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Robust Exponential Filtering for Uncertain Systems with Stochastic and Polytopic Uncertainties

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Abstract—**This paper addresses the problem of robust exponential filtering for discrete uncertain systems with mixed stochastic and deterministic uncertainties, in addition to unmodelled nonlinearities and measurement and process noises with bounded variances.**

keywords: Robust filter, stochastic uncertainties, Lyapunov function, error variance, exponential stability, asymptotic stability.

I. Introduction

State estimation is a fundamental problem in system theory, control and signal processing. As is well-known, the Kalman filter is the optimal linear least-mean-squares estimator for linear Markov models [1]. However, the central premise in Kalman filtering is that the underlying state space model is accurate. When this assumption is violated, the performance of the filter can deteriorate appreciably. This filter sensitivity to modeling errors has led to several works in the literature on the development of robust state-space filters; robust in the sense that they attempt to limit, in certain ways, the effect of model uncertainties on the overall filter performance; see for example [2]-[8].

In this paper, we move beyond earlier robust formulations and design a robust filter for linear systems with *mixed* stochastic and deterministic parametric uncertainties in the state-space model. Robustness is enforced by ensuring exponential stability of the error system in the mean square sense and by simultaneously minimizing an upper bound on the error variance. We pursue this objective by employing the stochastic framework of stability through the use of expectation decreasing martingales. Our formulation also allows for unmodelled nonlinearities and it incorporates process and measurement noises that are assumed to be white and uncorrelated but have unknown bounded variances.

II. Problem formulation

Consider an $n-$dimensional state-space model of the form:

$$x_{k+1} = (A+\Delta A_k)x_k + Bu_k + Df(x_k) \quad (1)$$
$$y_k = Cx_k + v_k, \quad k \geq 0 \quad (2)$$

where $\{u_k, v_k\}$ are uncorrelated white zero-mean random processes with unknown but bounded variances, say

$$Eu_k u_k^* \leq \rho_u I, \qquad Ev_k v_k^* \leq \rho_v I$$

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The initial condition x_0 is also a zero-mean random variable that is uncorrelated with $\{u_k, v_k\}$ for all k.

The state matrix A and the output matrix C are unknown but are assumed to lie inside a convex bounded polyhedral domain $\mathcal{K}$ described by p vertices as follows:

$$\mathcal{K} = \left\{ (A,C) = \sum_{i=1}^{i=p} \alpha_i (A_i, C_i), \;\; \alpha_i \geq 0, \;\; \sum_{i=1}^{i=p} \alpha_i = 1 \right\} \quad (3)$$

The vertices $A_i, i = 1,..,p$, are assumed to be bounded, say $\|A_i\| \leq \beta$. Note that although the A_i are constant, the coefficient matrix in (1) is itself time variant due to the presence of the uncertainties ΔA_k.

The multiplicative uncertainties ΔA_k are stochastic and are modelled as

$$\Delta A_k = E\Delta_k G \quad (4)$$

where E and G are known matrices, while Δ_k is a random matrix whose entries have zero mean and are uncorrelated with each other. The variances of the entries of Δ_k are assumed unknown but bounded by ρ so that

$$E\Delta_k \Delta_k^* \leq \rho_\Delta I$$

The function $f(\cdot)$ in (1) accounts for unmodelled nonlinearities. It is assumed to satisfy

$$\|f(x_k)\| \leq \|Ux_k\| \quad (5)$$

for some matrix U.

We thus see that the only parameters that are assumed known in model (1)–(2) are the matrices $\{B, D, U, E, G\}$. The other parameters

$$\{A, \; C, \; \Delta A_k, \; f, \; Eu_k u_k^*, \; Ev_k v_k^*, \; E\Delta_k \Delta_k^*\}$$

are all unknown but are either subject to randomness or lie in a bounded set.

