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Optimal State Regulation for Uncertain State-Space Models[1]

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Abstract

This paper studies the problem of state regulation for uncertain state-space models. It formulates a new weighted game-type cost function with bounds on the sizes of the uncertainties in the data. The cost function is of independent interest in its own right and its optimal solution is shown to satisfy an orthogonality condition similar to least-squares designs. When used in the context of state-space models, the solution leads to a control law with design equations that are similar in nature to LQR designs. The gain matrix, however, as well as the Riccati variable, turn out to be state-dependent in a certain way.

1 INTRODUCTION

This paper develops a technique for estimation and control purposes that is suitable for models with bounded data uncertainties. The technique will be referred to as a BDU design method for brevity. It is based on a constrained game-type formulation that allows the designer to explicitly incorporate into the problem statement a-priori information about bounds on the sizes of the uncertainties in the model. A key feature of the BDU formulation is that geometric insights (such as orthogonality conditions and projections), which are widely appreciated for classical quadratic-cost designs, can be pursued in this new framework (see [1]).

Consider the cost function

$$J(x) = x^T Q x + R(x) ,$$

where x^TQx is a regularization term, while the residual cost $R(x)$ is defined by (a more general case is treated in [2]):

$$R(x) \triangleq \Big((A+\delta A)x-(b+\delta b)\Big)^T W \Big((A+\delta A)x-(b+\delta b)\Big)$$

Here, $Q > 0$ and $W \geq 0$ are weighting matrices, x is an $n-$dimensional column vector, A is an $N \times n$ known or nominal matrix, b is an $N \times 1$ known or nominal vector, $\{\delta A\}$ denotes an $N \times n$ unknown perturbation matrix, and δb denotes an $N \times 1$ unknown perturbation vector. We then pose the problem of solving:

$$\hat{x} = \arg\min_{x} \max_{\substack{\|\delta A\| \leq \eta \\ \|\delta b\| \leq \eta_b}} J(x) , \tag{1}$$

where the notation $\|\cdot\|$ stands for the Euclidean norm of its vector argument or the maximum singular value of its matrix argument. In other words, we seek that solution $\hat{x}$ that performs "best" in a worst-possible scenario. The above problem can therefore be regarded as a constrained two-player game problem, with the designer trying to pick an $\hat{x}$ that minimizes the cost while the opponents $\{\delta A, \delta b\}$ try to maximize the cost. The game problem is constrained since it imposes a limit on how large (or how damaging) the opponents can be. The case $Q = 0$ and $W = I$, and variations thereof, were treated in detail in [1, 3, 4] with several applications in image processing, digital communications, and estimation in [1, 4]. It turns out that, unlike standard least-squares theory, solving a weighted problem of the form (1) is more complex (and also more rich) than solving the unweighted version (with $W = I$).

2 SOLUTION OF THE BDU PROBLEM

Note that for any given $\{\delta A, \delta b\}$, the residual cost $R(x)$ is convex in x. Therefore, the maximum

$$C(x) \triangleq \max_{\substack{\|\delta A\| \leq \eta \\ \|\delta b\| \leq \eta_b}} R(x) , \tag{2}$$

is a convex function in x. Now since x^TQx is strictly convex in x when $Q > 0$, we conclude that $x^TQx+C(x)$

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is strictly convex in x, which shows that problem (1) has a unique global minimum $\hat{x}$.[1]

To determine $\hat{x}$ we proceed in steps. We first show that the two variables $\{\delta A, \delta b\}$ in (2) can be replaced by a single variable y, which would therefore allow us to replace the maximization (2) over *two* constrained variables, by a maximization over a *single* constrained variable (see (4) below).

