Proceedings of the 38th
Conference on Decision & Control
Phoenix, Arizona USA • December 1999

A Feedback Analysis of the Tracking Performance of Blind Adaptive Equalization Algorithms[1]

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Abstract

This paper uses feedback and system theory concepts to study the tracking performance of a widely used class of (nonlinear) adaptive algorithms for blind equalization. These algorithms are normally used in highly nonstationary environments (e.g., wireless fast fading channels), and it is therefore important to study their ability to track changes in the environment characteristics. Due to their inherent nonlinear nature, there have been essentially no results in the literature on the tracking performance of these algorithms. The approach proposed in this paper is based on studying the energy flow through a feedback cascade that is induced by any such adaptive algorithm. A key feature of the feedback approach is that it bypasses the need for working directly with the nonlinear recursion for the weight error vector of the adaptive equalizer.

1 Introduction

There has been an increasing interest in adaptive algorithms for blind equalization purposes in digital communications. These algorithms avoid the initial training phase of an adaptive equalizer and, therefore, avoid potential losses in valuable channel capacity. Such algorithms are usually employed in highly nonstationary environments (e.g., wireless fast fading channels), and it is therefore important to study their ability to track changes in the environment characteristics. However, due to their inherent nonlinear nature, there have been essentially no results in the literature on the tracking performance of blind adaptive algorithms.

In this paper, we use feedback and system theory concepts to derive new expressions for the steady-state mean square error (MSE) of blind adaptive equalizers. Such expressions are useful for quantifying the ability of the adaptive equalizers to track variations in the environment.

We focus on the class of constant modulus (CM) algorithms, which are among the most widely used algorithms for (fractionally-spaced) blind equalization (see, e.g., [1, 2, 3]). The update equations for the weight-error vector in these algorithms are highly nonlinear (see Sec. 3), which makes it difficult to evaluate the steady-state MSE using conventional techniques. A major feature of the approach proposed herein is that it bypasses the need for working directly with the weight-error vector. This is achieved by exploiting a fundamental energy-preserving relation that in fact holds for a general class of adaptive algorithms (e.g., [4, 5, 6]).

The paper is organized as follows. In the next section, we describe the model for fractionally-space channel equalization. In Sec. 3, we describe two of the main CM algorithms that we study in this paper, and which are known by the names of CMA2-2 and CMA1-2. In Sec. 4 we motivate and derive the aforementioned fundamental energy-preserving relation, which we then apply to CMA2-2 and CMA1-2 in Secs. 5 and 6, respectively. Simulation results are presented in Sec. 7. A comparison between the tracking properties of the two algorithms is given in Sec. 8. Conclusions are given in Sec. 9.

2 The Channel-Equalizer Model

Equalization algorithms can be implemented in symbol-spaced form or in fractionally-spaced form (FSE). In this paper we concentrate on fractionally-spaced implementations due to their inherent advantages (see, e.g., [2, 3, 7]). Figure 1 shows the channel-equalizer model that arises when $\frac{T}{2}$-fractionally spaced equalization is used; a similar structure applies to more general $\frac{T}{P}$-FSEs.

[1]This work was supported in part by the National Science Foundation under Award No. CCR-9732376.

Thus consider an FIR channel $\mathbf{c}$ of length $2M$ and an FIR equalizer $\bar{\mathbf{w}}$ of length $2N$. We split the coefficients of the channel into even- and odd-indexed entries and denote them by

$$\begin{aligned} \mathbf{c}_e &= [c(0), c(2), ..., c(2M-2)]^T \\ \mathbf{c}_o &= [c(1), c(3), ..., c(2M-1)]^T . \end{aligned}$$

In a similar way, we define the two sub-equalizers

$$\begin{aligned} \mathbf{w}_e &= [\bar{w}(0), \bar{w}(2), ..., \bar{w}(2N-2)]^T \\ \mathbf{w}_o &= [\bar{w}(1), \bar{w}(3), ..., \bar{w}(2N-1)]^T . \end{aligned}$$

The system in Figure 1 then corresponds to what is called a multichannel model for a $\frac{T}{2}$-fractionally-spaced equalizer. This model is well motivated and explained in the survey article [3].