Our objective is to design a robust linear estimator for the state variable x_k of the form

$$\hat{x}_{k+1} = A_f \hat{x}_k + B_f y_k, \quad k \geq 0 \quad (6)$$

for some matrices A_f and B_f to be determined according to the criteria explained in the sequel.

First, we denote the state estimation error by:

$$\tilde{x}_k = x_k - \hat{x}_k \quad (7)$$

and use (1), (6) and (7) to note that the extended state vector $\eta_k = \text{col}\{x_k, \tilde{x}_k\}$ satisfies the recursion:

$$\boxed{\eta_{k+1} = (\bar{A} + \Delta\bar{A}_k)\eta_k + \bar{B}w_k + \bar{D}f(M\eta_k), \quad k \geq 0} \quad (8)$$

with the following definitions:

$$\begin{aligned}
\bar{A} &= \begin{pmatrix} A & 0 \\ A-A_f-B_fC & A_f \end{pmatrix} \\
\bar{B} &= \begin{pmatrix} B & 0 \\ B & -B_f \end{pmatrix} \\
\bar{D} &= \begin{pmatrix} D \\ D \end{pmatrix} \\
\bar{E} &= \begin{pmatrix} E \\ E \end{pmatrix} \\
\bar{G} &= \begin{pmatrix} G & 0 \end{pmatrix} \\
w_k &= \begin{pmatrix} u_k \\ v_k \end{pmatrix} \\
M &= \begin{pmatrix} I & 0 \end{pmatrix} \\
\Delta\bar{A}_k &= \bar{E}\Delta_k\bar{G}
\end{aligned} \quad (9)$$

We are interested in determining the filter parameters $\{A_f, B_f\}$ such that, for all admissible uncertainties in the model (1)–(2), the augmented system (8) is stable in the sense explained in the sequel. Since (8) includes both deterministic and stochastic quantities, we need to resort to stochastic notions of stability, defined as follows.

Definition 1 The stochastic process η_k of (8) is said to be stable with probability 1 if, and only if, for any $\delta > 0$ and $\varepsilon > 0$, there exists a $\sigma(\delta, \varepsilon) > 0$ such that if $\|\eta_0\| \leq \sigma(\delta, \varepsilon)$, then $P[\sup \|\eta_k\| \geq \varepsilon] \leq \delta$. If $P[\sup \|\eta_k\| \geq \varepsilon] \leq \delta$ holds for all η_0, then we say that the origin is stable at large.

Definition 2 The stochastic process η_k of (8) is said to be *asymptotically* stable at large with probability 1 if, and only if, it is stable at large with probability 1 and $\|\eta(k, \eta_0)\| \to 0$ with probability 1 as $k \to \infty$ for any η_0.

Definition 3 The stochastic process η_k of (8) is said to be exponentially bounded in mean-square, if there are real numbers $\mu, \nu > 0$ and $0 < \varsigma < 1$ such that $E\|\eta_k\|^2 \leq \mu\|\eta_0\|^2\varsigma^k + \nu$ for every $k \geq 0$.

With these definitions in mind, our objective is to determine $\{A_f, B_f\}$ in order to guarantee that for all admissible uncertainties in the model (1)–(2), the augmented system (8) is asymptotically stable with probability 1 in the absence of noises. And, when noises are present, we would also like the state estimation error $\tilde{x}_k$ to be exponentially bounded in mean-square sense. Actually, we shall seek $\{A_f, B_f\}$ in order to minimize a bound on the error variance, $E\|\tilde{x}_k\|^2$.

III. Robust Filter Design

Assume initially that the matrices $\{A, C\}$ are known (i.e., ignore the polytope $\mathcal{K}$ for now). Assume further that the noise component w_k is absent from (8) so that

$$\eta_{k+1} = (\bar{A} + \Delta\bar{A}_k)\eta_k + \bar{D}f(M\eta_k), \quad k \geq 0 \quad (10)$$

In order to determine $\{A_f, B_f\}$ to ensure asymptotic stability of η_k we proceed as follows. We first construct a Lyapunov function $V(\eta_k)$ that satisfies the inequality

$$\boxed{E(V(\eta_{k+1})/\eta_k) - V(\eta_k) \leq -\gamma(\|\eta_k\|^2) < 0} \quad (11)$$

for all η_0 and for all k. Such a $V(.)$ would ensure asymptotic stability in view of the following result.