Indeed, for any fixed value of x, let $\mathcal{Z}_x$ denote the set of all vectors z that are generated as follows:

$$\mathcal{Z}_x = \{z : \ z = \delta A x - \delta b \ , \ \|\delta A\| \le \eta \ , \ \|\delta b\| \le \eta_b\} \ ,$$

for all possible $\{\delta A, \delta b\}$ within the prescribed bounds. Let also $\mathcal{Y}_x$ denote the set of all vectors y that are generated as follows:

$$\mathcal{Y}_x = \{y : \ \|y\| \le \eta\|x\| + \eta_b\} \ .$$

Then $\mathcal{Z}_x = \mathcal{Y}_x$. That is, if $z \in \mathcal{Z}_x$ then $z \in \mathcal{Y}_x$ (this direction is immediate and follows from the triangle inequality of norms). Conversely, if $y \in \mathcal{Y}_x$ then $y \in \mathcal{Z}_x$. To establish the result for $x \neq 0$, define for a given y the perturbations:

$$\delta A(y) = \frac{\eta}{\eta\|x\| + \eta_b} \ \frac{yx^T}{\|x\|} \ , \quad \delta b(y) = -\frac{\eta_b y}{\eta\|x\| + \eta_b} \ . \qquad (3)$$

Then $\{\delta A(y), \delta b(y)\}$ are valid perturbations and $y = \delta A(y)x - \delta b(y)$ so that $y \in \mathcal{Z}_x$, which justifies our claim. [When $x = 0$, we select $\delta b = -y$ and δA arbitrary.]

The above argument shows that we can replace the maximization problem (2) by the equivalent problem:

$$C(x) = \max_{\|y\| \le \phi(x)} \left(Ax - b + y\right)^T W \left(Ax - b + y\right), \qquad (4)$$

where we are defining $\phi(x) = \eta\|x\| + \eta_b$.

The Maximization Problem

We now solve (4) for any fixed x. Note first that the cost

$$\left(Ax - b + y\right)^T W \left(Ax - b + y\right)$$

is convex in y, so that the maximum over y is achieved at the boundary, $\|y\| = \phi(x)$. We can therefore replace the inequality constraint in (4) by an equality. Introducing a Lagrange multiplier λ, the solution to (4) can then be found from the unconstrained problem:

$$\max_{y,\lambda} \left[\left(Ax - b + y\right)^T W \left(Ax - b + y\right) - \lambda\left(\|y\|^2 - \phi^2\right)\right]. \quad (5)$$

[1]It is easy to see that in the special case $\eta_b = 0$ *and* $Wb = 0$, the unique solution of (1) is $\hat{x} = 0$. In the sequel we shall therefore assume that η_b and Wb are not zero simultaneously.

Note that since the original problem has an inequality constraint, the Lagrange multiplier must be nonnegative: $\lambda \ge 0$ [5]. Differentiating (5) with respect to y and λ, and denoting the optimal solutions by $\{y^o, \lambda^o\}$, we obtain the equations

$$(\lambda^o I - W)y^o = W(Ax - b) \ , \ \|y^o\| = \phi(x) \ . \qquad (6)$$

It turns out that the solution λ^o should satisfy $\lambda^o \ge \|W\|$. This is because the Hessian of the cost in (5) w.r.t. y must be nonpositive-definite [5]. We should further stress that the solutions $\{y^o, \lambda^o\}$ of (6) are functions of x and we shall therefore sometimes write $\{y^o(x), \lambda^o(x)\}$.

At this stage, we do not need to solve the equations (6) for $\{y^o, \lambda^o\}$. It is enough to know that the optimal $\{y^o, \lambda^o\}$ satisfy (6).[2] Using this fact, we can verify that the maximum cost in (4) is equal to (where $X^\dagger$ denotes the pseudo-inverse of X):

$$\begin{aligned} C(x) \ &= \ (Ax - b)^T \left[W + W(\lambda^o(x) I - W)^\dagger W\right](Ax - b) \\ &\quad + \lambda^o(x)\phi^2(x) \ . \end{aligned} \qquad (7)$$

The Minimization Problem

We are now in a position to address the original problem (1), which is equivalent to the minimization problem below:

$$\min_x \ \left[x^T Q x \ + \ C(x)\right] \ . \qquad (8)$$

However, rather than minimizing the above cost over n variables, which are the entries of the vector x, we shall instead show how to reduce the problem to one of minimizing a certain cost function over a single scalar variable (see (13) further ahead).

