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LATTICE STRUCTURES FOR TIME-VARIANT INTERPOLATION PROBLEMS *

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ABSTRACT

We derive a recursive solution for a general time-variant interpolation problem of the Hermite-Fejér type, based on a fast algorithm for the recursive triangular factorization of time-variant structured matrices. The solution follows from studying the properties of an associated transmission-line. The line can be drawn as a cascade of first-order lattice sections, where each section is composed of a rotation matrix followed by a storage element and a tapped-delay filter. An application is made to a problem that arises in model validation.

I. INTRODUCTION

The successful application of interpolation problems in control and circuit theory has inspired the study of generalizations to the time-variant setting [1, 2, 3, 4, 5]. We describe here a computationally oriented solution for a general time-variant interpolation problem of the Hermite-Fejér type, based on a fast algorithm for the recursive triangular factorization of time-variant structured matrices [5, 6, 7]. We use the interpolation data to construct a convenient so-called generator for the factorization algorithm. The recursive algorithm then leads to a transmission-line cascade of first-order sections that makes evident the interpolation property. This is due to the fact that transmission lines have "transmission zeros": certain inputs at certain frequencies yield zero outputs. In the time-invariant case for example [6, 8, 9], each section of the cascade can be characterized by a $p \times q$ rational transfer matrix $\Theta_i(z)$ say, that has a left zero-direction vector g_i at a frequency f_i, viz.,

$$g_i\Theta_i(f_i) \equiv \begin{bmatrix} a_i & b_i \end{bmatrix} \begin{bmatrix} \Theta_{i,11} & \Theta_{i,12} \\ \Theta_{i,21} & \Theta_{i,22} \end{bmatrix} (f_i) = \mathbf{0},$$

which makes evident (with the proper partitioning of the row vector g_i and the matrix function $\Theta_i(z)$) the following interpolation property: $a_i\Theta_{i,12}\Theta_{i,22}^{-1}(f_i) = -b_i$. We shall extend this picture to the time-variant setting and describe the associated recursive solution.

II. TIME-VARIANT HERMITE-FEJÉR

We first introduce some notation and extend the notion of *"derivatives"* to the time-variant setting. We consider a finite-dimensional *time-variant* state-space model with a bounded upper triangular operator $\mathcal{T}$. The matrix entries of $\mathcal{T}$ are denoted by T_{ij} (of dimensions $r(i) \times r(j)$), and constitute the time-variant Markov parameters of the underlying state-space model: $\mathcal{T} =$

$$\begin{bmatrix} \ddots & \ddots & & & & \\ & T_{-1,-1} & T_{-1,0} & T_{-1,1} & \dots & \\ & & \boxed{T_{00}} & T_{01} & T_{02} & \dots \\ & \mathbf{O} & & T_{11} & T_{12} & T_{13} & \dots \\ & & & & \ddots & \ddots \end{bmatrix}$$

where $\boxed{T_{00}}$ denotes the $(0,0)$ entry of $\mathcal{T}$. We also introduce the symmetric functions $s_k^{(n)}$ of n variables (taken k at a time). That is, $s_0^{(n)} = 1$, and

$$s_k^{(n)}(x_1, x_2, \dots, x_n) = \sum_{1 \le i_1 < \dots < i_k \le n} x_{i_1} x_{i_2} \dots x_{i_k}$$

Let $\{f(t)\}_{t \in \mathbf{Z}}$ ($\mathbf{Z}$ is the set of integers) be a uniformly bounded sequence (over t) of scalar points inside the open unit disc. We then write $\mathcal{T}(f(t))$ to refer to the following expression:

$$\begin{aligned} \mathcal{T}(f(t)) \equiv\ & T_{tt} + f(t)T_{t-1,t} + f(t)f(t-1)T_{t-2,t} + \\ +\ & f(t)f(t-1)f(t-2)T_{t-3,t} + \dots \end{aligned}$$

In general, we define the p^{th} order time-variant derivative at $f(t)$ by $\frac{1}{p!}\mathcal{T}^{(p)}(f(t)) \equiv$

$$\sum_{m=0}^{\infty} s_m^{(m+p)}\left[f(t), f(t-1), \dots, f(t-m-p+1)\right] T_{t-m-p,t}$$

