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Design of Robust Filters with Improved Robustness Margins via Parameter Scaling

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Abstract

The paper describes a procedure for improving the robustness margins of robust filters via parameter scaling. The scaling parameter is chosen as the square-root factor of the inverse of a positive-definite solution to certain matrix inequalities. This choice is motivated by the desire to generate an estimator dynamics with a stable closed-loop matrix whose maximum singular value is bounded by unity; a step that enhances the robustness of the filters.

Keywords - Estimation, parametric uncertainty, $\mathcal{H}_\infty$-filter, guaranteed-cost design, steady-state filter, scaling, regularization.

1   Introduction

This paper focuses on three classes of robust filtering algorithms and describes a procedure for improving their robustness via parameter scaling, along the lines employed in [1, 2] for the design of stabilizing robust controllers.

The first class of robust filters we consider is based on the $\mathcal{H}_\infty$ criterion. In this framework, the designer constructs filters that bound the 2-induced norm of the operator mapping the disturbances to the estimation error (see, e.g., [3, 4, 5]). The second class of robust filters we consider is based on the guaranteed-cost criterion. In this approach, the designer constructs filters that guarantee that the steady-state variance of the state estimation error is upper bounded by a certain constant value for all admissible uncertainties in the model (see, e.g., [6, 7]). Both classes of filters involve certain parameters that need to be adjusted and that define

the robustness levels of the filters, e.g., the parameter γ in $\mathcal{H}_\infty$ filtering and the parameter ϵ in guaranteed-cost designs (see also [8]). For $\mathcal{H}_\infty$ filters it is necessary to decrease the value of γ for increased robustness, while for guaranteed-cost filters it is necessary to increase the value of ϵ for increased robustness. However, there are limits on how far these parameters can be adjusted without violating certain existence conditions that are associated with such filters.

The third class of robust filters we study is the one developed in [8]; it is based on minimizing the worst-case residual energies at each iteration subject to bounds on data uncertainties. The filters of [8] differ from $\mathcal{H}_\infty$ and guaranteed-cost filters in that they perform data regularization as opposed to data de-regularization. In this way, they are particularly suitable for on-line operations since they do not require continuous testing of existence conditions. Simulations suggest that this class of filters tends to lead to closed-loop estimators with larger robustness margins.

The contribution of this work is to show how the robustness of $\mathcal{H}_\infty$ filters, guaranteed-cost filters, and the filters of [8] could be further improved via scaling. The scaling parameter is chosen as the square-root factor of the inverse of a positive-definite solution to certain matrix inequalities. This choice is motivated by the desire to generate an estimator dynamics with a stable closed-loop matrix whose maximum singular value is bounded by unity. Accordingly, the design procedure attempts to decrease the value of the structured singular value of the closed-loop dynamics in the presence of model perturbations; a step that enhances the robustness of the resulting filters. We explain the procedure by examining first the $\mathcal{H}_\infty$ filtering problem and later study the other classes of filters.

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2 $\mathcal{H}_\infty$ Filtering

Consider a state-space model of the form

$$
\begin{aligned}
x_{i+1} &= Fx_i + Gu_i, \quad x_0, \\
y_i &= Hx_i + v_i, \quad i \geq 0,
\end{aligned} \tag{1}
$$

where $F \in C^{n \times n}$, $G \in C^{n \times m}$, and $H \in C^{p \times n}$ are known nominal matrices; x_0 (initial condition), u_i (process noise), and v_i (measurement noise) are unknown quantities; and y_i is the measured output. Let

$$
s_i = Lx_i, \tag{2}
$$

where $L \in C^{q \times n}$ is given, denote a desired linear combination of the state vector that we wish to estimate using the observations $\{y_j, 0 \leq j \leq i-1\}$. We denote the estimate by

$$
\hat{s}_i = \mathcal{F}(y_0, y_1, \ldots, y_{i-1})
$$

for some operator $\mathcal{F}$, and introduce the corresponding prediction error