For this purpose, we introduce the following function of two *independent* variables x and λ,

$$\begin{aligned} C(x, \lambda) \ &= \ (Ax - b)^T \left[W + W(\lambda I - W)^\dagger W\right](Ax - b) \\ &\quad + \lambda\phi^2(x) \ . \end{aligned}$$

Then it can be verified, by direct differentiation with respect to λ and by using the expression for $\lambda^o(x)$ from (6), that

$$\lambda^o(x) \ = \ \arg \min_{\lambda \ge \|W\|} \ C(x, \lambda) \ .$$

This means that problems (1) or (8) are equivalent to

$$\min_{\lambda \ge \|W\|} \ \min_x \left[x^T Q x \ + \ C(x, \lambda)\right] \ . \qquad (9)$$

[2]In fact, we can show that the solution λ^o is always unique while there might be several y^o.

Note that the cost function in the above expression, viz., $J(x, \lambda) = x^T Qx + C(x, \lambda)$, is now a function of two independent variables $\{x, \lambda\}$. This should be contrasted with the cost function in (8).

Now define, for compactness of notation, the quantities

$$\begin{aligned} M(\lambda) &= Q + A^T \left[W + W(\lambda I - W)^{\dagger} W\right] A, \\ d(\lambda) &= A^T \left[W + W(\lambda I - W)^{\dagger} W\right] b . \end{aligned}$$

To solve problem (9), we first search for the minimum over x for every fixed value of λ, which can be done by setting the derivative of $J(x, \lambda)$ w.r.t. x equal to zero. This shows that any nonzero minimum x must satisfy the equality

$$x = \left[M(\lambda) + \lambda\eta \left(\eta + \frac{\eta_b}{\|x\|}\right)\right]^{-1} d(\lambda) . \quad (10)$$

Note that x appears on both sides of the equality (except when $\eta_b = 0$, in which case the expression for x is complete in terms of $\{M, \lambda, \eta, d\}$). To solve for x in the general case we define $\alpha = \|x\|^2$ and square the above equation to obtain the scalar equation in α:

$$\alpha^2 - d^T(\lambda) \left[M(\lambda) + \lambda\eta \left(\eta + \frac{\eta_b}{\alpha}\right)\right]^{-2} d(\lambda) = 0 . \quad (11)$$

It can be shown that a unique solution $\alpha^o(\lambda) > 0$ exists for this equation if, and only if, $\lambda\eta\eta_b < \|d(\lambda)\|^2$. Otherwise, $\alpha^o(\lambda) = 0$. In the former case, the expression for x, which is a function of λ, becomes

$$x^o(\lambda) = \left[M(\lambda) + \lambda\eta \left(\eta + \frac{\eta_b}{\alpha^o(\lambda)}\right)\right]^{-1} d(\lambda) . \quad (12)$$

In the latter case we clearly have $x^o(\lambda) = 0$.

We thus have a procedure that allows us to determine the minimizing x^o for every λ. This in turn allows us to re-express the resulting cost $J(x^o(\lambda), \lambda)$ as a function of λ alone, say $G(\lambda) = J(x^o(\lambda), \lambda)$. In this way, we conclude that the solution $\hat{x}$ of the original optimization problem (1) can be solved by determining the $\hat{\lambda}$ that solves

$$\min_{\lambda \geq \|W\|} G(\lambda) , \quad (13)$$

and by taking the corresponding $x^o(\hat{\lambda})$ as $\hat{x}$. That is,

$$\hat{x} = \left[M(\hat{\lambda}) + \hat{\lambda}\eta \left(\eta + \frac{\eta_b}{\alpha^o(\hat{\lambda})}\right)\right]^{-1} d(\hat{\lambda}) .$$

We summarize the solution in the following statement.