Lemma 1: If for a stochastic process $V(\eta_k)$, there exists a continuous non-negative function $\gamma(.)$ of real numbers vanishing only at zero and

$$E(V(\eta_{k+1})/\eta_k) - V(\eta_k) \leq -\gamma(\|\eta_k\|^2) < 0 \quad (12)$$

for all η_0, then η_k is asymptotically stable with probability 1.

Proof: See [10].

◇

We shall construct $V(.)$ in the form

$$V(\eta_k) = \eta_k^T P \eta_k \quad (13)$$

for some positive-definite matrix P.

Theorem 1: *For any positive-definite matrix P and positive scalars $\{\epsilon, \xi\}$, and for given scalars $\{\gamma_1 > 1, \gamma_2 > 1\}$, define the matrix*

$$\begin{aligned}
R &= NPN^T - \rho_\Delta \bar{G}^T\bar{E}^T P\bar{E}\bar{G} - \bar{A}^T P\bar{A} \\
&\quad -\lambda_{\max}(\bar{D}^T P\bar{D})\bar{U}^T\bar{U} - \xi\epsilon^{-1}I \\
&\quad -\epsilon\lambda_{\max}(\bar{D}^T P^2\bar{D})\bar{U}^T\bar{U}
\end{aligned} \quad (14)$$

with $\bar{U} = UM$ and $N = \text{diag}\{\gamma_1^{-1/2}, \gamma_2^{-1/2}\}$. If there exist matrices $\{A_f, B_f, P\}$ and positive scalars $\{\epsilon, \xi\}$ such that

$$\begin{pmatrix} \xi I & \bar{A}^T \\ \bar{A} & I \end{pmatrix} > 0 \quad (15)$$

and

$$R > 0 \quad (16)$$

then $V(\eta_k) = \eta_k^T P \eta_k$ satisfies (11) and consequently, the process $\{\eta_k\}$ of (8) will be asymptotically stable in the absence of noise.

Proof: Note that

$$\begin{aligned}
&E(V(\eta_{k+1})/\eta_k) - V(\eta_k) \leq \\
&\eta_k^T\bar{A}^T P\bar{A}\eta_k - \eta_k^T P\eta_k \\
&+ \rho_\Delta\eta_k^T\bar{G}^T\bar{E}^T P\bar{E}\bar{G}\eta_k \\
&+ \eta_k^T\bar{A}^T P\bar{D}f(M\eta_k) + f^T(M\eta_k)\bar{D}^T P\bar{A}\eta_k \\
&+ f^T(M\eta_k)\bar{D}^T P\bar{D}f(M\eta_k)
\end{aligned} \quad (17)$$

Now it is known [9] that for any real matrices $\{X, Y, J\}$ with $J^TJ \leq \mu I$, it holds for any scalar $\varepsilon > 0$ that

$$XJ^TY + Y^TJX^T \leq \varepsilon^{-1}\mu XX^T + \varepsilon Y^TY \quad (18)$$

Using this result along with the condition (15) gives

$$
\begin{aligned}
&\eta_k^T \bar{A}^T P \bar{D} f(M\eta_k) + F^T(M\eta_k)\bar{D}^T P \bar{A}\eta_k \\
&\leq \epsilon^{-1}\xi\eta_k^T\eta_k + \epsilon F^T(M\eta_k)\bar{D}^T P^2 \bar{D} f(M\eta_k) \qquad (19)\\
&\leq \xi\epsilon^{-1}\eta_k^T\eta_k + \epsilon\lambda_{\max}(\bar{D}^T P^2 \bar{D})\eta_k^T \bar{U}^T\bar{U}\eta_k
\end{aligned}
$$

for some $\epsilon > 0$ and

$$
F^T(M\eta_k)\bar{D}^T P \bar{D} f(M\eta_k) \leq \lambda_{\max}(\bar{D}^T P \bar{D})\eta_k^T \bar{U}^T \bar{U}\eta_k \quad (20)
$$

