A Robust Minimum-Variance Filter for Time Varying Uncertain Discrete-Time Systems

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Abstract—**A robust filter is designed for uncertain discrete time models. The filter is based on a regularized solution and guarantees minimum state error variance. Simulation results confirm its superior performance over other robust filter designs.**

keywords: regularization, least-squares, robust filter, regularization parameter, parametric uncertainty.

I. Introduction

Most robust filtering designs perform de-regularization and involve existence conditions (see, e.g., [1]-[4]). This property can be problematic for real time/online filtering operations on time varying models because it entails checking the conditions on a regular basis. When the conditions fail, the filter robustness is lost. In [5], a robust filter was proposed that performs regularization as opposed to de-regularization. The design procedure in [5] involved choosing a certain Ricatti variable so as to enforce a local optimality property. In this paper, we show how to determine the weighting regularization matrices in order to minimize the state error covariance matrix globally. Simulation results are included to illustrate the performance of the proposed filter in comparison to other robust designs.

II. Least-Squares with Uncertainties

Many estimation and control problems rely on solving regularized least-squares problems of the form (see, e.g., [6]):

$$\min_{x} \left[x^TQx + (Ax-b)^TW(Ax-b)\right] \tag{1}$$

where x^TQx is a regularization term, $Q > 0$ and $W \geq 0$ are Hermitian weighting matrices, x is an unknown $n-$dimensional column vector, A is a known $N \times n$ data matrix, and b is a known $N \times 1$ measurement vector. The solution of (1) is

$$\hat{x} = \left[Q + A^TWA\right]^{-1} A^TWb \tag{2}$$

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When the nominal data $\{A, b\}$ are subject to uncertainties, especially large ones, the performance of the optimal estimator (2) can degrade appreciably. In [8], a generalization of (1) that accounts for uncertainties in $\{A, b\}$ was introduced. Let $J(x, y)$ denote a cost function of the form $J(x,y) = x^TQx + R(x,y)$ with

$$R(x,y) = \Big((A+\delta A)x-(b+\delta b)\Big)^T W\Big((A+\delta A)x-(b+\delta b)\Big) \tag{3}$$

where δA denotes an $N \times n$ perturbation to A, δb denotes an $N \times 1$ perturbation to b, and $\{\delta A, \delta b\}$ are assumed to satisfy a model of the form

$$\begin{bmatrix} \delta A & \delta b \end{bmatrix} = H\Delta \begin{bmatrix} E_a & E_b \end{bmatrix} \tag{4}$$

where Δ is an arbitrary contraction, $\|\Delta\| \leq 1$, and $\{H, E_a, E_b\}$ are known quantities of appropriate dimensions. Consider then the constrained two player game problem

$$\hat{x} = \arg\min_{x} \max_{\{\delta A, \delta b\}} \; J(x,y) \tag{5}$$

subject to (4). The following result is proven in [8].

Theorem 1: The problem (4)–(5) has a unique solution $\hat{x}$ that is given by

$$\hat{x} = \left[\widehat{Q} + A^T\widehat{W}A\right]^{-1}\left[A^T\widehat{W}b + \widehat{\beta}E_a^TE_b\right] \tag{6}$$

where

$$\widehat{Q} \triangleq Q + \widehat{\beta}E_a^TE_a \tag{7}$$

$$\widehat{W} \triangleq W + WH(\widehat{\beta}I - H^TWH)^{\dagger}H^TW \tag{8}$$

and the scalar $\widehat{\beta}$ is determined from the optimization

$$\widehat{\beta} = \arg \min_{\beta \geq \|H^TWH\|} G(\beta) \tag{9}$$

where the function $G(\beta)$ is defined as follows:

$$G(\beta) \triangleq x^T(\beta)Qx(\beta) + \beta\|E_ax(\beta) - E_b\|^2 + [Ax(\beta) - b]^TW(\beta)[Ax(\beta) - b] \tag{10}$$

with

$$W(\beta) \triangleq W + WH(\beta I - H^TWH)^{\dagger}H^TW \tag{11}$$

$$Q(\beta) \triangleq Q + \beta E_a^TE_a \tag{12}$$

and

$$x(\beta) \triangleq \left[Q(\beta)+A^T W(\beta) A\right]^{-1} \left[A^T W(\beta) b + \beta E_a^T E_b\right] \quad (13)$$

[The notation $X^\dagger$ denotes the pseudo-inverse of X.]