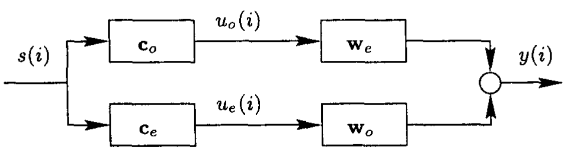

Figure 1: *A multichannel model for $\frac{T}{2}$-fractionally spaced equalization.*

We further define the equalizer weight (column) vector

$$\mathbf{w} = \begin{bmatrix} \mathbf{w}_e \\ \mathbf{w}_o \end{bmatrix} ,$$

and the (regressor) input row vector $\mathbf{u}_i = [\ \mathbf{u}_{o,i}\ \ \mathbf{u}_{e,i}\ ]$, where

$$\begin{aligned} \mathbf{u}_{o,i} &= [u_o(i), u_o(i-1), ..., u_o(i-N+1)] , \\ \mathbf{u}_{e,i} &= [u_e(i), u_e(i-1), ..., u_e(i-N+1)] . \end{aligned}$$

Then $y(i) = \mathbf{u}_i \mathbf{w}$.

Perfect Equalization. An important result for such fractionally-spaced equalizers is the following (see, e.g., [3]). Let $\mathbf{c}_e(z)$ and $\mathbf{c}_o(z)$ denote the polynomials associated with the even- and odd-indexed sub-channels,

$$\begin{aligned} \mathbf{c}_e(z) &\triangleq c(0) + c(2)z + \ldots + c(2M-2)z^{M-1} , \\ \mathbf{c}_o(z) &\triangleq c(1) + c(3)z + \ldots + c(2M-1)z^{M-1} . \end{aligned}$$

Then it can be shown that if these polynomials do not have common zeros, and if $N \geq M-1$, then there exists an equalizer $\mathbf{w}$ that leads to an overall channel-equalizer impulse response of the form

$$h_D = e^{j\theta} \operatorname{col}[0, ..., 0, 1, 0, ..., 0] , \quad j = \sqrt{-1} , \qquad (1)$$

for some constant phase shift $\theta \in [0, 2\pi]$, and where the unit entry is in some position D, $D \leq M+N-1$. Equalizers $\mathbf{w}$ that result in overall impulse responses of the above form are called *zero-forcing* equalizers and will be denoted by $\mathbf{w}^o$. Thus under such conditions, the output of the channel-equalizer system will be of the form $y(i) = s(i-D)e^{j\theta}$, for some $\{D, \theta\}$.

3 Constant Modulus Algorithms

A blind adaptive equalizer is one that attempts to approximate a zero forcing equalizer without knowledge of the channel impulse response $\mathbf{c}$ and without direct access to the transmitted sequence $\{s(\cdot)\}$ itself. This is achieved by seeking to minimize certain cost functions that are carefully chosen so that their global minima occur at zero forcing equalizers.

The most popular adaptive blind algorithms are the so-called constant modulus algorithms [8, 9]. They are derived as stochastic gradient methods for minimizing the cost function:

$$J_{CM}(\mathbf{w}) = \mathrm{E}(|y(i)|^p - R_p|)^2 ,$$

where R_p is suitably chosen in order to guarantee that the global minima of $J_{CM}(\mathbf{w})$ occur at zero forcing solutions (see, e.g., [8, 10]). In this paper we focus on the following two stochastic gradient variants, known as CMA2-2 and CMA1-2.

CMA2-2. In this case, we select $p = 2$,

$$R_2 = \frac{\mathrm{E}\,|s(i)|^4}{\mathrm{E}\,|s(i)|^2} ,$$

and the update equation for the weight estimates is given by

$$\mathbf{w}_{i+1} = \mathbf{w}_i + \mu \mathbf{u}_i^* \, y(i) \, \left[R_2 - |y(i)|^2\right] , \qquad (2)$$

with a step-size μ. The *row* vector $\mathbf{u}_i$ is the input data regressor to the adaptive equalizer and $y(i) = \mathbf{u}_i \mathbf{w}_i$ is the output of the adaptive equalizer. The symbol $*$ denotes complex conjugate transposition.

