An Unbiased and Cost-Effective Leaky-LMS Filter *

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Abstract

We propose a modified leaky-LMS filter that ensures stability of the estimates $\mathbf{w}(k)$ *in the presence of bounded noise, without introducing any bias term and with the added cost of only a comparison and a multiplication per iteration when compared to the classical LMS algorithm. The new algorithm is further shown to converge for* l_p *noise and persistently exciting regressors. It also provides bounded estimates even in finite precision arithmetic. The stability and convergence properties of the new algorithm are established through a deterministic analysis that is based on the Lyapunov theory for the stability of nonlinear difference equations.*

1. Introduction

The leaky-LMS algorithm is a widely used adaptive algorithm [1]. It was proposed to stabilize the weight drift problem (i.e., the possibility of unbounded weight estimates) that may occur in LMS in the presence of noise or in finite wordlength implementations, causing overflow and degraded performance in many applications.

However, as stated on p. 746 of [5], the prevention of the weight drift problem by the leaky LMS algorithm "is attained at the expense of an <u>increase in hardware</u> cost and at the expense of a <u>degradation in performance</u> compared to the infinite-precision form of the conventional LMS algorithm."

The purpose of this paper is to illustrate these varied effects and to propose a modification to leaky-LMS that eliminates the above expenses. More specifically, the algorithm proposed herein solves the weight drift problem *without* performance degradation and essentially at the *same* computational cost per iteration as LMS itself. Its added cost is only a single comparison and a single multiplication per iteration, which is negligible when compared to an overall cost of approximately $4M$ multiplications and additions that are required per iteration by LMS for an $M-$th order filter.

1.1. The Weight Drift and Bias Problems

In this section, we demonstrate the weight drift problem, and the performance degradation incurred by leaky LMS, by constructing a particular example.

To begin with, recall that the standard LMS recursion is given by

$$\mathbf{w}(k) = \mathbf{w}(k-1) + \mu(k)\mathbf{x}(k)\epsilon(k) \, , \tag{1}$$

$$\epsilon(k) = y(k) - \mathbf{x}(k)^T\mathbf{w}(k-1) \, , \tag{2}$$

where $\mathbf{w}(k)$ is the weight vector estimate, $\mathbf{x}(k)$ is the regressor vector (assumed to be a bounded sequence throughout this paper), $\mu(k)$ is the positive step-size parameter, $y(k)$ is the measurement variable, and $\epsilon(k)$ is the output estimation error. The measurements $\{y(k)\}$ are assumed to arise from a linear model of the form

$$y(k) = \mathbf{x}(k)^T\mathbf{w} + v(k) \tag{3}$$

where $\mathbf{w}$ is the true weight vector that we wish to estimate, and $v(k)$ is a bounded noise sequence.

The weight drift problem of LMS can be seen from the following contrived example. Assume, at each time instant k, the regression vector $\mathbf{x}(k)$ is orthogonal to the weight error vector $\tilde{\mathbf{w}}(k-1) = \mathbf{w} - \mathbf{w}(k-1)$. It then follows that $y(k) - \mathbf{x}(k)^T\mathbf{w}(k-1) = v(k)$. Consequently, the weight error vector satisfies the update equation $\tilde{\mathbf{w}}(k) = \tilde{\mathbf{w}}(k-1) + \mu(k)\mathbf{x}(k)v(k)$, and taking norms,

$$\|\tilde{\mathbf{w}}(k)\|_2^2 = \|\tilde{\mathbf{w}}(k-1)\|_2^2 + \mu^2(k)\|\mathbf{x}(k)\|_2^2 v^2(k).$$

Solving this recursion for $\|\tilde{\mathbf{w}}(N)\|_2^2$, we get

$$\|\tilde{\mathbf{w}}(N)\|_2^2 = \sum_{k=1}^{N} \mu^2(k)\|\mathbf{x}(k)\|_2^2 v^2(k) + \|\tilde{\mathbf{w}}(0)\|_2^2 \, .$$

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Algorithm	Stability	Convergence for l_p-noise and PE	Complexity	Condition on step-size	Rate of convergence (with same step-size)
LMS	NO	YES	O($4M$)	$\beta < 2$	—
Leaky LMS	YES	NO	O($5M$)	$\beta < 2 - 2\alpha_0$	does not converge
Switching-σ	YES	YES	O($6M$)	$\beta < 2 - 6\alpha_0$	comparable to LMS
Max-Tap Leaky	YES	YES	O($5M$)	$\beta < 2 - \frac{20}{3}\alpha_0$	comparable to LMS
Circular Leaky	YES	YES	O($4M$)	$\beta < \frac{1-\alpha_0}{\sqrt{MM}}$	comparable to LMS

Table 1. Comparison of various algorithms (the conditions for stability are for infinite-precision arithmetic; for finite-precision results, see Section 5).