$$
e_{p,i} = \hat{s}_i - Lx_i. \tag{3}
$$

The suboptimal $\mathcal{H}_\infty$ problem is defined as follows. Given a scalar $\gamma > 0$ and $Q > 0$, one seeks an estimation strategy $\mathcal{F}$ that satisfies the robustness bound

$$
\sup_{\left(\begin{array}{c} x_0 \\ u_j \\ v_j \end{array}\right)} \frac{\sum_{i=0}^{N} |\hat{s}_i - Lx_i|^2}{x_0^T \Pi_0^{-1} x_0 + \sum_{i=0}^{N-1} u_i^T Q^{-1} u_i + \sum_{i=0}^{N-1} v_i^T v_i} \leq \gamma^2
$$

for all $0 \leq i \leq N$. It is well-known that an estimator that satisfies the above requirement is given by (see, e.g., [3, 5]):

$$
\begin{aligned}
\hat{x}_{i+1} &= F\hat{x}_i + F\check{P}_i H^T (I + H\check{P}_i H^T)^{-1} [y_i - H\hat{x}_i] \\
\check{P}_i^{-1} &= P_i^{-1} - \gamma^{-2} L^T L \\
P_{i+1} &= FP_i F^T + GQG^T - K_i R_{e,i}^{-1} K_i^T \\
R_{e,i} &= \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix} + \begin{bmatrix} H \\ L \end{bmatrix} P_i \begin{bmatrix} H^T & L^T \end{bmatrix} \\
K_i &= FP_i \begin{bmatrix} H^T & L^T \end{bmatrix} \\
\hat{s}_i &= L\hat{x}_i.
\end{aligned} \tag{4}
$$

This filter guarantees the robustness bound for all $0 \leq i \leq N$ if, and only if, the following conditions are satisfied

$$
P_i^{-1} - \gamma^{-2} L^T L > 0 \quad \text{for} \quad 0 \leq i \leq N. \tag{5}
$$

It is also known that large values for γ may be necessary to satisfy (5). However, larger values of γ correspond to decreased robustness. To further improve the robustness of the filter, we apply parameter scaling as explained below.

Let P denote the steady-state stabilizing solution of the Riccati recursion (4), when it exists. We then rewrite the steady-state estimator from (4) in the equivalent form

$$
\hat{x}_{i+1} = F_p \hat{x}_i + K_p y_i \tag{6}
$$

where

$$
\begin{aligned}
F_p &= F \left[I - \check{P} H^T (I + H\check{P}H^T)^{-1} H \right] \\
K_p &= F\check{P}H^T (I + H\check{P}H^T)^{-1}.
\end{aligned}
$$

Here $F_p \in C^{n \times n}$ is a stable matrix and $K_p \in C^{n \times p}$. The matrix F_p determines the dynamics of the state estimator. It is also the same matrix that determines the dynamics of the state estimation error, $\tilde{x}_i = x_i - \hat{x}_i$, since we also have

$$
\tilde{x}_{i+1} = F_p \tilde{x}_i + Gu_i - K_p v_i. \tag{7}
$$

Now note that in the $\mathcal{H}_\infty$ formulation described above, the robustness of the filter is attained by treating the noise processes $\{u_i, v_i\}$ as unknown disturbances. This formulation requires exact knowledge of the nominal model parameters $\{F, G, H\}$. We could envision, however, modeling errors in these parameters. The immediate consequence of such errors would be the fact that the actual measurements that are available for filtering are not the $\{y_i\}$, which are the outputs of (1), but some other values $\{z_i\}$ that are the outputs of a perturbed model with unknown perturbations to $\{F, G, H\}$. We can then seek to further improve the robustness of the filter by attempting to account for this additional source of uncertainty.