$\diamond$

It is shown in [8], [9] that the function $G(\beta)$ has a unique global minimum (and no local minima) inside this interval, which means that the determination of $\widehat{\beta}$ can be pursued by employing standard search procedures without worrying about convergence to undesired local minima. Actually, it was argued in [5] that a reasonable approximation for $\widehat{\beta}$ is to choose it as $\widehat{\beta} = (1+\alpha)\|H^T W H\|$ for some $\alpha > 0$.

III. The Data Model

Now consider an $n-$dimensional state-space model of the form :

$$x_{k+1} = F_k x_k + G_k w_k \quad (14)$$
$$y_k = (H_k + \Delta H_k) x_k + v_k, \quad k \geq 0 \quad (15)$$

where $\{w_k, v_k\}$ are uncorrelated white zero-mean random processes with variances

$$E w_k w_k^* = W_k, \qquad E v_k v_k^* = V_k$$

and x_0 is a zero-mean random variable that is uncorrelated with $\{w_k, v_k\}$ for all k. The uncertainties ΔH_k are modelled as

$$\Delta H_k = M_k \Delta_k E_k \quad (16)$$

where M_k and E_k are known matrices, while Δ_k is an arbitrary contraction, $\Delta_k^T \Delta_k < I$.

We consider two types of uncertainty descriptions for the state matrices F_k: one is in terms of polytopic uncertainties and the other is in terms of norm bounded uncertainties. In the first case, we assume that the F_k lie inside convex bounded polyhedral domains $\mathcal{K}_k$ described by p vertices as follows:

$$\mathcal{K}_k = \left\{ F_k = \sum_{i=1}^{i=p} \alpha_{i,k} F_{i,k}, \;\; \alpha_{i,k} \geq 0, \;\; \sum_{i=1}^{i=p} \alpha_{i,k} = 1 \right\} \quad (17)$$

In the second case, we assume that the F_k are described by

$$F_k = F_k^0 + \Delta F_k, \quad \Delta F_k = N_k \Delta_k J_k \quad (18)$$

for some known F_k^0.

IV. Robust State Space Filtering

When uncertainties are not present in the model (14)–(15), it is known that the optimal linear estimator for the state vector is the Kalman filter [10]. This filter admits a deterministic interpretation as the solution to a regularized least-squares problem as follows. Let

$$\hat{x}_{k|k-1} \triangleq \text{an estimate of } x_k \text{ given } \{y_0, y_1, \ldots, y_{k-1}\}$$
$$\hat{x}_{k|k} \triangleq \text{an estimate of } x_k \text{ given } \{y_0, y_1, \ldots, y_{k-1}, y_k\}$$

Given the predicted estimate $\hat{x}_{k|k-1}$ and an observation y_k, the filtered estimate $\hat{x}_{k|k}$ computed by the Kalman filter is the solution of

$$\min_x \left[\|x - \hat{x}_{k|k-1}\|^2_{P_k^{-1}} + \|y_k - H_k x\|^2_{R_k^{-1}} \right] \quad (19)$$

where P_k and R_k are the state-error and the measurement noise covariance matrices, respectively. When uncertainties are present in $\{H_k, F_k\}$, we could formulate a robust version of (19), by solving instead a min-max problem of the form:

$$\min_x \max_{\delta H_k, \delta F_k} \left(\|x - \hat{x}_{k|k-1}\|^2_{P_k^{-1}} + \|y_k - (H_k + \delta H_k) x\|^2_{R_k^{-1}} \right) \quad (20)$$

This formulation was proposed in [5]. Compared with other robust designs it has the advantage of performing regularization as opposed to de-regularization. This property is useful for on-line/real-time opertaion. In [5], however, the weighing matrices P_k in (19) were determined through Ricatti equations that enforce a local optimality criterion. In the sequel, we shall determine P_k to minimize the state error covariance matrix *globally*. We do so by re-parametrizing P_k and R_k in terms of a single parameter Q_k, over which the global minimization of the error covariance is shown to reduce to a linear convex problem.