Theorem 1 (Solution). *The unique global minimum of (1) can be determined as follows. Introduce the function* $G(\lambda) = x^{oT}(\lambda) Q x^o(\lambda) + C[x^o(\lambda), \lambda]$, *where* $x^o(\lambda)$ *is given by (12) if* $\lambda\eta\eta_b < \|d(\lambda)\|^2$ *(and zero otherwise), and* $\alpha^o(\lambda)$ *in (12) is the unique positive root of (11). Let* $\hat{\lambda}$ *denote the minimum of* $G(\lambda)$ *over the interval* $\lambda \geq \|W\|$. *Then*

$$\hat{x} = \left[\hat{Q} + A^T \hat{W} A\right]^{-1} A^T \hat{W} b , \quad (14)$$

if $\hat{\lambda}\eta\eta_b < \|d(\hat{\lambda})\|^2$ *(and zero otherwise), where*

$$\begin{aligned} \hat{Q} &= Q + \hat{\lambda}\eta \left(\eta + \frac{\eta_b}{\alpha^o(\hat{\lambda})}\right) I , \\ \hat{W} &= W + W(\hat{\lambda} I - W)^{\dagger} W . \end{aligned}$$

■

We thus see that the solution of (1) requires that we determine an optimal scalar parameter $\hat{\lambda}$, which corresponds to the minimizing argument of a certain nonlinear function $G(\lambda)$ (or, equivalently, to the root of its derivative function). This step can be carried out very efficiently by any root finding routine, especially since the function $G(\lambda)$ is well defined and, moreover, $\hat{\lambda}$ is unique.

The Orthogonality Condition

Observe that, when $\hat{\lambda}\eta\eta_b < \|d(\hat{\lambda})\|^2$, the optimal solution $\hat{x}$ satisfies the orthogonality condition

$$\hat{Q}\hat{x} + A^T \hat{W}(A\hat{x} - b) = 0 .$$

Compared with the solution to the standard regularized least-squares problem,

$$\min_x \left[x^T Qx + (Ax - b)^T W (Ax - b)\right] ,$$

whose unique solution satisfies

$$Q\hat{x} + A^T W(A\hat{x} - b) = 0 ,$$

we see that the solution to the BDU problem satisfies a similar orthogonality condition, with the given weighting matrices $\{Q, W\}$ replaced by new matrices $\{\hat{Q}, \hat{W}\}$! To determine the necessary corrections to $\{Q, W\}$, one determines the optimal $\hat{\lambda}$ from the minimization (13). The power of such a geometric viewpoint is demonstrated in [1]. We omit further details here for brevity.

3 APPLICATION TO STATE REGULATION

We now provide one application for the weighted BDU problem in the context of state regulation for uncertain state-space models. Thus consider the linear state-space model $x_{i+1} = F_i x_i + G_i u_i$, where x_0 denotes the value of the initial state, and the $\{u_i\}$ denote the control (input) sequence. The classical linear quadratic regulator (LQR) problem seeks a control sequence $\{u_i\}$ that regulates the state vector towards zero while keeping the control cost low. This is achieved as follows. Introduce, for compactness of notation, the local cost

$$V_i(x_{i+1}, u_i) \triangleq (x_{i+1}^T R_{i+1} x_{i+1} + u_i^T Q_i u_i)$$

with $R_{N+1} = P_{N+1}$. Then the optimal control is determined by solving the nested minimizations:

$$x_0^T R_0 x_0 + \min_{u_0}\left\{V_0 + \min_{u_1}\left\{V_1 + \ldots + \min_{u_N}\{V_N\}\right\}\right\},$$

with $Q_j > 0$, $R_j \geq 0$, and $P_{N+1} \geq 0$ (note that x_0 does not really affect the solution). Only the innermost minimization in the above problem is dependent on u_N (through the state-equation for x_{N+1}). Hence we can determine $\hat{u}_N$ by solving

$$\min_{u_N} V_N \ , \quad \text{given } x_N \ , \tag{15}$$

and then progress backwards in time to determine the other control values. By carrying out this argument one finds the well-known LQR state-feedback solution:

$$\begin{aligned} &\hat{u}_i = -K_i x_i \ , \\ &K_i = (Q_i + G_i^T P_{i+1} G_i)^{-1} G_i^T P_{i+1} F_i \ , \\ &P_i = R_i + K_i^T Q_i K_i + (F_i - G_i K_i)^T P_{i+1} (F_i - G_i K_i) \ . \end{aligned}$$