CMA1-2. In this case, we select $p = 1$,

$$R_1 = \frac{\mathrm{E}\,|s(i)|^2}{\mathrm{E}\,|s(i)|} ,$$

and the update equation for the weight estimates is given by

$$\mathbf{w}_{i+1} = \mathbf{w}_i + \mu \mathbf{u}_i^* \left[R_1 \frac{y(i)}{|y(i)|} - y(i)\right] . \qquad (3)$$

Since these algorithms are based on instantaneous approximations of the true gradient vector of the cost function $J_{CM}(\mathbf{w})$, the equalizer output $y(i)$ need not converge to a zero forcing solution of the form $s(i-D)e^{j\theta}$ due to the presence of gradient noise. Therefore, the steady-state MSE,

$$\lim_{i \to \infty} \mathrm{E}\left|y(i) - s(i-D)e^{j\theta}\right|^2 ,$$

is usually used as a performance index of the adaptive equalization algorithm.

Moreover, in nonstationary environments, the channel actually varies with time and therefore the zero forcing weight vector $\mathbf{w}^o$ itself may also assumed to vary with time, say $\mathbf{w}_i^o$. It is customary to assume a first-order random walk model for $\mathbf{w}_i^o$ of the form (e.g., [11]):

$$\mathbf{w}_{i+1}^o = \mathbf{w}_i^o + \mathbf{q}_i \ , \tag{4}$$

for a sequence of i.i.d. random vectors $\{\mathbf{q}_i\}$, with a positive-definite covariance matrix, $\mathbf{Q} = \mathrm{E}\,\mathbf{q}_i\mathbf{q}_i^*$. We are therefore interested in evaluating the steady-state MSE under the assumption (4). For this purpose, we shall propose a method that evaluates the MSE without directly using the weight error vector $\tilde{\mathbf{w}}_i = \mathbf{w}_i^o - \mathbf{w}_i$.

4 A Fundamental Energy Relation

Introduce the so-called a-priori and a-posteriori estimation errors,

$$\begin{aligned} e_a(i) &= s(i-D)e^{j\theta} - y(i) = \mathbf{u}_i\mathbf{w}_i^o - \mathbf{u}_i\mathbf{w}_i = \mathbf{u}_i\tilde{\mathbf{w}}_i \ , \\ e_p(i) &= \mathbf{u}_i\left(\tilde{\mathbf{w}}_{i+1} - \mathbf{q}_i\right) \ . \end{aligned}$$

Now consider a general stochastic algorithm of the generic form

$$\mathbf{w}_{i+1} = \mathbf{w}_i + \mu\, \mathbf{u}_i^*\, f_e(i) \ , \tag{5}$$

where $f_e(i)$ denotes an instantaneous error function. CM algorithms are special cases of the above for different choices of $f_e(i)$. If we subtract $\mathbf{w}_i^o$ from both sides of (5) and multiply by $\mathbf{u}_i$ from the left, we find that the errors $\{e_p(i), e_a(i)\}$ are related via:

$$e_p(i) = e_a(i) - \frac{\mu}{\bar{\mu}(i)} f_e(i) \ , \tag{6}$$

where we defined, for compactness of notation, $\bar{\mu}(i) = 1/\|\mathbf{u}_i\|^2$. Using (4)–(6), we easily find that $\tilde{\mathbf{w}}_i$ satisfies

$$\tilde{\mathbf{w}}_{i+1} = \tilde{\mathbf{w}}_i - \bar{\mu}(i)\mathbf{u}_i^*[e_a(i) - e_p(i)] + \mathbf{q}_i \ .$$

By evaluating the energies of both sides of this equation we obtain the energy preserving relation [4, 5, 6]:

$$\|\tilde{\mathbf{w}}_{i+1} - \mathbf{q}_i\|^2 + \bar{\mu}(i)|e_a(i)|^2 = \|\tilde{\mathbf{w}}_i\|^2 + \bar{\mu}(i)|e_p(i)|^2 \tag{7}$$