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For a uniformly bounded sequence (over t) of row vectors $\{u(t)\}_{t\in \mathbf{Z}}$, we define the time-variant tangential evaluation: $u(t)\frac{1}{p!}\mathcal{T}^{(p)}(f(t)) \equiv$

$$\sum_{m=0}^{\infty} s_m^{(m+p)}[f(t),\ldots,f(t-m-p+1)]\,u(t-m-p)T_{t-m-p,t}$$

We shall also use the notation $\mathcal{H}_{\mathcal{T}}^{r}(f(t))$ to refer to the following block-Toeplitz upper-triangular matrix (where $r \geq 1$ is a positive integer)

$$\begin{bmatrix} \mathcal{T}(f(t)) & \frac{1}{1!}\mathcal{T}^{(1)}(f(t)) & \cdots & \frac{1}{(r-1)!}\mathcal{T}^{(r-1)}(f(t)) \\ & \mathcal{T}(f(t)) & \cdots & \frac{1}{(r-2)!}\mathcal{T}^{(r-2)}(f(t)) \\ & & \ddots & \vdots \\ & \mathbf{O} & & \frac{1}{1!}\mathcal{T}^{(1)}(f(t)) \\ & & & \mathcal{T}(f(t)) \end{bmatrix}$$

We also denote by $e_i \equiv \begin{bmatrix} \mathbf{0}_{1\times i} & 1 & \mathbf{0} \end{bmatrix}$ the i^{th} basis vector of the n-dimensional space of complex numbers $\mathbf{C}^{1\times n}$.

We now introduce and state a general time-variant Hermite-Fejér problem, which includes as special cases the time-variant versions of the Carathéodory-Fejér and Nevanlinna-Pick problems studied in [3, 4]. We consider m uniformly bounded (over t) time-variant points $\{\alpha_i(t)\}_{i=0}^{m-1}$ (not necessarily distinct) inside the open unit disc, and we associate with each point $\alpha_i(t)$ a positive integer $r_i \geq 1$ and uniformly bounded row vectors $\mathbf{a}_i(t)$ and $\mathbf{b}_i(t)$ partitioned as follows

$$\mathbf{a}_i(t) = \begin{bmatrix} u_1^{(i)}(t) & u_2^{(i)}(t) & \ldots & u_{r_i}^{(i)}(t) \end{bmatrix}$$

$$\mathbf{b}_i(t) = \begin{bmatrix} v_1^{(i)}(t) & v_2^{(i)}(t) & \ldots & v_{r_i}^{(i)}(t) \end{bmatrix}$$

where $u_j^{(i)}(t)$ and $v_j^{(i)}(t)$ $(j = 1,\ldots,r_i)$ are $1 \times p(t)$ and $1 \times q(t)$ row vectors respectively.

Tangential Hermite-Fejér Problem: *Given m uniformly bounded points $\{\alpha_i(t)\}$ with the associated data r_i, $\mathbf{a}_i(t)$, and $\mathbf{b}_i(t)$, describe all upper triangular strictly contractive transfer operators $\mathcal{S}$ ($\|\mathcal{S}\|_\infty < 1$) that satisfy $\mathbf{b}_i(t) = \mathbf{a}_i(t)\mathcal{H}_{\mathcal{S}}^{r_i}(\alpha_i(t))$.* ∎

The first step in the solution consists in constructing three matrices $F(t), G(t)$, and $J(t)$ directly from the interpolation data: we define $J(t) \equiv (I_{p(t)} \oplus -I_{q(t)})$, and associate with each $\alpha_i(t)$ a Jordan block $\bar{F}_i(t)$ of size $r_i \times r_i$,

$$\bar{F}_i(t) = \begin{bmatrix} \alpha_i(t) & & & \\ 1 & \alpha_i(t) & & \\ & \ddots & \ddots & \\ & & 1 & \alpha_i(t) \end{bmatrix}$$

and two $r_i \times p(t)$ and $r_i \times q(t)$ matrices $U_i(t)$ and $V_i(t)$ respectively, which are composed of the row vectors associated with $\alpha_i(t)$,

$$U_i(t) = \begin{bmatrix} u_1^{(i)}(t) \\ \vdots \\ u_{r_i}^{(i)}(t) \end{bmatrix} \quad \text{and} \quad V_i(t) = \begin{bmatrix} v_1^{(i)}(t) \\ \vdots \\ v_{r_i}^{(i)}(t) \end{bmatrix}$$