Using (19) and (20) we can write (17) as

$$
\begin{aligned}
&E(V(\eta_{k+1})/\eta_k) - V(\eta_k) \leq \\
&\eta_k^T \bar{A}^T P \bar{A}\eta_k - \eta_k^T P \eta_k \\
&+ \rho_\Delta \eta_k^T \bar{G}^T \bar{E}^T P \bar{E}\bar{G}\eta_k \qquad (21)\\
&+ \xi\epsilon^{-1}\eta_k^T\eta_k + \epsilon\lambda_{\max}(\bar{D}^T P^2 \bar{D})\eta_k^T \bar{U}^T\bar{U}\eta_k. \\
&+ \lambda_{\max}(\bar{D}^T P \bar{D})\eta_k^T \bar{U}^T \bar{U}\eta_k
\end{aligned}
$$

or, more compactly,

$$
E(V(\eta_{k+1})/\eta_k) - V(\eta_k) \leq -\eta_k \bar{R}\eta_k^T \quad (22)
$$

where

$$
\begin{aligned}
\bar{R} = {} & P - \rho_\Delta \bar{G}^T \bar{E}^T P \bar{E}\bar{G} - \bar{A}^T P \bar{A} \\
& - \lambda_{\max}(\bar{D}^T P \bar{D})\bar{U}^T\bar{U} - \xi\epsilon^{-1} I - \epsilon\lambda_{\max}(\bar{D}^T P^2 \bar{D})\bar{U}^T\bar{U}
\end{aligned}
\quad (23)
$$

Now the inequality (16), and the fact that $R < \bar{R}$, imply that $\bar{R} > 0$ and we have

$$
E(V(\eta_{k+1}/\eta_k)) - V(\eta_k) \leq -\alpha(\|\eta_k\|^2) < 0 \quad (24)
$$

as desired.

◇

Now assume that we restrict our search over positive-definite matrices P with block diagonal structure, say

$$
P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} \quad (25)
$$

and let

$$
Q_1 = A_f^T P_2, \qquad Q_2 = B_f^T P_2 \quad (26)
$$

We can then rewrite the requirement (16) as below:

$$
\left(\begin{array}{cc|cc}
Z & 0 & A^T P_1 & \hat{J} \\
0 & \gamma_2^{-1}P_2 - \epsilon^{-1}\xi I & 0 & Q_1 \\ \hline
P_1 A & 0 & P_1 & 0 \\
\hat{J}^T & Q_1^T & 0 & P_2
\end{array}\right) > 0 \quad (27)
$$

where

$$
\begin{aligned}
Z = {} & \gamma_1^{-1}P_1 - \rho_\Delta G^T E^T (P_1 + P_2) E G \\
& -\lambda_{\max}(D^T(P_1+P_2)D)U^T U - \epsilon^{-1}\xi I \\
& -\epsilon\lambda_{\max}(D^T(P_1^2 + P_2^2)D)U^T U
\end{aligned}
$$

and

$$
\hat{J} = -C^T Q_2 - Q_1 + A^T P_2 \quad (28)
$$

It can be seen that condition (27) can be satisfied by seeking matrices $\{P_1, P_2, Q_1, Q_2\}$ and positive scalars $\{\sigma_1, \sigma_2, \epsilon, \xi\}$ in order to satisfy the following inequalities:

$$
\begin{pmatrix}
\sigma_1 I & D^T P_1 & D^T P_2 \\
P_1 D & P_1 & 0 \\
P_2 D & 0 & P_2
\end{pmatrix} > 0 \quad (29)
$$

