greater than x. Thus, under a desired probability of false alarm, the optimal threshold and the corresponding probability of detection can be evaluated as:

$$\tau_{opt} = \frac{\sigma^2}{2} Q_{2N}^{-1}(P_{FA}) \tag{5}$$

$$P_D = Q_{2N}\left(\frac{2\tau_{opt}}{\sigma^2 + Gd^{-\alpha}}\right) \tag{6}$$

B. Food Position

To locate the position of food, each fish needs to estimate the direction and distance of the food with respect to itself. To do this, each fish constructs its own coordinate system with the origin at the current position of the fish and the x-axis coincident with the swimming direction of that fish.

It is reasonable to assume that the fish are capable of inferring the general direction of food. Suppose the ambient area of a fish is divided equally into Z regions, as shown in Fig. 1 for $Z = 8$. A fish senses the existence of food within each region. If there are multiple regions indicating the existence of food, it will select the region with maximum signal intensity, which corresponds to the nearest location of food in probability. Then, the fish selects the direction of food, ϕ, simply as the direction of the line that bisects that region. Therefore, we have

$$\hat{z} = \arg\max_{z}\{T_z\} \tag{7}$$

$$\phi = \phi_{\hat{z}} \tag{8}$$

where T_z denotes the received signal intensity in the zth region and ϕ_z is the corresponding direction of bisection; $\phi_z = (z-1)\pi/4$ in this case. In general, we could consider non-uniform regions because a fish should have higher sensitivity in the forward direction than in the lateral direction. Furthermore, a fish may not be able to sense signal intensity in a backward direction. However, our simple model is sufficient to capture the idea of fish behavior.

When a fish detects the existence of food, the distance of food, d, can be estimated under the maximum likelihood (ML) criterion as follows. Since the mean of $y(n)$ under the alternative hypothesis is a function of d (namely, $Gd^{-\alpha} + \sigma^2$), we use the ML estimator of the mean (i.e., $T_{\hat{z}}/N$) and estimate d as:

$$d = \left(\frac{G}{T_{\hat{z}}/N - \sigma^2}\right)^{1/\alpha} \tag{9}$$

Note that the result in parenthesis is always positive if we choose a reasonable probability of false alarm, specifically, $P_{FA} < Q_{2N}(2N)$. The two parameters (d, ϕ) are used to determine the rough position of food and are spread through the fish school via the diffusion algorithm.

C. Diffusion Adaptation

The desired parameters are the coordinates of the food source. Therefore, $\psi_{k,i}$ and $\omega_{k,i}$ are 2×1 vectors with x-coordinate in the first element and y-coordinate in the second element. In addition, the estimate $p_{k,i}$ is defined as the current estimated position of the food, i.e.

$$p_{k,i} = \left[d_{k,i}\cos\phi_{k,i} \quad d_{k,i}\sin\phi_{k,i}\right]^T \tag{10}$$

The subscripts of $d_{k,i}$ and $\phi_{k,i}$ represent the kth fish at time i. However, due to the movement of fish, the coordinate system of each fish changes over time. As figure 2 shows, **Coordinate 1** represents the coordinate system at the previous step while **Coordinate 2** is at the current step. Therefore, the diffusion result at time $i-1$ has to be adjusted via coordinate transformation. With the assumption that the speed of fish is a constant v and the time duration of one step is $\triangle t$, we have:

$$\omega_{k,i-1}^{(2)} = \mathbf{U}(\delta_{k,i-1})\omega_{k,i-1} - v\triangle t\mathbf{e}_1 \tag{11}$$

where $\delta_{k,i-1}$ denotes the swimming direction adjustment of the kth fish between time $i-1$ and time i, which is specified in (18) below, $\mathbf{e}_1$ is the 2×1 basis vector with 1 in the first element, and $\mathbf{U}(x)$ represents the rotation matrix:

$$\mathbf{U}(x) = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \tag{12}$$

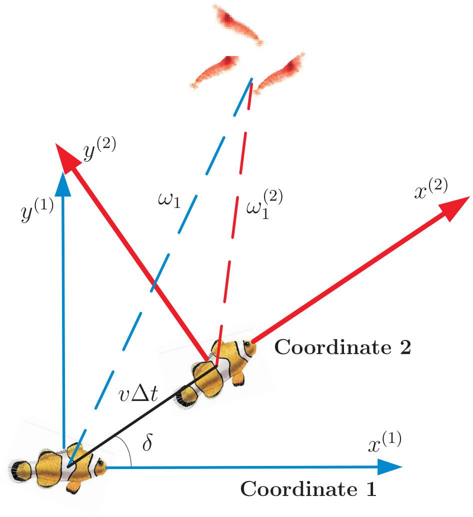

Fig. 2. Coordinate systems in adaptation. Coordinate 1 and coordinate 2 represent the coordinate system in the previous and current steps, respectively.

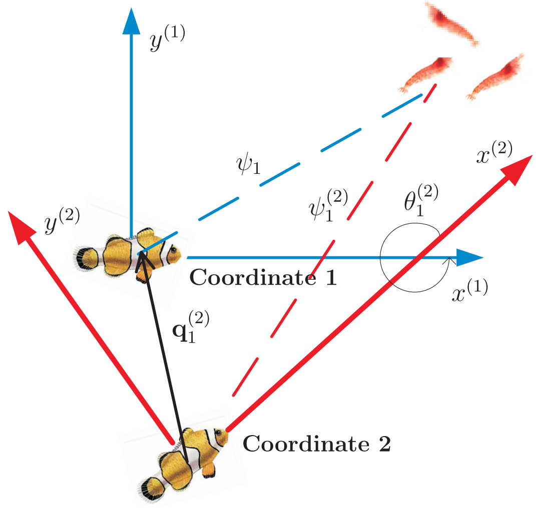

Fig. 3. Coordinate systems in diffusion. Coordinate 1 and coordinate 2 represent the coordinate systems of a neighboring fish and the reference fish, respectively.

In addition, the superscript of $\omega_{k,i-1}^{(2)}$ represents the diffusion result with respect to **Coordinate 2**. Then

$$\begin{aligned}\psi_{k,i} &= \omega_{k,i-1}^{(2)} + \mu\left(p_{k,i} - \omega_{k,i-1}^{(2)}\right) \\ &= (1-\mu)\omega_{k,i-1}^{(2)} + \mu p_{k,i}\end{aligned} \tag{13}$$

where μ is the step size of the adaptation. The second equality shows that the adaptation step is simply the convex combination of the previous diffusion result with the current estimate.

Before discussing the diffusion step, we have to specify the neighbors of a fish. Let r_1 represent the maximum distance within which two fish can be assumed to "communicate" successfully. All fish within a radius r_1 of one fish are candidate neighbors. However, due to limited computational ability and communication overhead, the number of neighbors will be constrained, say to B. There are different criteria for a fish to choose its neighbors, such as, front priority, distance priority [5], etc. In this work, we adopt front priority because it helps the fish to swim in the same direction and to form a school. Specifically, let $q_{l,i}^{(k)}$ be the position of the lth fish at time i with respect to the kth fish. Then, fish k chooses its neighbors from the smallest $|\angle(q_{l,i}^{(k)})|$ under the constraint $\|q_{l,i}^{(k)}\| \le r_1$, where $\angle$ and $\|\cdot\|$ denote the angle and the Euclidean norm of a vector. Note that the range of the operation $\angle$ is $(-\pi, \pi]$. Since the neighboring relationship is asymmetric, in general, this leads to a directional topology. In addition, due to the movement of fish, the topology is highly dynamic.

Now, after the adaptation step (13), each fish broadcasts its estimate $\psi_{k,i}$ and receives information from its neighbors, $\mathcal{N}_{k,i}$. Assume a fish is capable of knowing the coordinate vector, $q_{l,i}^{(k)}$, and the swimming direction, $\theta_{l,i}^{(k)}$, of its neighbors (see Fig. 3). Note that these values are with respect to the kth fish. Fig. 3 shows the coordinate system of the kth fish (**Coordinate 2**) and that of its neighbor (**Coordinate 1**). Therefore, fish k has to perform coordinate transformation on the adaptation information from its neighbors as follows:

$$\psi_{l,i}^{(k)} = \mathbf{U}(-\theta_{l,i}^{(k)})\psi_{l,i} + \mathbf{q}_{l,i}^{(k)} \tag{14}$$