Then $F(t) \equiv$ diagonal $\{\bar{F}_0(t),\ldots,\bar{F}_{m-1}(t)\}$ and

$$G(t) = \begin{bmatrix} U_0(t) & V_0(t) \\ \vdots & \vdots \\ U_{m-1}(t) & V_{m-1}(t) \end{bmatrix} \equiv \begin{bmatrix} \mathbf{U}(t) & \mathbf{V}(t) \end{bmatrix}$$

Let $n = \sum_{i=0}^{m-1} r_i$ and $r(t) = p(t) + q(t)$, then $F(t)$ and $G(t)$ are $n \times n$ and $n \times r(t)$ matrices respectively. We shall denote the diagonal entries of $F(t)$ by $\{f_i(t)\}_{i=0}^{n-1}$ (for example, $f_0(t) = \ldots = f_{r_0-1}(t) = \alpha_0(t)$). We also associate with the interpolation problem the time-variant displacement equation

$$R(t) - F(t)R(t-1)F^*(t) = G(t)J(t)G^*(t) \qquad (1)$$

and we shall further assume that the interpolation data satisfy the following nondegeneracy condition, which is automatically satisfied in many special cases [5, 6, 10].

$$\mathcal{U}(t)\mathcal{U}^*(t) > \mu > 0 \quad \text{for all } t \qquad (2)$$

where μ is a fixed constant and $\mathcal{U}(t) \equiv$

$$\begin{bmatrix} \ldots & F(t)F(t-1)\mathbf{U}(t-2) & F(t)\mathbf{U}(t-1) & \mathbf{U}(t) \end{bmatrix}$$

The above construction allows us to prove the following result [6, 10].

Theorem 1: *The tangential Hermite-Fejér problem is solvable if, and only if, there exists a fixed constant $\epsilon > 0$ such that $R(t) > \epsilon I$ for all t.* ∎

We shall say that $R(t)$ has a time-variant Toeplitz-like structure [5, 6, 7] with respect to $(F(t), G(t), J(t))$ and $G(t)$ is called its generator matrix. We should stress at this point that we only know $F(t), G(t)$, and $J(t)$, whereas the matrix $R(t) \equiv [r_{ij}(t)]_{i,j=0}^{n-1}$ is not known a priori. In fact, the recursive solution described in the next section does not need $R(t)$ explicitly. It <u>only</u> uses $F(t), G(t)$, and $J(t)$.

III. RECURSIVE ALGORITHM

Let $l_0(t)$ and $d_0(t)$ denote the first column and the $(0,0)$ entry of $R(t)$ respectively. If we subtract from $R(t)$ the outer product $l_0(t)d_0^{-1}(t)l_0^*(t)$, then we clearly obtain a new matrix whose first column and row are zero,

$$R(t) - l_0(t)d_0^{-1}(t)l_0^*(t) = \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & R_1(t) \end{bmatrix} \quad (3)$$

The matrix $R_1(t)$ is called the Schur complement of $r_{00}(t)$ in $R(t)$. We now verify that $R_1(t)$ is also a time-variant Toeplitz-like matrix, *i.e.*, it satisfies a displacement equation similar to (1). To check this, we let $g_0(t)$ denote the first row of $G(t)$. It then follows from (1) that

$$l_0(t) = F(t)l_0(t-1)f_0^*(t) + G(t)J(t)g_0^*(t)$$

$$d_0(t) = |f_0(t)|^2 d_0(t-1) + g_0(t)J(t)g_0^*(t)$$

Let $F_1(t)$ be the submatrix obtained after deleting the first row and column of $F(t)$. Using (3) we can readily check that [5, 6, 7]

$$R_1(t) - F_1(t)R_1(t-1)F_1^*(t) = G_1(t)J(t)G_1^*(t)$$

where $G_1(t)$ is related to $G(t)$ as follows

$$\begin{bmatrix} \mathbf{0}_{1\times r(t)} \\ G_1(t) \end{bmatrix} = F(t)l_0(t-1)h_0^*(t)J(t) + G(t)J(t)k_0^*(t)J(t)$$

and $h_0(t)$ and $k_0(t)$ are arbitrary $r(t)\times 1$ and $r(t)\times r(t)$ matrices respectively chosen so as to satisfy the embedding relation

$$\begin{bmatrix} f_0(t) & g_0(t) \\ h_0(t) & k_0(t) \end{bmatrix} \begin{bmatrix} d_0(t-1) & \mathbf{0} \\ \mathbf{0} & J(t) \end{bmatrix} \begin{bmatrix} f_0(t) & g_0(t) \\ h_0(t) & k_0(t) \end{bmatrix}^* =$$

$$\begin{bmatrix} d_0(t) & \mathbf{0} \\ \mathbf{0} & J(t) \end{bmatrix} \quad (4)$$