This relation shows that $\|\tilde{\mathbf{w}}(N)\|_2^2 \rightarrow \infty$ with N, if $\mu(k)\|\mathbf{x}(k)\|_2 v(k)$ is not a finite-energy sequence.

This situation does not occur when leaky-LMS is used. This is because the leaky-LMS algorithm employs a leakage parameter $0 < \alpha < 1$ (see, e.g., [1, 2, 10, 12]):

$$\mathbf{w}(k) = (1-\alpha)\mathbf{w}(k-1) + \mu(k)\mathbf{x}(k)\epsilon(k) \tag{4}$$

In this case, and for the same example, we obtain

$$\|\tilde{\mathbf{w}}(k)\|_2^2 = (1-\alpha)\|\tilde{\mathbf{w}}(k-1)\|_2^2 + \mu^2(k)\|\mathbf{x}(k)\|_2^2 v^2(k)\ ,$$

so $\|\tilde{\mathbf{w}}(k)\|_2^2$ remains bounded for $0 < \alpha < 1$.

More generally, by a small modification of the arguments of this paper (and also the arguments of [6]), the following fact can be established. Assume the sequence $\mu(k)\mathbf{x}(k)^T\mathbf{x}(k)$ has a finite upper bound β, viz.,

$$\beta = \sup_{k\geq 0}\left\{\mu(k)\mathbf{x}(k)^T\mathbf{x}(k)\right\}. \tag{5}$$

Lemma 1 *For the leaky-LMS algorithm, if $\beta < 2-2\alpha$ then $\|\tilde{\mathbf{w}}(k)\|_2$ remains bounded for a bounded noise sequence $v(k)$.*

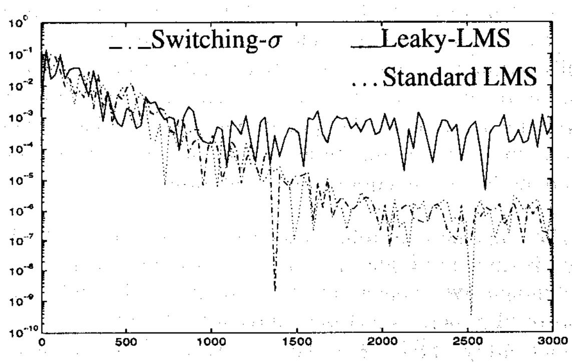

Figure 1. Learning curves ($\epsilon(k)$) for LMS, leaky-LMS and switching-σ algorithms. Adaptive equalization example with gaussian noise, averaged over 100 runs. In all cases $\frac{\mu(k)}{\mathbf{x}(k)^T\mathbf{x}(k)} = 0.03$, $\alpha_0 = 5\cdot 10^{-5}$, and $M = 7$.

However, the leaky-LMS algorithm adds a bias to the solution, which can be seen as follows. The corresponding weight-error equation is

$$\begin{aligned}\tilde{\mathbf{w}}(k) &= \left((1-\alpha)I - \mu(k)\mathbf{x}(k)\mathbf{x}(k)^T\right)\tilde{\mathbf{w}}(k-1)\\ &+ \alpha\mu(k)\mathbf{w} - \mu(k)\mathbf{x}(k)v(k)\end{aligned} \tag{6}$$

Assuming zero noise and infinite precision arithmetic, the fixed point of the difference equation (6) must satisfy

$$\left(\alpha I + \mu(k)\mathbf{x}(k)\mathbf{x}(k)^T\right)\tilde{\mathbf{w}} = \alpha\mathbf{w} \tag{7}$$

for all k, which clearly has no solution in general (except for very specific $\mathbf{x}(k)$) when α is nonzero. This means that for the leaky-LMS algorithm, the weight error $\tilde{\mathbf{w}}(k-1)$ will not converge to zero, even under persistence of excitation. Only when α is zero does (6) have a fixed-point at $\tilde{\mathbf{w}} = 0$.