where $\psi_{l,i}^{(k)}$ denotes the adaptation result of the lth fish with respect to the kth fish. Note that $\psi_{k,i}^{(k)} = \psi_{k,i}$ because both $\theta_{k,i}^{(k)}$ and $q_{k,i}^{(k)}$ are equal to zero. Finally, the results are linearly fused together as:

$$\omega_{k,i} = \sum_{l \in \mathcal{N}_{k,i}} a_{l,i}^{(k)} \psi_{l,i}^{(k)} \tag{15}$$

where the $a_{l,i}^{(k)}$'s are nonnegative combination coefficients from the lth fish to the kth fish at time i with $\sum_{l\in\mathcal{N}_{k,i}} a_{l,i}^{(k)} = 1$. Note that if the kth fish or its neighbors have no information about the position of food, the corresponding combination coefficient is set to 0. In addition, $a_{l,i}^{(k)}$ can be set depending on the estimated distance of food. However, we adopt equal weights in the simulations and this results in the desired behavior.

D. Swimming Direction Adjustment

In above discussion, we developed a method to estimate the position of food. Therefore, a fish can easily adjust its swimming direction as follows:

$$\delta_{k,i}^{food} = \angle(\omega_{k,i}) \tag{16}$$

In order to protect itself from predators or to benefit from its peers, a fish tries to swim together with other fish and to form a school even without the existence of food. Therefore, a fish will align its swimming direction with its neighbors.

Alternatively, reference [3] proposed another model to explain the grouping behavior. As we mentioned before, a fish adjusts its swimming direction by averaging the directions of those fish that are within a certain radius. In this paper, we adopt this method but here a fish only takes a fraction of communicable fish into consideration. The direction adjustment due to neighbors then becomes

$$\delta_{k,i}^{nb} = \angle\left(\sum_{l\in\mathcal{N}_{k,i}} b_{l,i}^{(k)} \exp\left(j\theta_{l,i}^{(k)}\right)\right) \tag{17}$$

Like $\{a_{l,i}^{(k)}\}$, the $b_{l,i}^{(k)}$'s are nonnegative combination coefficients with $\sum_{l\in\mathcal{N}_{k,i}} b_{l,i}^{(k)} = 1$.

Now, if there is a fish falsely claiming existence of food and diffusing the information to the other fish, soon, the group will be trapped into an erroneous position. To solve this problem, each fish must have the capability to verify the authenticity of the information from other fish. Thus, note first that if the distance of food is less than a certain value, say r_2, food can be detected with high probability. Therefore, if a fish is informed of the existence of food, it can estimate the position of the food by (15). However, if the distance of food is less than r_2 (i.e., $\|\omega_{k,i}\| < r_2$) and the fish does not detect the existence of food, the fish just discards the false information and resets the estimation of food position.

After checking the truth of the received diffusion information, a fish adjusts its swimming direction by convexly combining the effects of the neighbors and of food, i.e.,

$$\delta_{k,i} = \nu\delta_{k,i}^{nb} + (1-\nu)\delta_{k,i}^{food} + \xi_{k,i} \tag{18}$$

where $\xi_{k,i}$ is used to model uncertain effects, such as temperature and flow, and is modeled as a Gaussian random variable with zero mean and variance ρ^2. The choice of coefficient ν is important. It should be set large enough such that the school will not disperse in foraging. In addition, if a fish has no information about the position of food or treats it as false alarm, it simply follows the swimming direction of its neighbors, i.e., $\nu = 1$ in (18).

E. Summary of Algorithm

We summarize the procedure of the algorithm in this section. Assume there are K fish in a school and we introduce three indicator vectors:

1) $\mathbf{I}_{INFO}$: obtaining information of the food position.
2) $\mathbf{I}_{FOOD}$: detecting the existence of food.
3) $\mathbf{I}_{DIFF}$: detecting diffusion from other neighbors.