This shows that $R_1(t) \equiv \left[r_{ij}^{(1)}(t)\right]_{i,j=0}^{n-2}$ is indeed a time-variant Toeplitz-like matrix with respect to $(F_1(t), G_1(t), J(t))$. This process can now be repeated by defining the Schur complement $R_2(t)$ of $r_{00}^{(1)}(t)$ in $R_1(t)$ and so on. In summary, if we let $l_i(t)$ and $G_i(t)$ denote the first column and the generator of the i^{th} Schur complement $R_i(t)$ respectively, then we can compactly write

$$\begin{bmatrix} l_i(t) & \begin{matrix} \mathbf{0} \\ G_{i+1}(t) \end{matrix} \end{bmatrix} = \quad (5)$$

$$\begin{bmatrix} F_i(t)l_i(t-1) & G_i(t) \end{bmatrix} \begin{bmatrix} f_i^*(t) & h_i^*(t)J(t) \\ J(t)g_i^*(t) & J(t)k_i^*(t)J(t) \end{bmatrix}$$

where $g_i(t)$ is the first row of $G_i(t)$, and $h_i(t)$ and $k_i(t)$ are arbitrary $r(t)\times 1$ and $r(t)\times r(t)$ matrices respectively such that $\{f_i(t), g_i(t), h_i(t), k_i(t), d_i(t)\}$

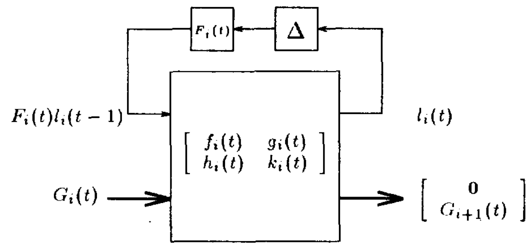

Figure 1: Time-variant transmission-line structure of the recursive algorithm.

satisfy an embedding relation similar to (4), with $d_i(t) = |f_i(t)|^2 d_i(t-1) + g_i(t)J(t)g_i^*(t)$, and $F_i(t)$ is the $(n-i)\times(n-i)$ submatrix obtained after deleting the first row and column of $F_{i-1}(t)$.

The generator recursion (5) has a transmission-line picture in terms of a cascade of elementary (first-order) sections as shown in figure 1, where each section depends on the parameters $\{f_i(t), g_i(t), h_i(t), k_i(t)\}$ and appears in ***state-space form*** on the right-hand side of (5), viz.,

$$\begin{bmatrix} \mathbf{x}_i(t+1) & \mathbf{y}_i(t) \end{bmatrix} =$$

$$\begin{bmatrix} \mathbf{x}_i(t) & \mathbf{w}_i(t) \end{bmatrix} \begin{bmatrix} f_i^*(t) & h_i^*(t)J(t) \\ J(t)g_i^*(t) & J(t)k_i^*(t)J(t) \end{bmatrix}$$

where $\mathbf{x}_i(t)$ is the state, $\mathbf{y}_i(t)$ is the output, and $\mathbf{w}_i(t)$ is a $1\times r(t)$ *row* input vector at time t (the Δ block represents a storage element where the present value of $l_i(t)$ is stored for the next time instant).

Let $\mathcal{T}_i \equiv \left[T_{lj}^{(i)} \right]$ denote the corresponding upper triangular transfer matrix, where $T_{lj}^{(i)}$ are the $r(l)\times r(j)$ time-variant Markov parameters defined by $T_{ll}^{(i)} = J(l)k_i^*(l)J(l)$, $T_{l,l+1}^{(i)} = J(l)g_i^*(l)h_i^*(l+1)J(l+1)$, and

$$T_{lj}^{(i)} = J(l)g_i^*(l)f_i^*(l+1)\dots f_i^*(j-1)h_i^*(j)J(j) \quad \text{for } j > l+1$$

After n recursive steps (recall that $G(t)$ has n rows) we obtain a cascade of sections $\mathcal{T} = \mathcal{T}_0\mathcal{T}_1\dots\mathcal{T}_{n-1}$.