$$
\begin{pmatrix}
\sigma_2 I & D^T P_1 & D^T P_2 \\
P_1 D & I & 0 \\
P_2 D & 0 & I
\end{pmatrix} > 0 \quad (30)
$$

$$
\left(\begin{array}{cc|cc}
Z' & 0 & A^T P_1 & \hat{J} \\
0 & \gamma_2^{-1}P_2 - \epsilon^{-1}\xi I & 0 & Q_1 \\ \hline
P_1 A & 0 & P_1 & 0 \\
\hat{J}^T & Q_1^T & 0 & P_2
\end{array}\right) > 0 \quad (31)
$$

where

$$
\begin{aligned}
Z' = {} & \gamma_1^{-1}P_1 - \rho_\Delta G^T E^T (P_1 + P_2) E G \\
& -(\sigma_1 + \epsilon\sigma_2)U^T U - \epsilon^{-1}\xi I
\end{aligned}
$$

Once $\{Q_1, Q_2\}$ are determined, the desired filter matrices $\{A_f, B_f\}$ can be found from (26).

IV. Performance Analysis

Let us now examine the performance of any filter $\{A_f, B_f\}$ that results from the above construction in the presence of measurement and process noises. It turns out that the process $\{\eta_k\}$ is not only asymptotically stable, but is also exponentially stable. In order to prove this result, we rely on the following lemma.

Lemma 2: If there exist real numbers $\lambda, \mu, \nu > 0$ and $0 < \alpha \leq 1$ such that

$$
\mu\|\eta_k\|^2 \leq V(\eta_k) \leq \nu\|\eta_k\|^2 \quad (32)
$$

and

$$
EV(\eta_{k+1}|\eta_k) - V(\eta_k) \leq \lambda - \alpha V(\eta_k) \quad (33)
$$

then the process $V(\eta_k)$ is exponentially bounded and, moreover,

$$
E\|\eta_k\|^2 \leq \frac{\nu}{\mu}E\|\eta_0\|^2(1-\alpha)^k + \frac{\lambda}{\mu\alpha} \quad (34)
$$

Proof: This result is a combination of Thm. 1 from [11] and Thm. 2 from [12].

◇

We now have the following result.

Theorem 2: *For any given $\{\gamma_1 > 1, \gamma_2 > 1\}$, let $\{A_f, B_f\}$ be a solution to (15) and (29)–(31). Then the resulting process $\{\eta_k\}$ is exponentially stable in the presence of measurement and process noises. Moreover, its variance is bounded as follows:*

$$
\boxed{E\|\eta_k\|^2 \leq \frac{1}{\lambda_{\min}(P)}\sup\left\{L\frac{\gamma}{\gamma-1}, \eta_0^T P \eta_0\right\}} \quad (35)
$$

where

$$L = \rho_u \mathrm{Tr}(B^T(P_1+P_2)B) + \rho_v \mathrm{Tr}(B_f^T P_2 B_f) \quad (36)$$

and $\gamma = \min(\gamma_1, \gamma_2)$.

Proof: We first show the exponential stability of (8). In the presence of measurement and process noises, with $\{A_f, B_f\}$ a feasible solution of (15) and (29)–(31), we have the following for $\gamma > 1$:

$$\begin{aligned} &E(V(\eta_{k+1})/\eta_k) - \frac{V(\eta_k)}{\gamma} \le \eta_k^T \bar{A}^T P \bar{A} \eta_k \\ &- \frac{\eta_k^T P \eta_k}{\gamma} + \rho_\Delta \eta_k^T \bar{G}^T \bar{E}^T P \bar{E} \bar{G} \eta_k + \eta_k^T \bar{A}^T P \bar{D} f(M\eta_k) \\ &+ E(w_k^T \bar{B}^T P \bar{B} w_k) + F^T(M\eta_k) \bar{D}^T P \bar{A} \eta_k \\ &+ F^T(M\eta_k) \bar{D}^T P \bar{D} f(M\eta_k) \end{aligned} \quad (37)$$