This relation holds for all adaptive algorithms whose recursions are of the form given by (5) — for any choice of $f_e(\cdot)$; it shows how the energies of the weight error vectors at two successive time instants are related to the energies of the a-priori and a-posteriori estimation errors. It also establishes that the mapping from $\left\{\tilde{\mathbf{w}}_i, \sqrt{\bar{\mu}(i)}e_p(i)\right\}$ to $\left\{\tilde{\mathbf{w}}_{i+1} - \mathbf{q}_i, \sqrt{\bar{\mu}(i)}e_a(i)\right\}$ is energy preserving (or lossless). Furthermore, combining (7) with (6), we see that any adaptive algorithm of the form (5) induces a feedback structure of the form

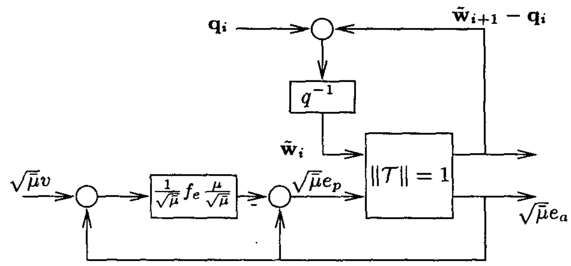

Figure 2: *Lossless mapping and a feedback loop.*

shown in Figure 2, where $\mathcal{T}$ denotes a lossless map and q^{-1} denotes the unit delay operator. It could be seen from the figure that the system nonstationarity vector $\mathbf{q}_i$ acts as a disturbance input to the system.

We now show the relevance of the energy relation (7) to the evaluation of the steady-state MSE of an adaptive equalizer. First, we impose the following modeling assumption, which is typical in the context of tracking analysis of adaptive filters (see, e.g., [11]).

M.1 The sequences $\{\mathbf{u}_i\}$ and $\{\mathbf{q}_i\}$ are mutually statistically independent.

Imposing the equality $\mathrm{E}\,\|\tilde{\mathbf{w}}_{i+1}\|^2 = \mathrm{E}\,\|\tilde{\mathbf{w}}_i\|^2$ in steady-state, and using (6) and M.1, it is straightforward to verify that the energy relation (7) leads to the following important equality, which we shall extensively use in the sequel:

$$\mathrm{E}\,\bar{\mu}(i)|e_a(i)|^2 = \mathrm{Tr}(\mathbf{Q}) + \mathrm{E}\,\bar{\mu}(i)\left|e_a(i) - \frac{\mu}{\bar{\mu}(i)}f_e(i)\right|^2 \tag{8}$$

This equation can now be solved for the steady-state MSE:

$$\mathsf{MSE} \triangleq \lim_{i\to\infty} \mathrm{E}\,|e_a(i)|^2 \ .$$

The procedure of finding the MSE through (8) completely avoids the need for evaluating $\mathrm{E}\,\|\tilde{\mathbf{w}}_\infty\|^2$. This is because in steady-state, and in view of the energy-preserving relation (7) and the feedback structure, the effect of the weight error vector is canceled out!

5 Tracking Analysis of CMA2-2

We now apply the above results to the CMA2-2 recursion (2). For mathematical tractability of the analysis, we impose the following two reasonable assumptions in *steady-state* ($i \to \infty$) — for more motivation and explanation on these two assumptions, see [13] and also [6, 12]:

I.1 The transmitted signal $s(i-D)$ and the estimation error $e_a(i)$ are independent in steady-state so that $\mathrm{E}\, s^*(i-D)e_a(i) = 0$, since $s(i-D)$ is assumed zero mean.

I.2 The scaled regressor energy $\mu^2\|\mathbf{u}_i\|^2$ is independent of $y(i)$ in steady-state.