A. Polytopic Uncertainties

We consider first the case of polytopic uncertainties in F_k as in (17). Our objective is then to design a robust linear estimator for the state variable x_k of the form

$$\hat{x}_{k|k} = F_{p,k} \hat{x}_{k|k-1} + K_{p,k} y_k, \quad k \geq 0 \quad (21)$$
$$\hat{x}_{k+1|k} = F_k^0 \hat{x}_{k|k} \quad (22)$$

for some matrices $F_{p,k}$ and $K_{p,k}$ to be determined in order to minimize the worst case error variance of the state for all uncertainties, and where F_k^0 denotes the centroid of the polytope $\mathcal{K}_k$:

$$F_k^0 = \frac{1}{p} \sum_{i=1}^{i=p} F_{i,k} \quad (23)$$

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Assume first that the F_k are fixed; we will incorporate the uncertainties in F_k soon. With uncertainties in the output matrices H_k alone, problem (20) becomes

$$\min_{x} \max_{\delta H_k} \left(\|x - \hat{x}_{k|k-1}\|^2_{P_k^{-1}} + \|y_k - (H_k + \delta H_k)x\|^2_{R_k^{-1}} \right) \quad (24)$$

which can be written more compactly in the form (3)–(5) with the identifications:

$$\begin{array}{rcl} x & \longleftarrow & \{x_k - \hat{x}_{k|k-1}\}, \quad b \longleftarrow y_k - H_k\hat{x}_{k|k-1} \\ \delta A & \longleftarrow & M_k \Delta_k E_k \\ \delta b & \longleftarrow & -M_k \Delta_k E_k \hat{x}_{k|k-1}, \; Q \longleftarrow P_k^{-1} \\ W & \longleftarrow & R_k^{-1}, \; H \longleftarrow M_k, \; E_a \longleftarrow E_k \\ E_b & \longleftarrow & -E_k \hat{x}_{k|k-1}, \; \Delta \longleftarrow \Delta_k, \; A \longleftarrow H_k \end{array}$$

From Theorem 1, the solution $\hat{x}_{k|k}$ of (24) is given by

$$\begin{aligned} \hat{x}_{k|k} = \hat{x}_{k|k-1} + &(P_k^{-1} + \hat{\beta} E_k^T E_k + H_k^T \hat{R}_k^{-1} H_k)^{-1} \\ &\{H_k^T \hat{R}_k^{-1}(y_k - H_k \hat{x}_{k|k-1}) - \hat{\beta} E_k^T E_k \hat{x}_{k|k-1}\} \end{aligned} \quad (25)$$

where

$$\hat{R}_k^{-1} = (R_k - \hat{\beta}^{-1} M_k M_k^T)^{-1} \quad (26)$$

If we now define Q_k as

$$Q_k \triangleq (P_k^{-1} + \hat{\beta} E_k^T E_k + H_k^T \hat{R}_k^{-1} H_k)^{-1} \quad (27)$$

then the expression for $\hat{x}_{k|k}$ becomes

$$\begin{aligned} \hat{x}_{k|k} = &(I - Q_k \beta E_k^T E_k - Q_k H_k^T \hat{R}_k^{-1} H_k)\hat{x}_{k|k-1} \\ &+ Q_k H_k^T \hat{R}_k^{-1} y_k \end{aligned} \quad (28)$$

in terms of the free parameter Q_k.

Noting that w_k is a zero-mean white random process, we let the following be an estimate of x_{k+1} given the measurement y_k:

$$\hat{x}_{k+1|k} \triangleq F_k^0 \hat{x}_{k|k} \quad (29)$$

We then get

$$\hat{x}_{k+1|k} = F_{p,k}\hat{x}_{k|k-1} + K_{p,k} y_k \quad (30)$$

where $F_{p,k}$ and $K_{p,k}$ are defined as

$$\begin{aligned} F_{p,k} &= F_k^0(I - Q_k \beta E_k^T E_k - Q_k H_k^T \hat{R}_k^{-1} H_k) \quad (31) \\ K_{p,k} &= F_k^0 Q_k H_k^T \hat{R}_k^{-1} \quad (32) \end{aligned}$$

Denoting $\tilde{x}_k = x_k - \hat{x}_{k|k-1}$, we define the extended weight vector

$$\eta_k \triangleq \begin{pmatrix} x_k \\ \tilde{x}_k \end{pmatrix} \quad (33)$$

Then η_k satisfies

$$\eta_{k+1} = \bar{F}_k \eta_k + \bar{G}_k \mu_k \quad (34)$$

where

$$\mu_k = \begin{pmatrix} w_k \\ v_k \end{pmatrix} \quad (35)$$

$$\bar{F}_k = \begin{pmatrix} F_k & 0 \\ F_k - F_{p,k} - K_{p,k} H_k & F_{p,k} \end{pmatrix} \quad (36)$$

$$\bar{G} = \begin{pmatrix} G & 0 \\ G & -K_{p,k} \end{pmatrix} \quad (37)$$

and the covariance matrix of η_k satisfies

$$\Pi_{k+1} = \bar{F}_k \Pi_k \bar{F}_k^T + \bar{G}_k S \bar{G}_k^T \quad (38)$$

where

$$S = \begin{pmatrix} W_k & 0 \\ 0 & V_k \end{pmatrix} \quad (39)$$

and Π_0 is the covariance of η_0.