IV. BLOCKING PROPERTIES

Our purpose is to prove that all solutions $\mathcal{S}$ to the Hermite-Fejér interpolation problem can be parametrized in terms of a linear fractional transformation based on $\mathcal{T}$. Before proceeding further, we first state [6, 10] the implications of the uniform boundedness of the interpolation data $\{f_i(t),\ \mathbf{a}_i(t),\ \mathbf{b}_i(t)\}$ on the boundedness of the quantities $d_i(t)$ and $g_i(t)$ that are needed in the recursive procedure.

Lemma 1: *There exist positive constants b_d, c_d, and c_g such that $0 < b_d < d_i(t) < c_d$ and $\|g_i(t)\| < c_g$ for all t.* ∎

Moreover, it is always possible (see [6, 10] and the next section) to choose uniformly bounded sequences $\{h_i(t), k_i(t)\}_{t \in \mathbf{Z}}$ so as to satisfy the embedding relation (4). It is then a standard result that the boundedness of $\{f_i(t), g_i(t), h_i(t), k_i(t)\}$ assures the boundedness of the corresponding operator $\mathcal{T}_i$ (see, *e.g.*, [11]).

If we define the direct sum $\mathcal{J} = \bigoplus_{t \in \mathbf{Z}} J(t)$, then it readily follows from the embedding relation (4) that each $\mathcal{T}_i$ also satisfies the $\mathcal{J}$-losslessness property $\mathcal{T}_i \mathcal{J} \mathcal{T}_i^* = \mathcal{T}_i^* \mathcal{J} \mathcal{T}_i = \mathcal{J}$. Furthermore, it is easy to check that each section $\mathcal{T}_i$ satisfies an important time-variant blocking property (which can be thought of as an extension of the notion of transmission zeros to the time-variant setting).

Theorem 2: *Each first-order section $\mathcal{T}_i$ satisfies*

$$\begin{bmatrix} \ldots & f_i(t)f_i(t-1)g_i(t-2) & f_i(t)g_i(t-1) & g_i(t) & ? \end{bmatrix} \mathcal{T}_i = \begin{bmatrix} 0 & ? \end{bmatrix}$$

and hence, $g_i(t)\mathcal{T}_i(f_i(t)) = 0$ (the ? symbol denotes irrelevant entries) ∎

The $\mathcal{J}$-losslessness and blocking properties of each section $\mathcal{T}_i$ reflect into the entire cascade $\mathcal{T}$, and it readily follows that $\mathcal{T}$ is a bounded upper triangular linear operator that satisfies $\mathcal{T}\mathcal{J}\mathcal{T}^* = \mathcal{T}^*\mathcal{J}\mathcal{T} = \mathcal{J}$. It also follows from the last theorem that $\mathcal{T}$ itself possesses a (global) blocking property.

Theorem 3: *The entire cascade $\mathcal{T}$ satisfies the global blocking property*

$$\begin{bmatrix} \ldots & F(t)F(t-1)G(t-2) & F(t)G(t-1) & G(t) & 0 \end{bmatrix} \mathcal{T} = \begin{bmatrix} 0 & ? \end{bmatrix} \tag{6}$$

That is, if we apply to $\mathcal{T}$ the block input

$$\begin{bmatrix} \ldots & F(t)F(t-1)G(t-2) & F(t)G(t-1) & G(t) & 0 \end{bmatrix}$$

then the output is zero up to and including time t. ∎

The global blocking property is closely related to the Hermite-Fejér interpolation conditions. To motivate this, we denote by $s_i = \sum_{p=0}^{i-1} r_p$, $s_0 = 0$, the total size of the Jordan blocks prior to $F_i(t)$. By comparing terms on both sides of (6) (and by using the Jordan structure of $F(t)$) we can verify that (6) can be rewritten in the following form

$$\begin{bmatrix} e_{s_i}G(t) & \ldots & e_{s_i+r_i-1}G(t) \end{bmatrix} \mathcal{H}_T^{r_i}(\alpha_i(t)) = 0, \tag{7}$$

where the row vector on the left hand-side of (7) is composed of the r_i row vectors in $\begin{bmatrix} U_i(t) & V_i(t) \end{bmatrix}$ associated with $\alpha_i(t)$,

$$\begin{bmatrix} u_1^{(i)}(t) & v_1^{(i)}(t) & \ldots & u_{r_i}^{(i)}(t) & v_{r_i}^{(i)}(t) \end{bmatrix}$$