Thus assume that the measurements that are available for filtering are not the actual $\{y_i\}$ that arise from (1), but rather a perturbed version, say $\{z_i\}$, that is assumed to be related to $\{y_i\}$ via (see Fig. 1)

$$
z_i = y_i + \Delta_o \hat{y}_i, \quad \hat{y}_i = H\hat{x}_i, \tag{8}
$$

for some $p \times p$ uncertainty Δ_o. Now the actual filtering operation should be described not by (6) but by
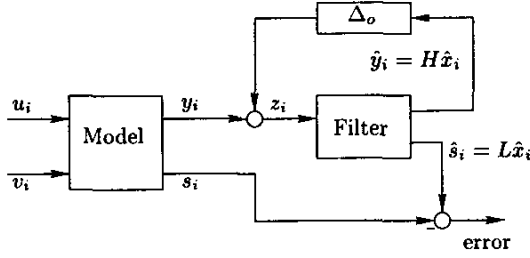
$$
\hat{x}_{i+1} = F_p \hat{x}_i + K_p z_i \tag{9}
$$

singular value (MSV) of the corresponding P. It is straighforward to verify that by decreasing the MSV of P, the gain K_p in (10) decreases and the influence of the perturbation on the filter can be reduced. Also, it is easy to verify that DPD^T is a solution of the Riccati equation of the $\mathcal{H}_\infty$ filter that is based on model (11). We will thus require two conditions for increasing the robustness of the $\mathcal{H}_\infty$ filter (i.e., for selecting D):

$$\bar{\sigma}(DPD^T) < \bar{\sigma}(P) \qquad (13)$$

and

$$\bar{\sigma}(DF_pD^{-1}) < 1 \qquad (14)$$

where $\bar{\sigma}(\cdot)$ denotes the largest singular value of its argument. The second condition is motivated by the desire to enhance the robustness and stability of the closed-loop system in the presence of uncertainties. The following statement shows how the choice of D relates to determining the solution Y of certain linear matrix inequalities.

Theorem 1. *Let $P > 0$ and F_p be given. A nonsingular scaling D satisfying (13) and (14) exists if, and only if, a matrix Y exists such that*

$$Y = Y^T > 0$$
$$\begin{bmatrix} Y & YF_p^T \\ F_pY & Y \end{bmatrix} > 0 \qquad (15)$$
$$\begin{bmatrix} aY & P \\ P & Y \end{bmatrix} > 0$$

where $a = [\bar{\sigma}(P)]^2$. Furthermore, if Y is a solution to (15), D is obtained from $Y^{-1} = D^T D$. That is, D^T is a square-root factor of Y^{-1}.

Proof: By definition, D must be such that

$$\bar{\sigma}(DF_PD^{-1}) < 1 \iff$$
$$D^{-T}F_P^T\left(D^TD\right)F_PD^{-1} < I \iff$$
$$D^TD - F_P^TD^TDF_P > 0 \iff$$
$$(D^TD)^{-1}(D^TD - F_P^TD^TDF_P)(D^TD)^{-1} > 0 \iff$$
$$Y - YF_P^TY^{-1}F_PY > 0 \iff$$
$$\begin{bmatrix} Y & Y^TF_p^T \\ F_pY & Y \end{bmatrix} > 0.$$

Figure 1: *Block diagram describing robust stability analysis on filtering problems*

with z_i replacing y_i. The perturbation model (8) assumes that the discrepancies between y_i and z_i can be modeled via output feedback.