We consider first the case of real-valued data $\{s(\cdot), y(\cdot), \mathbf{u}_i\}$. In this case, we can assume that the zero forcing response h_D that the adaptive equalizer attempts to achieve (*cf.* (1)) can be of either form $h_D = \pm[0, ..., 0, 1, 0, ..., 0]$. In the following, we continue with the choice $h_D = [0, ..., 0, 1, 0, ..., 0]$, which yields $e_a(i) = s(i-D) - y(i)$. A similar analysis holds for the case $h_D = [0, ..., 0, -1, 0, ..., 0]$.

Now relation (8) in the CMA2-2 context leads to the equality, for $i \to \infty$,

$$\mathrm{E}\,\bar{\mu}(i)|e_a(i)|^2 = \mathrm{Tr}(\mathbf{Q}) + \mathrm{E}\,\bar{\mu}(i)\left|e_a(i) - \frac{\mu}{\bar{\mu}(i)}y(i)(R_2 - |y(i)|^2)\right|^2 \quad (9)$$

We shall write more compactly (here and throughout the paper)

$$e_a \triangleq e_a(i),\ \bar{\mu} \triangleq \bar{\mu}(i),\ y \triangleq y(i),\ \mathbf{u} \triangleq \mathbf{u}_i,\ s \triangleq s(i-D),$$

for $i \to \infty$, so that (9) becomes, after expanding,

$$2\mu\,\mathrm{E}\,e_a y(R_2 - y^2) = \mathrm{Tr}(\mathbf{Q}) + \mu^2\,\mathrm{E}\,\|\mathbf{u}\|^2 y^2 (R_2 - y^2)^2 .$$

Using this equality we can now obtain an expression for the steady-state MSE, $\mathrm{E}\,|e_a|^2$. Replacing y by $s - e_a$, using assumptions I.1–I.2 and neglecting $2\mu\,\mathrm{E}\,e_a^4$, for sufficiently small μ and small e_a^2, it is straightforward to show that the steady-state MSE can be approximated by

$$\zeta^{\mathsf{CMA2-2}} \approx \frac{\mathrm{Tr}(\mathbf{Q})/\mu + \mu\,\mathrm{E}(s^2R_2^2 - 2R_2s^4 + s^6)\,\mathrm{E}\,\|\mathbf{u}\|^2}{2\,\mathrm{E}(3s^2 - R_2)}$$

This result implies that the steady-state MSE is composed of two terms. The first term decreases with μ and increases with the system nonstationarity variance $\mathrm{Tr}(\mathbf{Q})$. The second term increases with μ and the received signal variance, $\mathrm{E}\,\|\mathbf{u}\|^2$. Thus, unlike the stationary case (see, e.g., [12, 13]), the steady-state MSE is not a monotonically increasing function of μ. We can also see that in the noiseless case, and for non-constant modulus data $\{s(\cdot)\}$, there exists a finite optimal value of the step size, μ_o, that minimizes the above expression for the steady-state MSE, which is given by

$$\mu_o^{\mathsf{CMA2-2}} = \sqrt{\mathrm{Tr}(\mathbf{Q})/\left[\mathrm{E}(s^2R_2^2 - 2R_2s^4 + s^6)\,\mathrm{E}\,\|\mathbf{u}\|^2\right]} .$$

This expression shows that μ_o decreases with the signal variance, $E\|\mathbf{u}\|^2$, and increases with the system nonstationarity variance $\mathrm{Tr}(\mathbf{Q})$. The corresponding minimum value of the steady-state MSE is then given by

$$\zeta_o^{\mathsf{CMA2-2}} = \frac{\sqrt{\mathrm{Tr}(\mathbf{Q})\,\mathrm{E}(s^2R_2^2 - 2R_2s^4 + s^6)\,\mathrm{E}\,\|\mathbf{u}\|^2}}{\mathrm{E}(3s^2 - R_2)}$$