It is well known that the above LQR controller is sensitive to modeling errors. Robust design methods to ameliorate these sensitivity problems include the $\mathcal{H}_\infty$ design methodology (e.g., [6, 7, 8, 9]) and the so-called guaranteed-cost designs (e.g., [10, 11, 12]). We suggest below a procedure that is based on the BDU problem solved above. At the end of this exposition, we shall compare our result with a guaranteed-cost design. [A comparison with an $\mathcal{H}_\infty$ design is given in [1] for a special first-order problem.] We may mention that the BDU formulation can also treat more general classes of uncertainties than those treated so far in the literature (see [2]).

State Regulation

Consider the perturbed state-equation

$$x_{i+1} = (F_i + \delta F_i) x_i + (G_i + \delta G_i) u_i \ , \tag{16}$$

with known x_0, and where the uncertainties $\{\delta F_i, \delta G_i\}$ are assumed to satisfy

$$\|\delta F_i\| \leq \eta_{f,i} \ , \quad \|\delta G_i\| \leq \eta_{g,i} \ ,$$

for known bounds $\{\eta_{f,i}, \eta_{g,i}\}$. Note that, for generality, we are allowing for time-variant bounds. Note also that we are not restricting the perturbations to be related in any way. They are treated independently here. In the related work [2], we treat other classes of perturbations including some where there is common structure among the perturbations.

Consider the problem of determining a control sequence $\{\hat{u}_j, 0 \leq j \leq N\}$ that solves the nested min-max optimizations:

$$x_0^T R_0 x_0 \ + \tag{17}$$

$$\min_{u_0}\max_{\substack{\delta F_0 \\ \delta G_0}}\left\{V_0 + \min_{u_1}\max_{\substack{\delta F_1 \\ \delta G_1}}\left\{V_1 + \ldots + \min_{u_N}\max_{\substack{\delta F_N \\ \delta G_N}}\left\{V_N\right\}\right\}\right\}$$

where we are writing, for compactness of notation, $\{\delta F_i, \delta G_i\}$ under the max symbols instead of the complete notation.

Let $\{\hat{u}_j\}$ denote a solution of (17), and let us focus on the innermost optimization in (17):

$$\min_{u_N} \max_{\substack{\|\delta F_N\| \leq \eta_{f,N} \\ \|\delta G_N\| \leq \eta_{g,N}}} \left[u_N^T Q_N u_N + x_{N+1}^T P_{N+1} x_{N+1}\right] .$$

In order to determine an expression for $\hat{u}_N$ from the above, the state vector x_N has to be taken as $\hat{x}_N$, which is the value that would result had the earlier optimal control signals $\{\hat{u}_j, 0 \leq j \leq N-1\}$ been determined already and using the worst-case disturbances (as explained in the next section). Then expanding the term $x_{N+1}^T P_{N+1} x_{N+1}$ by using the state equation for x_{N+1}, the above problem reduces to a problem of the same form as the weighted BDU problem that we considered before with the identifications:

$$A \leftarrow G_N \ , \quad W \leftarrow P_{N+1} \ , \quad Q \leftarrow Q_N \ , \quad b \leftarrow -F_N \hat{x}_N \ ,$$

$$x \leftarrow u_N, \quad \eta \leftarrow \eta_{g,N}, \quad \eta_b \leftarrow \eta_{f,N}\|\hat{x}_N\| \ ,$$

and $\delta A \leftarrow \delta G_N$, $\delta b \leftarrow -\delta F_N \hat{x}_N$. Using (14), and the above identifications, we conclude that the optimal control value $\hat{u}_N$ is given by