Each indicator is a $K \times 1$ vector. The kth element is equal to 1 if the corresponding statement is true for the kth fish; otherwise, it is equal to 0. The detail of the algorithm is summarized as follows:

1) Initialization: $\mathbf{I}_{INFO} = \mathbf{0}$, $\mathbf{I}_{FOOD} = \mathbf{0}$, $\mathbf{I}_{DIFF} = \mathbf{0}$.
2) (Adaptation) At every time step, **for** $k = 1$ to K **do**:
   a) Determine neighbors and their positions and swimming directions (i.e., $\mathcal{N}_{k,i}, q_{l,i}^{(k)}, \theta_{l,i}^{(k)}, l \in \mathcal{N}_{k,i}$).
   b) Compute the direction adjustment due to neighbors, $\delta_{k,i}^{nb}$, by (17).
   c) Detect the existence of food, $\mathbf{I}_{FOOD}(k)$.
   d) **If** $\mathbf{I}_{FOOD}(k) = 1$, **then**
      i) Estimate the position of food by (7)-(10).
      ii) **If** $\mathbf{I}_{INFO}(k) = 1$, adapt $\psi_{k,i}$ by (11) and (13), **else**, determine it by (13) with $\mu = 1$ and set $\mathbf{I}_{INFO}(k) = 1$.
   e) **Else**
      i) **If** $\mathbf{I}_{INFO}(k) = 1$, adapt $\psi_{k,i}$ by (13) with $\mu = 0$.
3) (Diffusion) **For** $k = 1$ to K **do**:
   a) Determine the diffusion from other neighbors, $\mathbf{I}_{DIFF}(k)$.
   b) **If** $\mathbf{I}_{DIFF}(k) = 1$ **then**
      i) **If** $\mathbf{I}_{INFO}(k) = 1$, fuse $\omega_{k,i}$ by (14) and (15), **else**, fuse it by (14) and (15) with $a_{k,i}^{(k)} = 0$ and set $\mathbf{I}_{INFO}(k) = 1$.
   c) **Else**
      i) **If** $\mathbf{I}_{INFO}(k) = 1$, determine $\omega_{k,i}$ by (15) with $a_{k,i}^{(k)} = 1$.
   d) Compute the direction adjustment due to food, $\delta_{k,i}^{food}$, by (16).
   e) **If** $(\|\omega_{k,i}\| < r_2)$ and $(\mathbf{I}_{FOOD}(k) = 0)$, reset $\mathbf{I}_{INFO}(k) = 0$.
   f) **If** $\mathbf{I}_{INFO}(k) = 1$, determine $\delta_{k,i}$ by (18), **else**, determine it by (18) with $\nu = 1$.

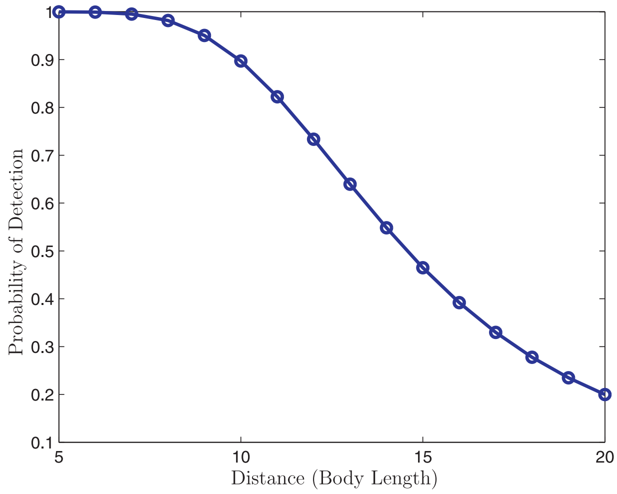

Fig. 4. Probability of detection versus distance between fish and food under the Neyman-Pearson criterion with probability of false alarm $P_{FA} = 0.01$.