Here we may add that for complex-valued data, the steady-state MSE will have a different expression than that in the real-valued case. Following the same derivation, and assuming signal constellations that satisfy the circularity condition $\mathrm{E}\,|s(i)|^2 = 0$, in addition to the condition $\mathrm{E}(2|s(i)|^2 - R_2) > 0$ (both of which hold for most constellations [8]), we can show that the steady-state MSE for complex-valued data, and for sufficiently small step-sizes, can be approximated by

$$\zeta^{\mathsf{CMA2-2}} \approx \frac{\mathrm{Tr}(\mathbf{Q})/\mu + \mu\,\mathrm{E}(|s|^2R_2^2 - 2R_2|s|^4 + |s|^6)\,\mathrm{E}\,\|\mathbf{u}\|^2}{2\,\mathrm{E}(2|s|^2 - R_2)}$$

In this case, the optimum value of the algorithm step size still has the same value as in the real-valued data case, while the minimum achievable steady-state MSE is given by

$$\zeta_o^{\mathsf{CMA2-2}} = \frac{\sqrt{\mathrm{Tr}(\mathbf{Q})\,\mathrm{E}(|s|^2R_2^2 - 2R_2|s|^4 + |s|^6)\,\mathrm{E}\,\|\mathbf{u}\|^2}}{\mathrm{E}(2|s|^2 - R_2)}$$

6 Tracking Analysis of CMA1-2

We now extend the earlier results to the CMA1-2 recursion (3). In this case, the expressions for the steady-state MSE for both real and complex-valued data will coincide. For this reason, we shall consider only the real-valued case. In addition to assumptions I.1–I.2, we need the following two assumptions (also in steady-state) — see [13]:

I.3 The output $y(i)$ of the equalizer is distributed symmetrically around the transmitted signal $s(i-D)$ in steady-state, so that $\mathrm{E}\,|y(i)| = \mathrm{E}\,|s(i-D)|$.

I.4 The a-priori error $e_a(i)$ is independent of $\mathrm{sign}\, y(i)$ in steady-state, and $\mathrm{E}\,\mathrm{sign}\, y(i) = 0$, so that $\mathrm{E}\, e_a(i)\,\mathrm{sign}\, y(i) = 0$.

For the CMA1-2 recursion (3), the basic equation (8) is given, in steady-state, by

$$2\mu\,\mathrm{E}\,e_a(R_1\,\mathrm{sign}(y) - y) = \mathrm{Tr}(\mathbf{Q}) + \mu^2\,\mathrm{E}\left[(R_1\,\mathrm{sign}(y) - y))^2 \cdot \|\mathbf{u}\|^2\right] .$$

Using assumptions I.1–I.4 and ignoring the term $\mu^2 \, \mathrm{E}\, \|\mathbf{u}\|^2 \cdot \mathrm{E}\, e_a^2$, when μ and e_a are sufficiently small, leads to

$$\zeta^{\mathsf{CMA1-2}} \approx 0.5\mu^{-1}\,\mathrm{Tr}(\mathbf{Q}) + 0.5\mu\,\mathrm{E}\left(R_1^2 + |s|^2 - 2R_1|s|\right) \cdot \mathrm{E}\,\|\mathbf{u}\|^2 \ .$$

Again we can see that the steady-state MSE is composed of two terms. The first term decreases with μ, while the second increases with μ. Therefore, for non-constant modulus data $\{s(\cdot)\}$, there exists a finite optimal value of the step size, μ_o, that minimizes the steady-state MSE, and is given by

$$\mu_o^{\mathsf{CMA1-2}} = \sqrt{\mathrm{Tr}(\mathbf{Q})/[\mathrm{E}\,(R_1^2 + |s|^2 - 2R_1|s|)\,\mathrm{E}\,\|\mathbf{u}\|^2]} \ .$$

The corresponding minimum achievable value of the steady-state MSE is then given by

$$\zeta_o^{\mathsf{CMA1-2}} = \sqrt{\mathrm{Tr}(\mathbf{Q})\,\mathrm{E}\,(R_1^2 + |s|^2 - 2R_1|s|)\,\mathrm{E}\,\|\mathbf{u}\|^2} \ .$$