Using (15) and (29)–(31) , equation (37) becomes

$$\begin{aligned} &EV(\eta_{k+1}|\eta_k) - \frac{V(\eta_k)}{\gamma} \\ &< E(w_k^T \bar{B}^T P \bar{B} w_k) \\ &< \rho_u \mathrm{Tr}(B^T(P_1+P_2)B) + \rho_v \mathrm{Tr}(B_f^T P_2 B_f) \end{aligned} \quad (38)$$

Now from this inequality, we see that there exists an $0 < \alpha < 1$ such that

$$\begin{aligned} &EV(\eta_{k+1}|\eta_k) - V(\eta_k) \\ &\le \rho_u \mathrm{Tr}(B^T(P_1+P_2)B) + \rho_v \mathrm{Tr}(B_f^T P_2 B_f) \\ &- \alpha V(\eta_k) \end{aligned} \quad (39)$$

The desired result now follows from Lemma 2. The bound on the state error variance can be shown using (38) and the following observation [11]. If $V(\eta_k)$ satisfies

$$E(V(\eta_{k+1}|\eta_k)) - \frac{V(\eta_k)}{\gamma} - L < 0 \quad \text{a.s.} \quad (40)$$

for some $\gamma > 1$ and $L > 0$, then $V(\eta_k)$ is bounded with probability 1, and $EV(\eta_k)$ remains bounded for all k with

$$EV(\eta_k) \; < \; \frac{V(\eta_0)}{\gamma^k} + L\frac{\gamma}{\gamma - 1}\left(1 - \frac{1}{\gamma^{k+1}}\right) \quad (41)$$

$$EV(\eta_k) \; \le \; \sup\left\{L\frac{\gamma}{\gamma-1}, V(\eta_0)\right\} \quad (42)$$

Applying this result to (38) we obtain the bound on the state error variance.

◇

V. Polytopic Uncertainties

Let us now incorporate the fact that the matrices $\{A, C\}$ are not known but lie within the polytopic set $\mathcal{K}$.

Theorem 3: *Any filter defined by the matrices*

$$A_f \; = \; (Q_1 P_2^{-1})^T, \qquad B_f = (Q_2 P_2^{-1})^T \quad (43)$$

where $\{Q_1, Q_2, P_2\}$ *are a feasible solution of the inequalities (15) and (29)-(31) for all* $\{A, C\}$ *taking values in* $[A_1, \ldots, A_p]$ *and* $[C_1, \ldots, C_p]$ *ensures the following:*

(i) *Asymptotic stability of (8) with probability 1 for all admissible parameters* $\{A, C\}$.

(ii) $E\|\eta_k\|^2$ *is bounded as given in Thm. 2.*

Proof: From the definition of Q_1 and Q_2 and from the fact that the inequalities (15) and (29)-(31) are linear in A and C, the result easily follows.

◇

The result (35) in Thm. 2 suggests that we can attempt to minimize the upper bound on the error variance by seeking filter coefficients $\{A_f, B_f\}$ that solve

$$\min_{\substack{\text{subject to conditions} \\ (15),(29)-(31) \\ \text{and } P > I}} \left(\frac{L\gamma}{\gamma - 1}\right)$$

One way to solve this optimization problem is as follows. Start with a value $\gamma > 1$ that is close to 1.

1. **Step 1**. Solve the following convex optimization problem over the variables $\{P_1, P_2, Q_1, Q_2, W, \sigma_1, \sigma_2, \epsilon, \xi\}$:

$$\min \; (\mathrm{Tr}(B^T(P_1+P_2)B + W)\frac{\gamma}{\gamma - 1} \quad (44)$$

subject to conditions (15) and (29)–(31) and

$$\begin{pmatrix} W & Q_2 \\ Q_2^T & P_2 \end{pmatrix} > 0 \quad (45)$$

with $P > I$

2. **Step 2**. Compute the resulting cost of (44)and compare it with the previous cost.
3. **Step 3**. If the new cost is less than the previous cost increment γ by $\delta\gamma$ (say .01) and go to step 1, otherwise stop.