$$\begin{aligned} \hat{u}_N &= \begin{cases} -K_N \hat{x}_N, & \text{if } \hat{\lambda}_N \eta_{g,N} \eta_{f,N} \|\hat{x}_N\| < \\ & \quad \|G_N^T \hat{W}_{N+1} F_N \hat{x}_N\|^2 \\ 0, & \text{otherwise} \end{cases} \\ K_N &= \left(\hat{Q}_N + G_N^T \hat{W}_{N+1} G_N\right)^{-1} G_N^T \hat{W}_{N+1} F_N \ , \\ \hat{Q}_N &= Q_N + \hat{\lambda}_N \eta_{g,N}\left(\eta_{g,N} + \frac{\eta_{f,N}\|\hat{x}_N\|}{\|K_N \hat{x}_N\|}\right) \ , \\ \hat{W}_{N+1} &= P_{N+1} + P_{N+1}\left(\hat{\lambda}_N I - P_{N+1}\right)^{\dagger} P_{N+1} \ . \end{aligned}$$

where $\hat{\lambda}_N$ is the optimal parameter that corresponds to the above data $\{A, b, W, Q, \eta, \eta_b\}$, and which can be found as explained in Thm. 1. [We can assume $K_N = 0$ when $\hat{u}_N = 0$.]

Moreover, using (7)–(8) and the above identifications again, we find that

$$\hat{x}_N^T R_N \hat{x}_N + \left(\min_{u_N}\max_{\substack{\delta F_N \\ \delta G_N}} V_N\right) = \hat{x}_N^T P_N \hat{x}_N \ ,$$

where $P_N = R_N$ when $\hat{x}_N = 0$, otherwise

$$P_N = R_N + K_N^T Q_N K_N +$$

$$+(F_N - G_N K_N)^T \hat{W}_{N+1}(F_N - G_N K_N) + \quad (18)$$

$$+\hat{\lambda}_N \left[\left(\eta_{f,N}^2 + 2\eta_{g,N}\eta_{f,N} \frac{\|K_N \hat{x}_N\|}{\|\hat{x}_N\|} \right) I + \eta_{g,N}^2 K_N^T K_N \right] .$$

We now proceed to determine an approximation for the optimal control value at time $N-1$ by solving

$$\min_{u_{N-1}} \max_{\substack{\|\delta F_{N-1}\| \le \eta_{f,N-1} \\ \|\delta G_{N-1}\| \le \eta_{g,N-1}}} \left[u_{N-1}^T Q_{N-1} u_{N-1} + x_N^T P_N x_N \right] ,$$

where we assume that $\hat{x}_{N-1}$ is available. We take the solution as $\hat{u}_{N-1}$, and so on. Note that this step is an approximation because we are employing the P_N found above, which is a function of $\hat{x}_N$. For optimality, we would need to determine the functional form $P_N(x_N)$ — this form is defined by the same equations as above with x_N replacing $\hat{x}_N$. It turns out that for single-state models, the value of P_N is independent of the state and therefore the above $\hat{u}_{N-1}$ agrees with the optimal value [2].

Compared with the solution to the LQR problem we see that there are three main differences in the recursions. First, the gain matrix K_N is not defined directly in terms of the original quantities $\{Q_N, P_{N+1}\}$ but in terms of modified quantities $\{\hat{Q}_N, \hat{W}_{N+1}\}$. Secondly, the term P_{N+1} in the LQR Riccati recursion for P_N is replaced by $\hat{W}_{N+1}$ in (18), in addition to a new correction term that is equal to $\hat{\lambda}_N \phi^2(\hat{u}_N)$. Finally, the above solution in fact has the form of a two-point boundary value problem (TPBVP). This is because the expressions for $\{K_N, P_N\}$ are dependent on the worst-case state vector $\hat{x}_N$. We can denote this dependency more explicitly by writing, for any i,

$$\hat{u}_i = -K_i(\hat{x}_i)\hat{x}_i \ . \quad (19)$$

A reasonable state-feedback implementation would be to choose $\hat{u}_i = -K_i(\hat{x}_i)x_i$ (see simulation further ahead).

An Iterative Solution to the TPBVP

We are currently studying the TPBVP more closely. An iterative solution that we found performs reasonably well is the following.

I. Initialization. Choose initial values for all variables P_0 to P_N (for example, by running the LQR Riccati recursion or by using a suboptimal guaranteed-cost design). Choose also initial values for all $\hat{\lambda}_i$, say $\hat{\lambda}_i > \|P_{i+1}\|_F$.