7 Simulation Results

We now provide some simulation results that compare the experimental performance with the one predicted by the derived expressions. The channel considered in this simulation is given by $\mathbf{c} = [0.1, 0.3, 1, -0.1, 0.5, 0.2]$. A 4-tap FIR filter is used as a $\frac{T}{2}$-fractionally spaced equalizer. The standard deviation of each element of the environment nonstationarity vector, $\mathbf{q}_i$, is 10^{-3}. In this simulation, the transmitted signal was 6-PAM constellated, $s(i) \in \{5, 3, 1, -1, -3, -5\}$ with $\mathrm{E}\, s^6 = 5451.7$, $\mathrm{E}\, s^4 = 235.7$, $\mathrm{E}\, s^2 = 11.67$, and $R_2 = 20.2$. The value of $\|\mathbf{u}_i\|^2$ is the norm of the received signal vector. The value of $\mathrm{E}\,\|\mathbf{u}_i\|^2$ was computed as the average over 10,000 realizations of $\|\mathbf{u}_i\|^2$. The value of experimental MSE was obtained as the average over 100 repeated runs. Figure 3 is a plot of the experimental MSE and the theoretical MSE versus the step-size μ for CMA2-2. It can be seen from the figure that the theoretical results match reasonably well the experimental results. We can also see that the experimental MSE reaches a minimum value of 0.017, which corresponds to an optimal value of μ that lies between 10^{-5} and 1.5×10^{-5}. These values reasonably match our theory, which predicted a minimum achievable MSE of 0.0201 at $\mu_o = 1.3448 \times 10^{-5}$.

Figure 4 is a plot of the experimental MSE and the theoretical MSE for CMA1-2 versus the step-size μ for the same simulation environment we used for CMA2-2. Again, we see from the figure that the theoretical and simulation results are in good match. We can also see that the experimental MSE reaches a minimum value of 0.0182, which corresponds to an optimal value of μ that lies between 1.8×10^{-4} and 2×10^{-4}. These values reasonably match our theory, which predicted a minimum achievable MSE of 0.021 at $\mu_o = 1.9016 \times 10^{-4}$. Comparing the minimum achievable MSE of the CMA1-2 with that of the CMA2-2, we note that CMA2-2 outperforms CMA1-2 by approximately 0.29 dB. This result will be verified by the comparison given in the next section.

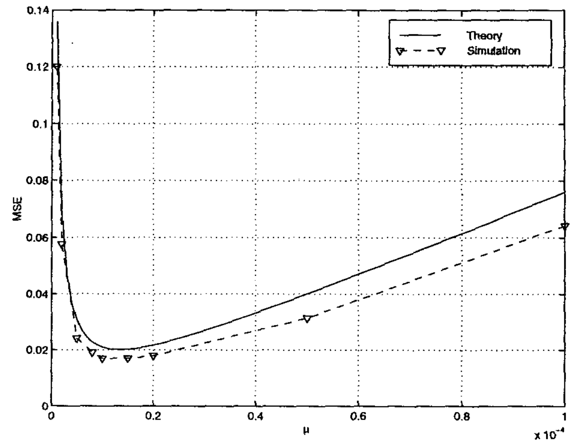

Figure 3: *Theoretical and simulation* MSE *of* CMA*2-2.*

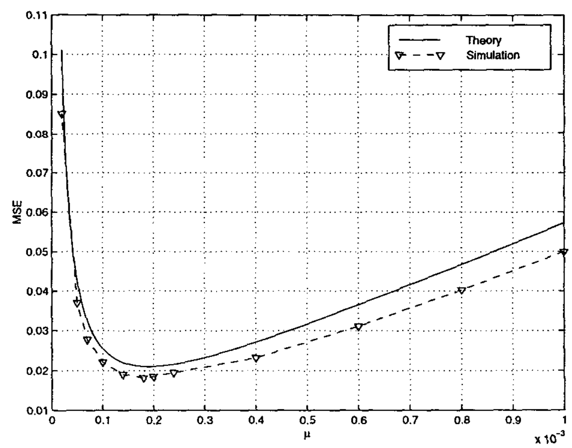

Figure 4: *Theoretical and simulation* MSE *of* CMA*1-2.*

8 Comparison of the Tracking Properties of the CM algorithms

We now compare the ability of the CMA2-2 and CMA1-2 to track variations in nonstationary environments for various types of transmitted sequence $\{s(\cdot)\}$. We use the ratio of the minimum achievable steady-state MSE of both algorithms, which is given by

$$\eta \triangleq \frac{\zeta_o^{\mathsf{CMA2-2}}}{\zeta_o^{\mathsf{CMA1-2}}} \ .$$