The first two lines of Table 1 compare the performances of LMS and leaky LMS. For an $M-$th order adaptive filter, the computational cost increases from $O(4M)$ to $O(5M)$ for leaky-LMS. Moreover, the simulation in Fig. 1 shows the degradation in performance due to the bias problem.

1.2. The Switching-σ Algorithm

The fact that the leakage term in (4) is only necessary when $\mathbf{w}(k)$ becomes too large suggests that we replace the constant leakage parameter α by a function $\alpha_s(\mathbf{w}(k))$ such that $\alpha_s(\mathbf{0}) = 0$ and $\alpha_s(\mathbf{w}(k)) = \alpha_0$ when $\|\mathbf{w}(k)\|_2$ is too large.

In fact, a solution of this kind to the weight drift and bias problems has been suggested in the adaptive control literature and is known as the switching-σ algorithm [6]. It employs a time-variant leakage parameter that is defined as a function of the weight estimate:

$$\mathbf{w}(k) = [1-\alpha_s(\mathbf{w}(k-1))]\mathbf{w}(k-1) + \mu(k)\mathbf{x}(k)\epsilon(k)\ , \tag{8}$$

where α_s is a function of $\mathbf{w}(k-1)$ defined as follows:

$$\alpha_s(\mathbf{w}(k)) = \begin{cases}\alpha_0 & \text{if } \|\mathbf{w}(k)\|_2 > \Omega_s\\ 0 & \text{otherwise}\end{cases} \tag{9}$$

for a bound Ω_s that must be larger than $\|\mathbf{w}\|_2$ to guarantee $\alpha_s(\mathbf{w}(k)) = 0$ when $\mathbf{w}(k) = \mathbf{w}$. The following result can be established by a modification of the arguments of [6]. It assures bounded and unbiased estimates.

Lemma 2 *For the switching-σ algorithm, if $\beta < 2-6\alpha$ and $\Omega_s > \|\mathbf{w}\|_2$, then $\|\tilde{\mathbf{w}}(k)\|_2$ remains bounded for a bounded noise sequence $v(k)$. Moreover, in the noiseless case and with persistently exciting regressors, $\tilde{\mathbf{w}}(k) \to 0$.*

Moreover, the arguments of the current paper also can be used to establish other properties of the switching-σ algorithm, viz., that it also provides bounded weight estimates in finite precision implementations and that it guarantees convergence of the weight estimates to the true weight vector for PE regressors and noise in l_p. We shall not establish these facts here, but rather focus on the new modifications that we propose. The arguments for these new algorithms are more involved and they can be specialized for the case of the switching-σ algorithm.

In any case, note that while the switching-σ modification solves the weight drift and bias problems (see Fig. 1 for a comparison of the performances of standard LMS, leaky-LMS, and switching-σ algorithms), it does not solve the increased computational (hardware) cost problem. The third line of Table 1 shows that the cost per iteration for switching-σ is $O(6M)$.

2. New Leaky LMS Algorithms

In this section we propose new leakage algorithms, which we refer to as circular leaky and max-tap leaky LMS, with the following properties: i) they solve the weight drift problem, ii) they solve the bias problem, iii) they are stable in finite precision implementations, iv) they guarantee convergence of the weight estimates to the true weight vector for PE regressors and noise in l_p (i.e., $\sum_0^\infty |v(k)|^p < \infty$ for any $1 < p < \infty$), v.1) circular leaky has the same computational requirements as LMS (i.e., $O(4M)$ computations per iteration), and v.2) max-tap leaky has the same computational requirements as leaky-LMS ($O(5M)$ computations per iteration).

2.1. Circular Leaky LMS Algorithm

The circular version operates as follows: it first updates the weight estimate through a standard LMS recursion and obtains an intermediate weight vector estimate $\mathbf{q}(k)$, say

$$\mathbf{q}(k) = \mathbf{w}(k-1) + \mu(k)\mathbf{x}(k)\epsilon(k)\ . \tag{10}$$

It then modifies a single entry of $\mathbf{q}(k)$ to obtain $\mathbf{w}(k)$. The choice of which entry to modify is done sequentially. The algorithm starts by modifying the top entry of $\mathbf{q}(0)$, followed by the second entry of $\mathbf{q}(1)$ and so on, until the last entry of $\mathbf{q}(M-1)$, at which point the algorithm returns to the top entry of $\mathbf{q}(M)$ and repeats the process. This can be described as follows:

$$\mathbf{w}(k) = [I - \alpha_s(\mathbf{w}(k-1))\mathbf{e}_{\bar{k}}\mathbf{e}_{\bar{k}}^T]\mathbf{w}(k-1) + \mu(k)\mathbf{x}(k)\epsilon(k)\ , \tag{11}$$

where $\bar{k} = k \bmod M$ (the remainder of k/M) and

$$\alpha_s(\mathbf{w}(k)) = \begin{cases} \alpha_0 & \text{if } |e_{\bar{k}}^T\mathbf{w}(k)| > \Omega \\ 0 & \text{otherwise,} \end{cases} \tag{12}$$