If we partition the matrix entries T_{lj} of the cascade $\mathcal{T}$ accordingly with $J(l)$ and $J(j)$,

$$T_{lj} = \begin{bmatrix} T_{11}^{lj} & T_{12}^{lj} \\ T_{21}^{lj} & T_{22}^{lj} \end{bmatrix},$$

and consider the triangular operators

$$\mathcal{T}_{12} = \left[T_{12}^{lj}\right]_{l,j=-\infty}^{\infty} \quad \text{and} \quad \mathcal{T}_{22}^{(i)} = \left[T_{22}^{lj}\right]_{l,j=-\infty}^{\infty}$$

Then it can be shown [6, 10] that $\mathcal{S} \equiv -\mathcal{T}_{12}\mathcal{T}_{22}^{-1}$ is an upper triangular strictly contractive operator. It also follows from Theorem 3 that $\mathcal{S}$ satisfies the required interpolation conditions. Moreover, we can describe all solutions.

Theorem 4: *All solutions $\mathcal{S}$ of the tangential Hermite-Fejér problem are given through a linear fractional transformation of a strictly contractive upper triangular operator $\mathcal{K}$,*

$$\mathcal{S} = -\left[\mathcal{T}_{11}\mathcal{K} + \mathcal{T}_{12}\right]\left[\mathcal{T}_{21}\mathcal{K} + \mathcal{T}_{22}\right]^{-1}$$

V. LATTICE STRUCTURES

We now show how to further simplify the generator recursion (5) and derive a cascade of lattice sections. To begin with, recall that the generator recursion (5) requires knowledge of the quantities $h_i(t)$ and $k_i(t)$. Using the embedding relation (4) we can verify the following result.

Lemma 2: *All choices of $h_i(t)$ and $k_i(t)$ are*

$$h_i(t) = \Theta_i^{-1}(t) \left\{ \frac{\left[1 - \tau_i^*(t)f_i(t)\right] J(t) g_i^*(t)}{\tau_i^*(t)d_i(t) - d_i(t-1)f_i^*(t)} \right\}$$

$$k_i(t) = \Theta_i^{-1}(t) \left\{ I_{r(t)} - \frac{\tau_i^*(t)J(t)g_i^*(t)g_i(t)}{\tau_i^*(t)d_i(t) - d_i(t-1)f_i^*(t)} \right\}$$

where $\Theta_i(t)$ is an arbitrary $J(t)-$unitary matrix ($\Theta_i(t)J(t)\Theta_i^*(t) = J(t)$), and $\tau_i(t)$ is an arbitrary complex number chosen on the circle of radius $\sqrt{\frac{d_i(t-1)}{d_i(t)}}$. ∎

We claimed earlier that it is always possible to choose uniformly bounded sequences (over t) $\{h_i(t), k_i(t)\}_{t\in\mathbf{Z}}$. One possibility is to choose $\Theta_i(t) = I_{r(t)}$ and $\tau_i(t)$ on the circle of radius $\sqrt{\frac{d_i(t-1)}{d_i(t)}}$ but in the opposite direction of $f_i(t)$ [10]. We shall discuss here an alternative choice for $\Theta_i(t)$ that leads to a substantial simplification of the generator recursion (5), and provides a cascade structure of lattice sections: we choose $\Theta_i(t)$ (using elementary rotations, Householder transformations, or other possible implementations) such that the first row of $G_i(t)$ is reduced to either form:

$$g_i(t)\Theta_i(t) = \left[\begin{array}{cccc} \delta_i(t) & 0 & \dots & 0 \end{array}\right] \text{ if } g_i(t)J(t)g_i^*(t) > 0$$

$$g_i(t)\Theta_i(t) = \left[\begin{array}{cccc} 0 & \dots & 0 & \delta_i(t) \end{array}\right] \text{ if } g_i(t)J(t)g_i^*(t) < 0$$