III. SIMULATION RESULTS

In this section, we compare two fish schools with and without the diffusion mechanism. The fish without diffusion mechanism simply estimate the position of food individually and the information will not be broadcast. More precisely, the diffusion step of the algorithm just applies (15) with $a_{k,i}^{(k)} = 1$. The parameters are set as follows. The radius of successful communication between two fish is $r_1 = 5$ with the unit equal to the body length of a fish and the maximum number of neighbors $B = 6$. In modeling the food source, let $G = 100$, $\alpha = 2$, and $\sigma^2 = 0.5$. In addition, the number of samples in detection is $N = 10$ and the probability of false alarm is $P_{FA} = 0.01$, which results in an optimal threshold under the Neyman-Pearson criterion of $\tau_{opt} = 9.39$. We set $r_2 = 2r_1 = 10$ in checking the truth of the information from other fish. This results in a probability of detection of about 0.9 at this distance, as shown in Fig. 4. In estimating the direction of food, the number of regions is $Z = 8$. For the diffusion algorithm, the step size is $\mu = 0.7$ and the adaptation results are equally weighted at the diffusion step. In direction adjustment, the combination coefficient is $\nu = 0.6$ and the standard deviation of uncertainty effects is $\rho = \pi/12$. Finally, the speed of fish is $v = 1$, and the time step of the algorithm is $\triangle t = 0.5$ sec. The coefficients, $a_{l,i}^{(k)}$ and $b_{l,i}^{(k)}$, are specified as follows. Let $\mathcal{D}_{k,i}$ be the set of neighbors that diffuse the adaptation result to fish k. Obviously, $\mathcal{D}_{k,i} \subseteq \mathcal{N}_{k,i}$. Then we set

$$
\begin{aligned}
a_{l,i}^{(k)} &= \frac{1}{|\mathcal{D}_{k,i}|} \quad \forall l \in \mathcal{D}_{k,i} \\
b_{l,i}^{(k)} &= \frac{1}{|\mathcal{N}_{k,i}|} \quad \forall l \in \mathcal{N}_{k,i}
\end{aligned}
\tag{19}
$$

where $|\mathcal{N}|$ denotes the number of elements in the set $\mathcal{N}$.

In Figs. 5 and 6, we construct a global coordinate system, whose unit length is body length of a fish. Initially, there are 40 fish, which are uniformly distributed in a square region centered at the origin and with length 10 and their swimming directions are set moving towards the x-axis. The position and the swimming direction of a fish are indicated by "•" and "$-$", respectively. In addition, one food source is located at $(50, 30)$ and is marked "■". Two schools with (Fig. 5) and without (Fig. 6) the diffusion mechanism are compared over time.

For the school with the diffusion mechanism, the whole group searches for food and moves together. We observe that the school always moves within a 15×15 square range. In addition, the arrows in Fig. 5(a) (c) indicate the moving direction of the school. Eventually, the school successfully finds food and moves around the food source. However, for the school without the diffusion mechanism, the group spreads out and even separates into fragments (see Fig. 6(b)). This is because some fish falsely detect the existence of food and then these fish may move toward the wrong position of food. Since the information is not diffused, fish in the front of a false-alarm fish do not notice the deviation of this fish and keep moving forward. However, fish in the rear of the false-alarm fish follow this fish and form a subgroup. This fault suggests that the diffusion mechanism is essential for fish to form a group in foraging.

IV. Conclusion

In this paper, we proposed an algorithm for fish to form a school while foraging for food. The diffusion algorithm is applied to illustrate self-organization in fish schooling. A fish detects the existence of food, estimates the position of food, and broadcasts the information. To release the burden of communication and computation in diffusion, a fish only selects a limited number of candidate fish as its neighbors. Furthermore, due to the constant movement of fish, this results in a distributed algorithm over a dynamic and directional network topology. The simulations indicate that fish foraging without a diffusion mechanism does not result in schooling.