$\mathbf{e}_{\bar{k}}$ is the $\bar{k}$-th canonical basis vector, and Ω is a positive constant whose value will be given later. We note from the above description that the leakage term is applied to one entry in the weight estimate at a time. Moreover, this process is repeated circularly.

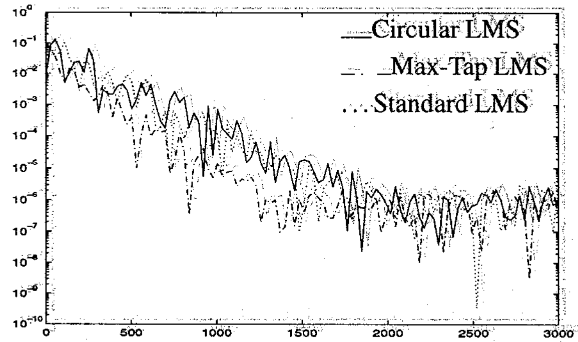

Figure 2. Learning curves ($\epsilon(k)$) for LMS, Circular Leaky and Max-Tap Leaky. Adaptive equalization example with gaussian noise, averaged over 100 runs. In all cases $\frac{\mu(k)}{\mathbf{x}(k)^T\mathbf{x}(k)} = 0.03$, $\alpha_0 = 5 \cdot 10^{-5}$, and $M = 7$.

2.2. Max-Tap Leaky LMS Algorithm

The Max-Tap version applies the leakage correction to the tap entry that corresponds to $\|\mathbf{w}\|_\infty$, i.e. to the largest entry in magnitude. The update form in this case can be written as follows. Let $\mathbf{e}_{\max}$ be the basis vector with a one at the position of the largest entry in magnitude of $\mathbf{w}(k-1)$. Then

$$\mathbf{w}(k) = [I - \alpha_s(\mathbf{w}(k-1))\mathbf{e}_{\max}\mathbf{e}_{\max}^T]\mathbf{w}(k-1) + \mu(k)\mathbf{x}(k)\epsilon(k)\ , \tag{13}$$

where

$$\alpha_s(\mathbf{w}(k)) = \begin{cases} \alpha_0 & \text{if } \|\mathbf{w}(k)\|_\infty > \Omega_1 \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

and Ω_1 is a positive constant.

2.3. Comparison with LMS

The last two lines of Table 1 compare the new leaky algorithms (circular and max-tap) to the standard LMS algorithm. We see that the circular version has the same computational cost, while max-tap requires $O(5M)$ computations per iteration. Moreover, the performances of both algorithms are comparable to LMS as shown in Fig. 2. We should add though that the condition listed in Table 1 for the step-size of circular leaky, viz., $\mu(k)\|\mathbf{x}(k)\|_2^2 < (1-\alpha_0)/M^{1.5}$, is in terms of the filter size M.

3. Stability in Infinite-Precision Arithmetic

In the remaining parts of this paper, we outline the main arguments that establish the properties mentioned earlier about the modified leaky LMS algorithms.

The derivation relies on the Lyapunov theory for the stability of nonlinear difference equations. It involves some tedious calculations that we omit for obvious reasons of brevity.

To study the stability of (11), we define the Lyapunov function candidate $V(k) = V(\tilde{\mathbf{w}}(k)) = \tilde{\mathbf{w}}(k)^T\tilde{\mathbf{w}}(k)$ and compute its one-step difference $\Delta V(k) = V(k) - V(k-1)$.