VI. Simulation

To illustrate the developed filter, we choose a state-space model of order 2 with parameters :

$$A_1 = \begin{pmatrix} .62 & 0 \\ 0 & .61 \end{pmatrix}, A_2 = \begin{pmatrix} .5 & -1 \\ .2 & .5 \end{pmatrix}, A_3 = \begin{pmatrix} .54 & 1 \\ 0 & .56 \end{pmatrix}$$

$$C_1 = \begin{pmatrix} 100 & 0 \\ 50 & 10 \end{pmatrix}, C_2 = \begin{pmatrix} 90 & 0 \\ 50 & 10 \end{pmatrix}$$

$$D = \begin{pmatrix} .1 & 0 \\ 0 & .1 \end{pmatrix}$$

$$E = G = \begin{pmatrix} .1 & .1 \\ .1 & .1 \end{pmatrix}$$

$$\begin{pmatrix} x_{1,k+1} \\ x_{2,k+1} \end{pmatrix} = (A + \Delta A_k) \begin{pmatrix} x_{1,k} \\ x_{2,k} \end{pmatrix} + w_k + D \begin{pmatrix} .1\sin(x_{1,k}) \\ .1\sin(x_{2,k}) \end{pmatrix}$$

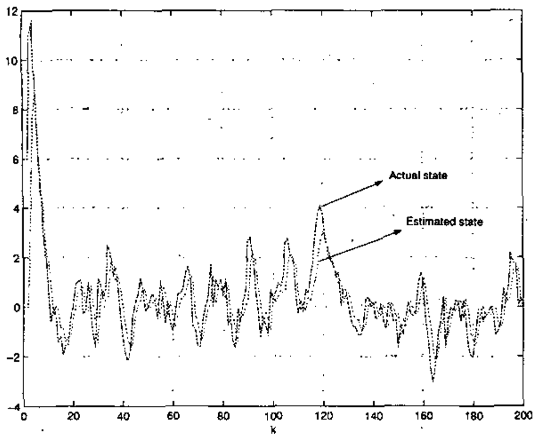

Fig. 1. *State tracking by the robust filter.*

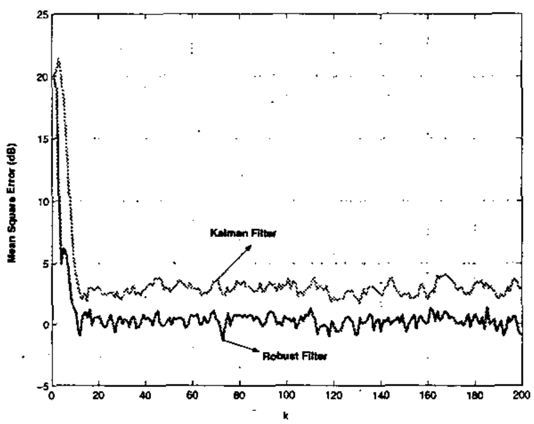

Fig. 2. *Mean square error behavior of Kalman filter and the robust filter*

$$y_k = C\begin{pmatrix} x_{1,k} \\ x_{2,k} \end{pmatrix} + v_k$$

The bound β on the norm of the state matrices A_i in the example is 1 and ξ is also 1. The performance of the filter is illustrated in the figures. Figure 1 shows a plot of the first element of the actual state vector and that of the estimated state vector. Figure 2 compares the mean square error in dB when the actual state matrix is A_3, for both the Kalman filter operating at the centroid of the polytopic region and the robust filter. In both Figures 1 and 2 the noise variances are 1.

VII. Concluding Remarks

In this paper we developed a procedure for designing a robust state estimator for uncertain discrete-time systems with mixed deterministic and stochastic uncertainties. The procedure guarantees almost-sure bounded error variance and exponential stability of the state error vector.

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