II. Forwards Iteration. Given values $\{x_0, P_{i+1}, \hat{\lambda}_i\}$, we evaluate the quantities $\{\hat{W}_{i+1}, \hat{Q}_i, K_i, \hat{y}_i, \hat{u}_i\}$ by using the recursions derived above, as well as propagate the state-vectors $\{\hat{x}_i\}$ by using $\hat{x}_{i+1} = F_i\hat{x}_i + G_i\hat{u}_i + \hat{y}_i$, where, from (6), $\hat{y}_i$ is found by solving the equation

$$\left(\hat{\lambda}_i I - P_{i+1}\right)\hat{y}_i = P_{i+1}(F_i\hat{x}_i + G_i\hat{u}_i) \ .$$

If the matrix $\left(\hat{\lambda}_i I - P_{i+1}\right)$ is singular, then among all possible solutions we choose one that satisfies

$$\|\hat{y}_i\|^2 = (\eta_{g,i}\|\hat{u}_i\| + \eta_{f,i}\|\hat{x}_i\|)^2 \ .$$

III. Backwards Iteration. Given values $\{P_{N+1}, \hat{u}_i, \hat{x}_i\}$ we find new approximations for $\{P_i, \hat{\lambda}_i\}$ by using the recursions derived above for the state regulation problem.

IV. Recursion. Repeat steps II and III.

■

We compare in Fig. 1 the performance of the above design with a guaranteed-cost design (using, as mentioned above, $\hat{u}_i = K_i(\hat{x}_i)x_i$ with the true state and with K_i computed from the earlier recursions — see [2] for a simulation with $\hat{u}_i = -K_i(\hat{x}_i)\hat{x}_i$. The example presented here is of a 2-state system with $\eta_f = 0$, $\eta_g = 0.4$, $G = \begin{bmatrix} 1 & -0.5 \end{bmatrix}^T$, and $N = 20$. The nominal model is stable with only one control variable. The central horizontal line is the worst-case cost that is predicted by our BDU construction (solid line). The upper horizontal line is an upper bound on the optimal cost. It is never exceeded by the guaranteed-cost design (dotted line). The situation at the right-most end of the graph corresponds to the worst-case scenario. Observe the improvement in performance in the worst-case (approx. 20% for this example – at the right-end of the graph).

4 CONCLUDING REMARKS

Earlier work in the literature on guaranteed-cost designs found either sub-optimal steady-state and finite-horizon controllers (e.g., [11]), or steady-state controllers over the class of linear control laws [10]. Our solution has the following properties: i) It has a geometric interpretation in terms of an orthogonality condition with modified weighting matrices, ii) it does not restrict the control law to linear controllers, iii) it allows for independent uncertainties and can also be extended to other classes of uncertainties [2], and iv)

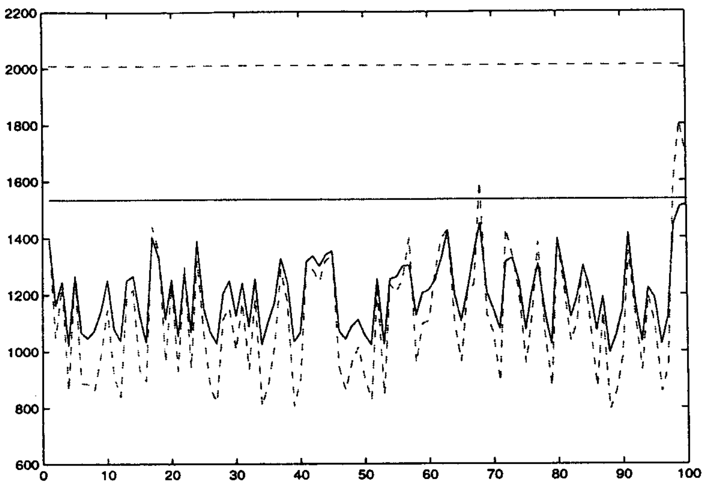

Figure 1: *100 random runs with* $\eta_g = 0.4$ *and a stable 2-dimensional model.*

it handles both regular and degenerate situations. We are currently studying these connections more closely, as well as the TPBVP.

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