Theorem 1 *Under bounded noise $v(k)$ and in infinite-precision arithmetic, the circular leaky LMS algorithm (11) guarantees bounded weight estimates $\mathbf{w}(k)$ if the following conditions are satisfied:*

$$\alpha_0 + \sqrt{M}M\beta < 1 \tag{15}$$

$$\tfrac{20}{3}\alpha_0 + \beta < 2 \tag{16}$$

$$\Omega > 3\|\mathbf{w}\|_\infty \tag{17}$$

Proof: Using (11), (17), and applying Young's inequality [7], we get after some algebra

$$V(k+M-1) - V(k-1) \leq$$

$$-c_1 \sum_{n=0}^{M-1} \alpha_s(\mathbf{w}(k+n))\tilde{w}_{k+n}^2(k+n) + c_2 \sum_{n=0}^{M-1} v^2(k+n+1) \,,$$

where $c_2 = \sup_{k\geq 0}\{\mu(k)\}\,[\beta + \frac{1}{K}(1+2\alpha_0\beta)]$,

$$c_1 = \frac{3}{8}[2 - \beta - \frac{20}{3}\alpha_0 - \frac{10}{3}K] \,, \tag{18}$$

and K is any positive number. Under the conditions stated in the body of the theorem, we can show that there exists a finite $V_0 > 0$ such that whenever $V(k-1)$ exceeds V_0, the difference $V(k+M-1) - V(k-1)$ will be negative. The details are omitted.

■

A similar result holds for the max-tap leaky LMS.

Theorem 2 *Under bounded noise $v(k)$ and in infinite-precision arithmetic, the max-tap leaky LMS algorithm (13) guarantees bounded weight estimates $\mathbf{w}(k)$ if the following conditions are satisfied:*

$$\tfrac{20}{3}\alpha_0 + \beta < 2 \tag{19}$$

$$\Omega_1 > 3\|\mathbf{w}\|_\infty \tag{20}$$

4. Convergence for l_p-Noise

In this section we assume that the regression vectors $\mathbf{x}(k)$ are persistently exciting (PE) and prove that, under this condition, the origin of the error equation of the circular leaky-LMS algorithm is exponentially stable, viz., the origin of the equation

$$\tilde{\mathbf{w}}(k) = [I - \alpha_s \mathbf{e}_{\bar{k}}\mathbf{e}_{\bar{k}}^T - \mu(k)\mathbf{x}(k)\mathbf{x}(k)^T]\tilde{\mathbf{w}}(k-1) +$$

$$+\alpha_s \mathbf{e}_{\bar{k}}\mathbf{e}_{\bar{k}}^T\mathbf{w} - \mu(k)\mathbf{x}(k)v(k) \,. \tag{21}$$

Using this result, we can establish the convergence of the algorithm under l_p noise for any $1 < p < \infty$. [With minor alterations of the arguments, the results also hold for max-tap and switching-σ LMS algorithms].

Theorem 3 *Assume the regression vectors $\mathbf{x}(k)$ are PE, and conditions (16) and (17) are satisfied. It then follows, in the noiseless case, that the origin of (21) is exponentially stable. Moreover, when $v(k)$ is present, the weight error vector satisfies a contractive relation of the form*

$$\frac{\alpha_1\|\tilde{\mathbf{w}}(N)\|_2^p + \gamma_1\sum_{k=0}^{N-1}\|\tilde{\mathbf{w}}(k)\|_2^p}{\alpha_2\|\tilde{\mathbf{w}}(0)\|_2^p + \gamma_2\sum_{k=0}^{N-1}|v(k+1)|^p} \leq 1 \,, \tag{22}$$

for any $1 < p < \infty$, and for some positive finite constants $\{\alpha_1, \alpha_2, \gamma_1, \gamma_2\}$ that depend on p. In particular, if $v \in l_p$, for some p, then the weight error converges to zero.

Proof: A brief outline of the proof is the following. Consider the standard LMS algorithm (1) with $v(k) = 0$. In this case, it is known that $\tilde{\mathbf{w}} = \mathbf{0}$ is an exponentially stable equilibrium point under a PE condition and if $\beta < 2$ [9]. It then follows that there should exist a Lyapunov function $U(k, \tilde{\mathbf{w}}(k))$ and positive constants A, A_1, A_2, and B satisfying [11]

$$A_1\|\tilde{\mathbf{w}}(k)\|_2^2 \leq U(k, \tilde{\mathbf{w}}(k)) \leq A_2\|\tilde{\mathbf{w}}(k)\|_2^2$$

$$U(k, \tilde{\mathbf{w}}(k)) - U(k-1, \tilde{\mathbf{w}}(k-1)) \leq -A\|\tilde{\mathbf{w}}(k-1)\|_2^2$$

$$\left\|\frac{\partial U}{\partial \tilde{\mathbf{w}}}(k, \tilde{\mathbf{w}}(k))\right\|_2 \leq B\|\tilde{\mathbf{w}}(k)\|_2$$

where $\tilde{\mathbf{w}}(k)$ is obtained from $\tilde{\mathbf{w}}(k-1)$ using the LMS recursion (1). This Lyapunov function can be used to establish the result of the theorem.