Such a measure is conventionally used as a comparison index of the tracking performance of various adaptive algorithms (see, e.g., [11]). For real valued-data, the ratio of the minimum achievable steady-state MSE of the CMA2-2 and CMA2-2, η_r, is given by

$$\eta_r = \frac{1}{\mathrm{E}(3s^2 - R_2)}\sqrt{\frac{\mathrm{E}(s^2R_2^2 - 2R_2s^4 + s^6)}{\mathrm{E}\,(R_1^2 + s^2 - 2R_1|s|)}} \quad . \tag{10}$$

On the other hand, the same ratio for complex-valued data, η_c, is given by

$$\eta_c = \frac{1}{\mathrm{E}(2|s|^2 - R_2)}\sqrt{\frac{\mathrm{E}(|s|^2R_2^2 - 2R_2|s|^4 + |s|^6)}{\mathrm{E}\,(R_1^2 + |s|^2 - 2R_1|s|)}} \quad , \tag{11}$$

We can see from (10) and (11) that the ratio of the minimum achievable steady-state MSE of the CMA2-2 and CMA1-2 depends *only* on the statistical properties of the transmitted sequence $\{s(\cdot)\}$. We now compare the tracking performance of both CM algorithms for the following classes of sequences that are frequently encountered in blind equalization applications.

4-PAM. In this case, the transmitted sequence, $\{s(\cdot)\}$, has 4 real-valued symbols $\{-3a, -a, a, 3a\}$, where a is a positive real number. For such transmitted sequence, we can verify from (10) that

$$\eta_r^{4-\mathsf{PAM}} = 12/17 \approx -1.5\,\mathrm{dB} \ .$$

This indicates that the minimum achievable value of steady-state MSE of CMA2-2 is less than that of CMA1-2 by approximately 1.5 dB, for all values of the parameter a. This reflects the superiority of CMA2-2 over the CMA1-2 in this case.

6-PAM. For 6-PAM transmitted symbols of the form $\{-5a, -3a, -a, a, 3a, 5a\}$, where a is a positive real number. In this case, we get from (10) that

$$\eta_r^{6-\mathsf{PAM}} \approx -0.2\,\mathrm{dB} \ ,$$

which indicates that the minimum achievable value of steady-state MSE of CMA2-2 is less than that of CMA1-2 by approximately 0.2 dB for 6-PAM signals.

16-QAM. In this case, the transmitted symbols lie on a grid of 16 complex-valued symbols that are separated by $2a$, where a is a positive real number. In this case we get from (11) that

$$\eta_c^{16-\mathsf{QAM}} \approx 2 \approx 3\,\mathrm{dB} \ .$$

Thus, unlike the 4-PAM and 6-PAM cases, CMA1-2 is superior to CMA2-2 by approximately 3 dB for all values of the parameter a, in the case of 16-QAM signals.

9 Conclusions

In this paper we studied the tracking performance of two blind adaptive algorithms of the constant modulus type, namely, CMA2-2 and CMA1-2. Analytical expressions for the steady-state MSE were derived and verified by computer simulations. It is found that there exists a step size that minimizes the steady-state MSE in each case. Expressions for the optimal step size values are derived along with the minimum achievable steady-state MSE for both algorithms. A comparison between the tracking properties of CMA2-2 and CMA1-2 is performed for various classes of the transmitted sequence. It is found that for the cases of 4-PAM and 6-PAM signals, the tracking performance of CMA2-2 is superior to that of CMA1-2 by 1.5 and 0.2 dB, respectively. On the other hand, for the case of 16-QAM signals, CMA1-2 is superior to CMA2-2 by 3 dB.

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