¿From (8) and (9) we arrive at the following description for the dynamics of the $\mathcal{H}_\infty$ estimator:

$$\hat{x}_{i+1} = [F_p + K_p\Delta_oH]\hat{x}_i + K_py_i. \qquad (10)$$

This relation shows that the presence of Δ_o affects the dynamics of $\hat{x}_i$ and can deteriorate the robustness of the filter. Now define a transformed state vector

$$\xi_i \triangleq Dx_i,$$

for some $n \times n$ nonsingular matrix D that we shall explain how to determine further ahead. It follows from (1) that y_i also satisfies the model

$$\begin{aligned}\xi_{i+1} &= DFD^{-1}\xi_i + DGu_i, \quad \xi_0, \\ y_i &= HD^{-1}\xi_i + v_i, \quad i \geq 0, \qquad (11)\end{aligned}$$

with the system matrices $\{F, G, H\}$ replaced by

$$\{DFD^{-1}, DG, HD^{-1}\}. \qquad (12)$$

We can thus proceed to design an $\mathcal{H}_\infty$ filter, using equations (4), with the modified parameters (12) in order to estimate

$$s_i = Lx_i = LD^{-1}\xi_i.$$

That is, the L in (2) should be replaced by LD^{-1} in this new design as well. Our choice of D will be motivated by the desire to increase the robustness of the resulting filter by decreasing the maximum

Furthermore, from (13),

$$
\begin{aligned}
\bar{\sigma}(DPD^T) &< \bar{\sigma}(P) \iff \\
(DPD^T)(DPD^T) &< aI \iff \\
aI - DPD^T DPD^T &> 0 \iff \\
D^{-1}(aI - DPD^T DPD^T)D^{-T} &> 0 \iff \\
a(D^T D)^{-1} - PD^T DP &> 0 \iff \\
aY - PY^{-1}P &> 0 \iff \\
\begin{bmatrix} aY & P \\ P & Y \end{bmatrix} &> 0.
\end{aligned}
$$

◇

Remark. We may add that additional structural constraints on D and the set of uncertainties could be added, such as requiring D to have a block diagonal structure.

◇

The result of the above theorem suggests the following procedure.

Algorithm 1. *The D-scaling procedure for $\mathcal{H}_\infty$ filtering involves the following steps:*

1. *Given the system (1), define a minimal γ such that the $\mathcal{H}_\infty$-suboptimal filter (4) has a steady-state solution with stable F_p.*

2. *If $\bar{\sigma}(F_p) < 1$, stop. If not, go to the next step:*

3. *If there exists a solution Y to the inequalities (15), compute D for the filter (4) applied to the modified parameters*
   $$\{DFD^{-1}, DG, HD^{-1}, LD^{-1}, D\Pi_0 D^T\}.$$

◇

We may remark that while the procedure defined by Algorithm 1 does not change the matrix Q, the value of γ can be decreased if we replace Q by DQD^T.

3 Guaranteed-Cost Filter

We apply a similar procedure to guaranteed-cost filtering and, in particular, to the guaranteed-cost filter of [6][p.44]. Thus consider a state-space model with parametric uncertainties of the form:

$$
\begin{aligned}
x_{i+1} &= (F + \delta F)x_i + u_i \\
y &= Hx_i + v_i \\
\delta F &= MSE_f
\end{aligned}
\tag{16}
$$

where M and E_f are known matrices and S is an arbitrary contraction, $\|S\| \le 1$. The GC filter is described by the following equations (where $\{Q, R\}$ are given positive-definite matrices):

$$
\begin{aligned}
\hat{x}_{i+1} &= F[I + \epsilon(P^{-1} + H_i^T R^{-1} H - \epsilon E_f^T E_f)^{-1} E_f^T E_f]\hat{x}_i + \\
&\quad + F(P^{-1} + H^T R^{-1} H - \epsilon E_f^T E_f)^{-1} H^T R^{-1}(y_i - H\hat{x}_i)
\end{aligned}
\tag{17}
$$

where P is taken as the positive-definite stabilizing solution of the Riccati equation

$$
\begin{aligned}
P &= F(P^{-1} + H^T R^{-1} H - \epsilon E_f^T E_f)^{-1} F^T + \\
&\quad Q + \epsilon^{-1} M M^T.
\end{aligned}
\tag{18}
$$

The value of ϵ is picked from within an open interval $(0, \epsilon^o)$, where $\epsilon^o > 0$ is chosen such that the following additional Riccati equation,

$$
\bar{P} = F(\bar{P}^{-1} - \epsilon^o E_f^T E_f)^{-1} F^T + Q + \epsilon^{-o} M M^T \tag{19}
$$

has a positive-definite stabilizing solution satisfying

$$
\bar{P}^{-1} - \epsilon^o E_f^T E_f > 0. \tag{20}
$$

This condition guarantees $P^{-1} - \epsilon E_f^T E_f > 0$ since it can be shown that $P \le \bar{P}$.