■

We may add that a contractive relation of the form (22) can be established for LMS for the special case $p = 2$ without the additional requirement of PE regressors [4, 8].

5. Stability in Finite-Precision Arithmetic

The stability results of the circular, max-tap, switching-σ, and leaky LMS algorithms can be extended to the finite-precision arithmetic case.

We argue here for the circular leaky LMS case. Numerical errors will modify the error equation (21). In particular, to compute the new weight vector using (11) we first find

$$\bar{\epsilon}(k) = \mathrm{fl}\left(y(k) - \mathbf{x}(k)^T\mathbf{w}(k-1)\right). \qquad (23)$$

From [3], the above expression can be evaluated as

$$\bar{\epsilon}(k) = \left(y(k) - \mathbf{x}(k)^T\mathbf{w}(k-1) + \delta_1' \|\mathbf{x}(n)\| \|\mathbf{w}(n-1)\|\right)(1+\delta_2')$$

where $|\delta_1'| < 1.01M\varepsilon$, $|\delta_2'| < \varepsilon$, and ε is the machine precision. Similarly, let $\mathbf{z}(k)$ denote the computed value of $\mathbf{w}(k)$, viz.,

$$\mathbf{z}(k) = \mathrm{fl}\left[\left(I - \mathbf{e}_{\bar{k}}\mathbf{e}_{\bar{k}}^T\alpha_s\right)\mathbf{z}(k-1) + \mu(k)\bar{\epsilon}(k)\mathbf{x}(k)\right].$$

This equation describes another (nonlinear) dynamic system. The floating point arithmetic introduces some difficulties into the stability analysis, but it still can be performed leading to the following conclusion.

Define

$$\delta = [42 + (8.2 + 2.04 \sup_k \mu(k))M]\varepsilon, \ \kappa = [3.03 + 1.02M\beta]\varepsilon$$

$$\Gamma(M-1) = \frac{(1+\kappa)^M - 1}{\kappa}, \ \Gamma_1(M) = 2\frac{(1+\kappa)^{2M} - (1+\kappa)^2}{2\kappa + \kappa^2}$$

Note that for $M\varepsilon \ll 1$, we have $\Gamma(M-1) \approx M$ and $\Gamma_1(M) \approx 2M$.

Theorem 4 *The circular leaky LMS algorithm (11) still guarantees bounded weight estimate vectors $\mathbf{w}(k)$ under bounded noise and finite-precision arithmetic if, in addition to the conditions of Theorem 1, the following condition is satisfied:*

$$\frac{1-\alpha_0}{\sqrt{M}} - \beta\Gamma(M-1) - \left(\frac{\Gamma_1(M)\delta}{\alpha_0 c_1}\right)^{\frac{1}{2}} > 0, \qquad (24)$$

where, from (18), $c_1 \approx \frac{3}{8}[1 - \beta - \frac{20}{3}\alpha_0]$.

A similar conclusion holds for max-tap leaky LMS where instead of (24) we require

$$2 - \frac{20}{3}\alpha_0 - \beta - \frac{M\delta}{\alpha_0} > 0. \qquad (25)$$

Note that in finite-precision arithmetic, the conditions for stability show that α_0 should not be too small. This is also true for the switching-σ and the leaky-LMS algorithms, in which cases the condition will now read $2 - 6\alpha_0 - \beta - \frac{\delta}{\alpha_0} > 0$.

6. Conclusions

We have shown that it is possible to solve the weight drift and bias problems of LMS and leaky-LMS at an additional cost of only one comparison, one addition, and one multiplication per iteration.

We have also shown how the stepsize $\mu(k)$ and the leakage parameter α_0 should be chosen to guarantee stability in both finite and infinite-precision arithmetic.

In particular, our analysis for the finite-precision case suggests that choosing a very small leakage factor in the leaky-LMS algorithm with the intent of reducing the bias could in fact lead to instability.

We should note that the conditions on the step-size for the circular leaky LMS depend on the filter order M; hence the larger the filter order the smaller the μ.

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