In order to guarantee the uniform boundedness of the choices $\{\Theta_i(t)\}_{t\in\mathbf{Z}}$, we add the additional assumption that the sequence $\{g_i(t)J(t)g_i^*(t)\}_{t\in\mathbf{Z}}$ be uniformly bounded from *below* (it is clearly uniformly bounded from above because of Lemma 1). In the case $g_i(t)J(t)g_i^*(t) > 0$, expression (5) reduces to

$$l_i(t) = F_i(t)l_i(t-1)f_i^*(t) + G_i(t)\Theta_i(t)\left[\begin{array}{cc} \delta_i(t) & \mathbf{0} \end{array}\right]^T \quad (8)$$

$$\left[\begin{array}{c} \mathbf{0} \\ G_{i+1}(t) \end{array}\right] = G_i(t)\Theta_i(t)\left[\begin{array}{cc} -\phi_i(t)f_i(t) & \mathbf{0} \\ \mathbf{0} & I_{r(t)-1} \end{array}\right] + \frac{\phi_i(t)\delta_i(t)}{d_i(t-1)}F_i(t)l_i(t-1)\left[\begin{array}{cc} 1 & \mathbf{0} \end{array}\right] \quad (9)$$

where

$$\phi_i(t) = \frac{1 - \tau_i(t)f_i^*(t)}{1 - \tau_i^*(t)f_i(t)}\tau_i^*(t)$$

The last expression has a simple array interpretation. It shows that $G_{i+1}(t)$ can be obtained as follows: multiply $G_i(t)$ by $\Theta_i(t)$ and keep the *last* $(r(t) - 1)$ columns; the *first* column of the next generator is obtained as a linear combination of $F_i(t)l_i(t - 1)$ and the first column of $G_i(t)\Theta_i(t)$. In fact, this linear combination is obtained through an elementary *unitary* transformation. If we let $\bar{x}_i(t)$ and $x_{i+1}(t)$ denote the first columns of $G_i(t)\Theta_i(t)$ and $G_{i+1}(t)$ respectively, and define $\bar{l}_i(t) \equiv l_i(t)d_i^{-1/2}(t)$, then using (8) and (9) we write

$$\left[\begin{array}{cc} \bar{l}_i(t) & 0 \\ & x_{i+1}(t) \end{array}\right] = \left[\begin{array}{cc} F_i(t)\bar{l}_i(t-1) & \bar{x}_i(t) \end{array}\right] U_i(t)$$

where $U_i(t)$ is a 2×2 unitary matrix ($U_i(t)U_i^*(t) = I$) given by

$$U_i(t) = \left[\begin{array}{cc} f_i^*(t) & \rho_i(t) \\ \rho_i^*(t) & -f_i(t) \end{array}\right]\left[\begin{array}{cc} |\tau_i(t)| & 0 \\ 0 & \phi_i(t) \end{array}\right]$$

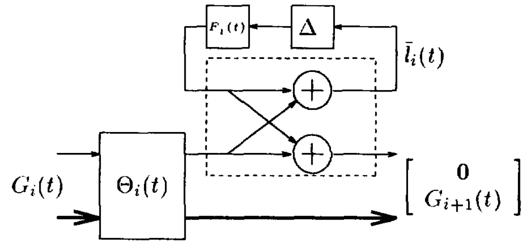

Figure 2: **Time-variant step of the generator recursion: $g_i(t)J(t)g_i^*(t) > 0$.**

and

$$\rho_i(t) = \frac{\delta_i(t)}{\sqrt{d_i(t-1)}}$$

This is depicted in figure 2: the first column of $G_i(t)$ goes through the top line and the last $(r(t) - 1)$ columns propagate through the bottom line. The output $\bar{x}_i(t)$ of the top line (which is the first column of $G_i(t)\Theta_i(t)$) goes through an elementary *unitary* rotation $U_i(t)$, along with $F_i(t)\bar{l}_i(t-1)$, and generates the first input of the next section ($x_{i+1}(t)$), as well as $\bar{l}_i(t)$.

The feedback line with $F_i(t)$ and Δ blocks is equivalent to a time-variant tapped-delay filter. To clarify this, observe that the columns of $G_i(t)$ are fed one row at a time through $\Theta_i(t)$, and that $F_i(t)$ has a bidiagonal structure of the form

$$F_i(t) = \left[\begin{array}{cccc} f_i(t) & & & \\ \xi_{i+1}(t) & f_{i+1}(t) & & \\ & \xi_{i+2}(t) & f_{i+2}(t) & \\ & & \ddots & \ddots \end{array}\right], \quad \xi_j(t) = 1, 0$$

If we denote the entries of $\bar{l}_i(t - 1)$ by $\left[\begin{array}{ccc} \bar{l}_{i,0}(t-1) & \bar{l}_{i,1}(t-1) & \dots \end{array}\right]^T$, then the computation of $F_i(t)\bar{l}_i(t-1)$ involves operations of the form

$$f_{i+j}(t)\bar{l}_{i,j}(t-1) + \xi_{i+j}(t)\bar{l}_{i,j-1}(t-1), \quad j \geq 0,$$

which can be implemented using a first-order time-variant tapped-delay (or FIR) structure [6, 10].