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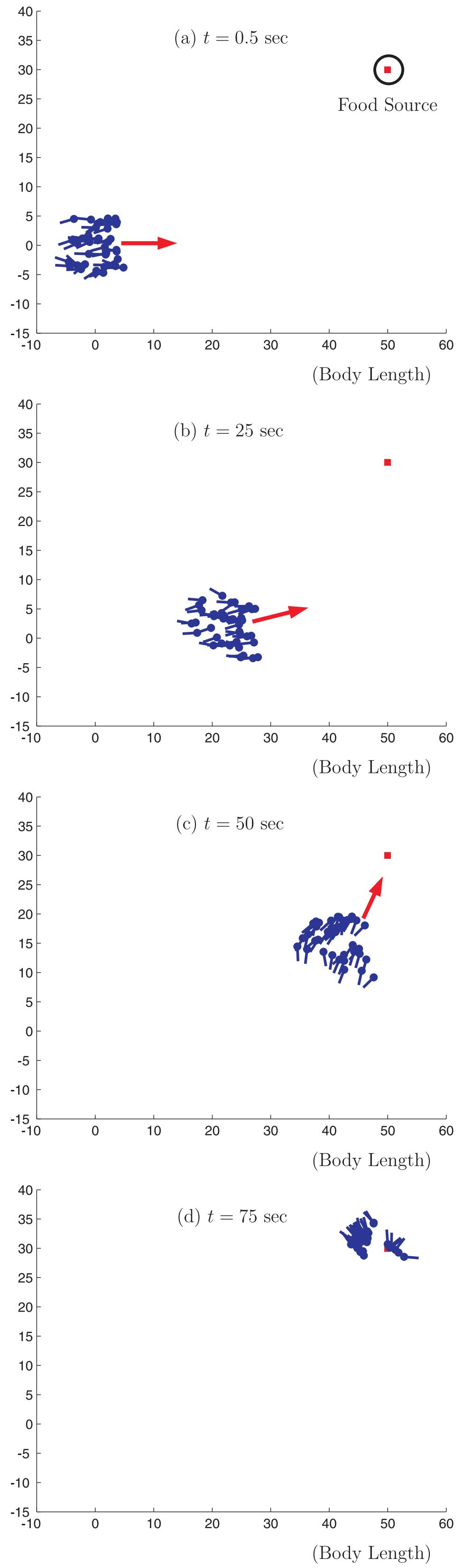

Fig. 5. Maneuvers of fish schools *with* diffusion over time: (a) $t = 0.5$ sec, (b) $t = 25$ sec, (c) $t = 50$ sec, and (d) $t = 75$ sec

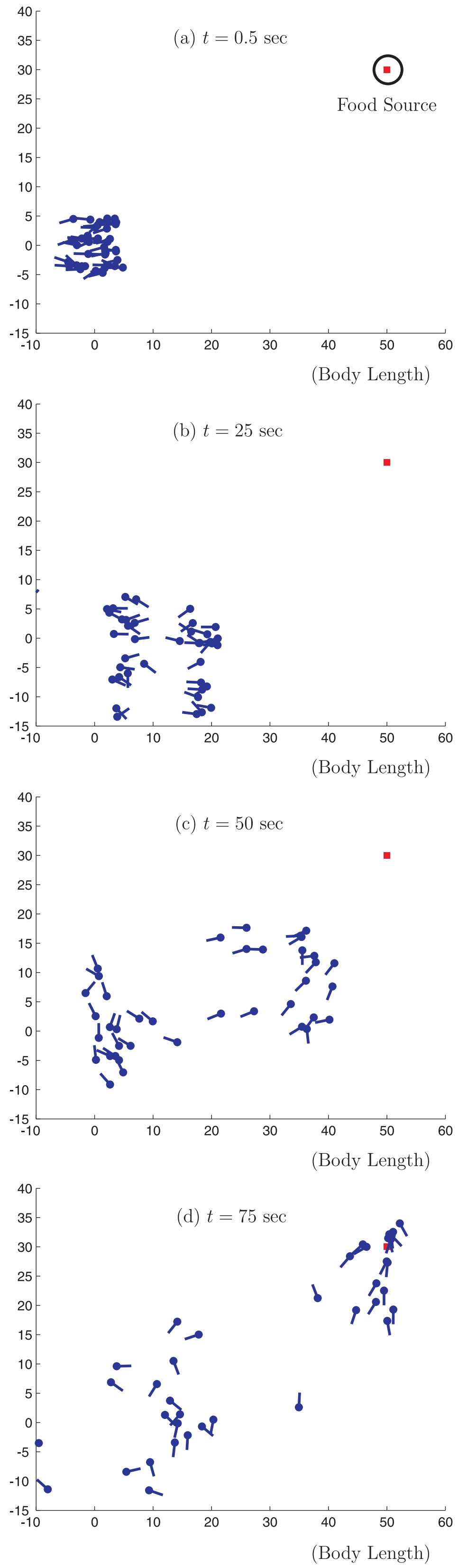

Fig. 6. Maneuvers of fish schools *without* diffusion over time: (a) $t = 0.5$ sec, (b) $t = 25$ sec, (c) $t = 50$ sec, and (d) $t = 75$ sec