Observe that the expressions for $\{F_{p,k}, K_{p,k}\}$ are parametrized in terms of the free parameter Q_k. We shall choose Q_k so as to minimize the covariance of η_k. In this way, the resulting filter will satisfy the robustness condition (24) in addition to minimizing the state error variance. This is achieved as follows. First note that Q_k in (27) is to problem (24) as the matrix $\widehat{Q} + A^T \widehat{W} A$ in (6) is to problem (3)–(5). Therefore, Q_k must be positive definite matrices so that the $\hat{x}_{k|k}$ are guaranteed to be minima of (24). Then we shall choose $Q_k > 0$ so as to minimize Π_{k+1} of (40). This can be obtained by solving

$$\min_{Q_k > 0} \quad \text{Trace}(\Pi_{k+1})$$

subject to the following inequality

$$\Pi_{k+1} \geq \bar{F}_k \Pi_k \bar{F}_k^T + \bar{G}_k S \bar{G}_k^T \quad (40)$$

or, equivalently,

$$\begin{pmatrix} -\Pi_{k+1} & \bar{F}_k \Pi_k & \bar{G}_k S^{1/2} \\ \Pi_k \bar{F}_k^T & -\Pi_k & 0 \\ S^{T/2} \bar{G}_k^T & 0 & -I \end{pmatrix} \leq 0 \quad (41)$$

In order to incorporate the polytopic uncertainties in the F_k, as defined by the sets $\mathcal{K}_k$, we need to solve the above optimization problem with F_k taking values at the vertices of the convex polytope $\mathcal{K}_k$, i.e., from the set $\{F_{1,k}, F_{2,k}, \ldots\ldots, F_{p,k}\}$. Since the inequality (41) is affine in F_k, the Q_k thus found will ensure minimum error covariance Π_k over all possible F_k in $\mathcal{K}_k$. Therefore, the

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time varying robust filter is given by (30)–(32) where Q_k is the positive definite solution of (41) with F_k taking values on the vertices of the convex polytope $\mathcal{K}_k$, and $\Pi_0 = \text{diag}\{\hat{P}_o, \epsilon I\}$.

Remark : To guarantee mean square stability in the infinite horizon case for a stable stationary model with constant matrices $\{F, G, H\}$, we may proceed as follows. We assume $\|F\| < 1$ and choose Q_k to satisfy (41) as well as $\|\bar{F}_k\| < 1$. This additional constraint is easily represented in terms of a linear matrix inequality in the variable Q_k as

$$\begin{pmatrix} I & \bar{F}_k^T \\ \bar{F}_k & I \end{pmatrix} > 0 \qquad (42)$$

B. Norm Bounded Uncertainties

We now consider the case of norm bounded uncertainties in F_k as in (18) for a finite horizon operation. In this case the covariance matrix of the extended state vector η_k will satisfy

$$\Pi_{k+1} = (\hat{F}_k + \hat{N}_k \Delta \hat{J}_k)\Pi_k(\hat{F}_k + \hat{N}_k\Delta\hat{J}_k)^T + \bar{G}_k S \bar{G}_k^T \qquad (43)$$

where

$$\hat{F}_k = \begin{pmatrix} F_k^0 & 0 \\ F_k^0 - F_{p,k} - K_{p,k}H_k & F_{p,k} \end{pmatrix} \qquad (44)$$

$$\hat{N}_k = \begin{pmatrix} N_k & 0 \\ N_k & 0 \end{pmatrix} \qquad (45)$$

$$\hat{J}_k = \begin{pmatrix} J_k & 0 \\ 0 & 0 \end{pmatrix} \qquad (46)$$

Let ϵ_k be any positive scalar and let (α_k, σ_k) be such that $\alpha_k I - \hat{J}_k\Pi_k\hat{J}_k^T > 0$ and $\hat{F}^k\Pi_k\hat{J}_k^T\hat{J}_k\Pi_k\hat{F}_k^T < \sigma_k I$. Then