The closed-loop matrix that determines the dynamics of the GC estimator is now given by

$$
F_p =
$$

$$
F[I + (P^{-1} + H^T R^{-1} H - \epsilon E_f^T E_f)^{-1}(\epsilon E_f^T E_f - H^T R^{-1} H)]
$$

with

$$
K_p = F(P^{-1} + H^T R^{-1} H - \epsilon E_f^T E_f)^{-1} H^T R^{-1}.
\tag{21}
$$

The D-scaling procedure that we employ is one that replaces the parameters $\{F, G, H, M, E_f, Q\}$ by

$$
\{DFD^{-1}, DG, HD^{-1}, DM, E_f D^{-1}, DQD^T\}
\tag{22}
$$

so that the corresponding state-space model becomes

$$
\begin{aligned}
\xi_{i+1} &= (DFD^{-1} + \delta F_D)\xi_i + DGu_i \\
y &= H_i D^{-1}\xi_i + v_i \\
\delta F_D &= DMSE_f D^{-1}
\end{aligned}
\tag{23}
$$

with $\xi_i = Dx_i$. Following the same arguments that we used in the $\mathcal{H}_\infty$ case we end up with the following algorithm:

Algorithm 2. *The D-scaling procedure for guaranteed-cost filtering involves the following steps:*

1. *Given the model (16), choose ϵ such that the guaranteed-cost filter (17) has a solution satisfying the requirements on (18), (19), and (20).*

2. *If $\bar{\sigma}(F_p) < 1$, stop. If not, go to the next step.*

3. *If there exists a solution Y to (15), with the F_P from (21), compute D and search for a new ϵ using the modified parameters*

$$\{DFD^{-1}, DG, HD^{-1}, DM, E_f D^{-1}, DQD^T\}.$$

◇

4 Robust Filter from [8]

We now apply a similar procedure to the bounded data uncertainty filter of [8]. As mentioned in the introduction, this filter is based on minimizing the worst-case residual energy at each iteration subject to bounds on data uncertainties. The filter differs from $\mathcal{H}_\infty$ and guaranteed-cost filters in that it performs data regularization and does not require existence conditions.

We again consider the uncertain state-space model (16), where only δF (and hence S) is allowed to change with time. The filter is described by the following equations [8]. A correction parameter $\hat{\lambda}_i$ is first chosen as

$$\hat{\lambda} = (1 + \alpha) \left\| M^T H^T R^{-1} H M \right\|$$

for some scalar parameter $\alpha > 0$ that is chosen freely by the designer. Here, the notation $\|\cdot\|$ denotes the maximum singular value of its matrix argument (or the Euclidean norm of a vector argument). Then the steady-state form of the filter is given by (again, $\{Q, R\}$ are given positive-definite matrices):

$$\hat{x}_{i+1} = \hat{F}[I - PH^T R_e^{-1} H]\hat{x}_i + \hat{F}PH^T R_e^{-1} H y_i$$
$$(24)$$

where

$$R_e = \hat{R} + HPH^T$$
$$\hat{R}^{-1} = R^{-1} - \hat{\lambda}^{-1} H M M^T H^T$$

and P is the stabilizing solution of the Riccati equation

$$P = FPF^T - \bar{K}\bar{R}_e^{-1}\bar{K}^T + GQG^T$$
$$\bar{K} = FP\bar{H}^T$$
$$\bar{R}_e = I + \bar{H}P\bar{H}^T$$
$$\bar{H}^T = \begin{bmatrix} H^T \hat{R}^{T/2} & \sqrt{\hat{\lambda}}E_f^T \end{bmatrix}.$$
$$(25)$$