A similar argument holds when $g_i(t)J(t)g_i^*(t) < 0$ and leads to figure 3. In this case however, the elementary unitary transformation $U_i(t)$ is replaced by an elementary hyperbolic ($(1 \oplus -1)$-unitary) transformation $V_i(t)$. Let $\bar{y}_i(t)$ and $y_{i+1}(t)$ denote the *last* columns of $G_i(t)\Theta_i(t)$ and $G_{i+1}(t)$ respectively. The generator recursion (5) then reduces to the following array picture: multiply $G_i(t)$ by $\Theta_i(t)$ and keep the *first* $(r(t) - 1)$ columns; the *last* column of the next generator $G_{i+1}(t)$ is obtained through the elementary transformation

$$\left[\begin{array}{cc} \bar{l}_i(t) & 0 \\ & y_{i+1}(t) \end{array}\right] = \left[\begin{array}{cc} F_i(t)\bar{l}_i(t-1) & \bar{y}_i(t) \end{array}\right] V_i(t)$$

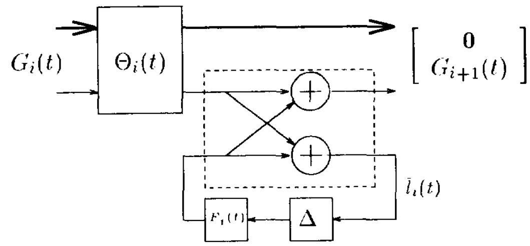

Figure 3: Time-variant step of the generator recursion: $g_i(t)J(t)g_i^*(t) < 0$.

where

$$V_i(t) = \begin{bmatrix} f_i^*(t) & \rho_i(t) \\ -\rho_i^*(t) & -f_i(t) \end{bmatrix} \begin{bmatrix} |\tau_i(t)| & 0 \\ 0 & \phi_i(t) \end{bmatrix}$$

VI. MODEL VALIDATION

To conclude this paper we briefly discuss an application of the recursive algorithm to a so-called model validation (or Carathéodory-Fejér) problem [12, 13] (see [5] for more details).

Carathéodory-Fejér Problem: *Given data points $\{u_i(t), v_i(t)\}_{t\in \mathbf{Z}}$, $i = 0, 1, \ldots, n-1$, it is required to find conditions for the existence of an upper triangular contraction $\mathcal{S} = [S_{ij}]$ such that*

$$\begin{bmatrix} u_0(t-n+1) & \ldots & u_{n-1}(t) \end{bmatrix} \begin{bmatrix} \ddots & & \vdots \\ & S_{t-1,t-1} & S_{t-1,t} \\ & & \mathcal{S}_{tt} \end{bmatrix} = \begin{bmatrix} v_0(t-n+1) & \ldots & v_{n-1}(t) \end{bmatrix}$$

■

This problem can be stated as imposing linear constraints on the "time-variant derivatives" of $\mathcal{S}$. We construct a displacement equation as in (1) with

$$\mathbf{U}(t) = \begin{bmatrix} u_0(t) \\ \vdots \\ u_{n-1}(t) \end{bmatrix}, \quad \mathbf{V}(t) = \begin{bmatrix} v_0(t) \\ \vdots \\ v_{n-1}(t) \end{bmatrix}$$

$$F(t) = \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}, \quad G(t) = \begin{bmatrix} \mathbf{U}(t) & \mathbf{V}(t) \end{bmatrix}$$

Theorem 5: *The tangential Carathéodory-Fejér problem has solutions if, and only if, $R(t) \geq 0$ for all t. This is equivalent to $\mathcal{U}(t)\mathcal{U}^*(t) \geq \mathcal{V}(t)\mathcal{V}^*(t)$ with*

$$\mathcal{U}(t) \equiv \begin{bmatrix} & & & u_0(t) \\ & & u_0(t-1) & u_1(t) \\ & \cdot & & \vdots \\ u_0(t-n+1) & \ldots & u_{n-2}(t-1) & u_{n-1}(t) \end{bmatrix}$$

and a similar expression for $\mathcal{V}(t)$ with $v_i(t)$ instead of $u_i(t)$. ■

We finally remark that the framework described in this paper can be extended to the operator setting, and can also be used to solve several matrix completion problems [5, 14].

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