$$\begin{aligned} \Pi_{k+1} \leq\ & \hat{F}_k\Pi_k\hat{F}_k^T + \bar{G}_k S \bar{G}_k^T \\ & + \alpha_k \hat{N}_k\hat{N}_k^T + \hat{N}_k\Delta\hat{J}_k\Pi_k\hat{F}_k^T \\ & + \hat{F}^k\Pi_k\hat{J}_k^T\Delta^T\hat{N}_k^T \\ \leq\ & \hat{F}_k\Pi_k\hat{F}_k^T + \bar{G}_kS\bar{G}_k^T + (\alpha_k + \epsilon_k)\hat{N}_k\hat{N}_k^T \\ & + \epsilon_k^{-1}\hat{F}^k\Pi_k\hat{J}_k^T\hat{J}_k\Pi_k\hat{F}_k^T \\ \leq\ & \hat{F}_k\Pi_k\hat{F}_k^T + \bar{G}_kS\bar{G}_k^T + (\alpha_k + \epsilon_k)\hat{N}_k\hat{N}_k^T \\ & + \epsilon_k^{-1}\sigma_k I = \bar{\Pi}_{k+1} \end{aligned}$$

where in the second inequality we used the fact that for any real matrices $\{X, Y, J\}$ with $J^TJ \leq \mu I$, it holds for any scalar $\epsilon > 0$,

$$XJ^TY + Y^TJX^T \leq \epsilon^{-1}\mu XX^T + \epsilon Y^TY \qquad (47)$$

We have thus found an upper bound on Π_{k+1} that is independent of Δ. We can now choose $Q_k > 0$ so as to solve

$$\min_{Q_k > 0, \alpha_k, \sigma_k} \quad \text{Trace}(\bar{\Pi}_{k+1})$$

TABLE I

ERROR VARIANCE WITH UNCERTAINTIES IN F_k ALONE.

Filters	error variance
Proposed filter	150.9
filter of [5]	175.5
Guaranteed-cost filter [1]	503.4
Set-valued filter [2]	1606.7
Kalman filter with nominal model	2404

$$\begin{pmatrix} L & \hat{F}_k\bar{\Pi}_k & \bar{G}_k S^{1/2} \\ \bar{\Pi}_k\hat{F}_k^T & -\bar{\Pi}_k & 0 \\ S^{T/2}\bar{G}_k^T & 0 & -I \end{pmatrix} \leq 0 \qquad (48)$$

$$\begin{pmatrix} \sigma_k I & \hat{F}^k\Pi_k\hat{J}_k^T \\ \hat{J}_k\Pi_k\hat{F}_k^T & I \end{pmatrix} > 0 \qquad (49)$$

and

$$\alpha_k I - \hat{J}_k\Pi_k\hat{J}_k^T > 0 \qquad (50)$$

where $L = -\bar{\Pi}_{k+1} + (\alpha_k + \epsilon_k)\hat{N}_k\hat{N}_k^T + \epsilon_k^{-1}\sigma_k I$.

V. SIMULATIONS

To illustrate the developed filter, we choose an implementation of order 2 with $E_k = [.12 \quad .12]$, $M_k = 1$ for all k. The uncertain state matrices F_k are assumed to lie inside the convex polytope

$$F_k = \begin{pmatrix} .9802 & .0196 + \delta \\ 0 & .5802 + \delta \end{pmatrix} \qquad (51)$$

with $|\delta| \leq 0.4982$

Table 1 shows the average squared state-error values (averaged over 50 experiments) for the Kalman filter, the proposed filter, the set-valued estimation filter [2], the guaranteed cost filter [1], and the filter of [5].

When there are uncertainties in the output matrices H_k, in addition to those in F_k, the proposed filter outperforms the other filters (which actually have not been designed to deal with output uncertainties). In the above example, if the uncertainties in H_k are determined by $M_k = 1$ and $E_k = .4$, the robust filter performs better as shown in table 2.

VI. CONCLUSION

In this paper we developed a robust filter for state-space estimation. The design procedure is through the solution of a regularized weighted recursive least squares problem and enforces a minimum state error variance.

TABLE II

ERROR VARIANCE WITH UNCERTAINTIES IN F_k AND H_k.

Filters	error variance
Proposed filter	190.9
filter of [5]	500.5
Guaranteed-cost filter [1]	1000.4
Set-valued filter [2]	2800.7
Kalman filter with nominal model	3904.6

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