Moreover,

$$\hat{F} = F[I - \hat{\lambda}(I + P\bar{H}^T\bar{H})^{-1}PE_f^T E_f]. \quad (26)$$

The closed-loop matrix that determines the dynamics of the filter is given by

$$F_p = \hat{F}[I - PH^T R_e^{-1} H] \quad \text{and}$$
$$K_p = \hat{F}PH^T R_e^{-1} H. \quad (27)$$

The D-scaling procedure that we employ to improve the robustness margin of the filter replaces the parameters $\{F, G, H, M, E_f\}$ by

$$\{DFD^{-1}, DG, HD^{-1}, DM, E_f D^{-1}\} \quad (28)$$

so that the corresponding (equivalent) state-space model becomes (23). Following the same arguments that we used in the $\mathcal{H}_\infty$ and GC cases we end up with the following construction.

Algorithm 3. *The D-scaling procedure for the filter of [8] involves the following steps:*

1. *Given the model (16), choose $\alpha > 0$ and computer the corresponding correction parameter $\hat{\lambda}_i$ through (24).*

2. *If $\bar{\sigma}(F_p) < 1$, stop. If not, go to the next step.*

3. *If there exists a solution Y to (15), with the F_p from (27), compute D and design a new filter with the modified parameters*

$$\{DFD^{-1}, DG, HD^{-1}, DM, E_f D^{-1}\}.$$

◇

5 A Numerical Example

Consider the 2-dimensional time-invariant model from [6]:

$$F = \begin{bmatrix} 0.9802 & 0.0196 \\ 0 & 0.9802 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$H = \begin{bmatrix} 1 & -1 \end{bmatrix},$$

$$Q = \begin{bmatrix} 1.9608 & 0.0195 \\ 0.0195 & 1.9605 \end{bmatrix}, \quad \Pi_0 = I, \ \hat{x}_0 = 0.$$

Applying the $\mathcal{H}_\infty$ solution (4), the minimal γ that satisfies the existence condition (5) for large N (e.g., 1000) is around $\gamma = 54$. With this value, we find $\bar{\sigma}(F_p) = 1.3318$, which is larger than unity. Solving the inequalitlies of Thm. 1 we redesign the $\mathcal{H}_\infty$ filter using the modified parameters given by Alg. 1 and we obtain $\bar{\sigma}(F_p) = 0.9811$. We used the same matrix Q and the smallest γ in both cases is the same, $\gamma = 54$.

Applying the guaranteed-cost filter with $R = 1$, $\epsilon^o = 0.8 \times 10^{-5}$, $\epsilon = 0.7 \times 10^{-5}$,

$$M = \begin{bmatrix} 0.0198 \\ 0 \end{bmatrix}, \quad E_f = \begin{bmatrix} 0 & 5 \end{bmatrix}$$

and using (17)–(20) we get $\bar{\sigma}(F_p) = 1.3951$. Solving the inequalities of Alg. 2, we redesign the GC filter using the parameters (22), and with the new solution for the Riccati equation we obtain $\bar{\sigma}(F_p) = 0.9954$.

Applying the robust Filter from [8], choosing $\alpha = 0.5$ and using (24) and (25), we get $\hat{\lambda} = 5.88 \times 10^{-4}$ and $\bar{\sigma}(F_p) = 0.8873$. We see that, in this case, the condition $\bar{\sigma}(F_p) < 1$ is attained prior to the application of the D-scaling procedure.

6 Concluding Remarks

In this work we described procedures for improving the robustness margins of robust filters via parameter scaling. The procedure was applied to three robust filters, $\mathcal{H}_\infty$ filter, guaranteed-cost filter and the filter of [8].

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