

---

---

# CONTENTS

<b>PREFACE</b>	<b>xi</b>
<b>1 INTRODUCTION AND MOTIVATION</b>	<b>1</b>
1.1 Some Historical Remarks	2
1.2 Some Motivation for Introducing Indefinite Metric Spaces	3
1.2.1 State-Space Estimation	4
1.2.2 Array Algorithms	9
1.2.3 Fast Array Algorithms	10
1.2.4 Infinite Horizon Problems	12
1.2.5 Factorization Approach to Infinite Horizon Problems	15
1.2.6 $H^2$ and $H^\infty$ Control Problems	21
1.3 Outline of the Contents	25
<b>2 LINEAR ESTIMATION IN KREIN SPACES</b>	<b>31</b>
2.1 Introduction	31
2.1.1 Notation	32
2.2 On Krein Spaces	33
2.2.1 A Geometric Interpretation	35
2.3 Projections in Krein Spaces	36
2.3.1 Vector-Valued Projections	37
2.3.2 Some Examples	39
2.4 Projections and Quadratic Forms	40
2.4.1 Stochastic Minimization in Hilbert and Krein Spaces	40
2.4.2 A Partially Equivalent Deterministic Problem	43
2.4.3 Alternative Inertia Conditions for Minima	46
2.5 Recursive Projections	47
2.6 Concluding Remarks	50
2.7 Notes and References	50
<b>3 STATE-SPACE MODELS IN KREIN SPACE</b>	<b>51</b>
3.1 Introduction	51
3.2 State-Space Structure	52

3.2.1	The Conditions for a Minimum	55
3.3	Recursive Formulas	57
3.3.1	The Krein Space Kalman Filter	58
3.3.2	Canonical Factorization	60
3.3.3	Some Variations	62
3.4	Recursive State-Space Estimation and Quadratic Forms	63
3.5	Concluding Remarks	67
3.6	Notes and References	67
3.A	Proof of the Time-Variant KYP Lemma	68
3.A.1	Statement of the Lemma	68
3.A.2	Computation of the Output Gramian	68
3.A.3	An Equivalence Class for the Input Gramians	71
3.A.4	The Proof	74
3.A.5	Geometric Interpretation	76
<b>4</b>	<b>FINITE HORIZON <math>H^\infty</math> FILTERING</b>	<b>80</b>
4.1	Introduction	80
4.2	$H^\infty$ Estimation	81
4.2.1	Formulation of the $H^\infty$ Filtering Problem	81
4.2.2	Solution of the Suboptimal $H^\infty$ Filtering Problem	84
4.3	Derivation of the $H^\infty$ Filters	85
4.3.1	The Suboptimal $H^\infty$ Problem and Quadratic Forms	85
4.3.2	A Krein Space State-Space Model	87
4.3.3	Proof of Theorem 4.2.1	87
4.3.4	Parametrization of All $H^\infty$ A Posteriori Filters	91
4.3.5	Derivation of the A Priori $H^\infty$ Filter	93
4.3.6	All $H^\infty$ A Priori Filters	97
4.4	The $H^\infty$ Smoother	98
4.5	1-step $H^\infty$ Prediction	100
4.6	Conclusion	106
4.7	Notes and References	106
<b>5</b>	<b>ARRAY ALGORITHMS</b>	<b>107</b>
5.1	Introduction	107
5.2	$H^2$ Square-Root Array Algorithms	108
5.3	$H^\infty$ Square-Root Array Algorithms	111
5.3.1	J-Unitary Transformations	112
5.3.2	The General Case	115
5.3.3	The Central Filters	118
5.4	$H^2$ Fast Array Algorithms	122
5.5	$H^\infty$ Fast Array Algorithms	125
5.5.1	The General Case	125
5.5.2	The Central Filters	129

---

5.6	Conclusion	130
5.7	Notes and References	131
<b>6</b>	<b>SEVERAL RELATED PROBLEMS</b>	<b>132</b>
6.1	Introduction	132
6.2	Risk-Sensitive Estimation	133
6.2.1	The Exponential Cost Function	134
6.2.2	Minimizing the Risk-Sensitive Criterion	136
6.2.3	Risk-Sensitive Control	141
6.3	Quadratic Dynamic Games	142
6.3.1	General Remarks	142
6.3.2	Specific Examples	145
6.4	Finite Memory Adaptive Filtering	148
6.4.1	The Standard Problem	148
6.5	Conclusion	151
6.6	Notes and References	151
<b>7</b>	<b><math>H^\infty</math> OPTIMALITY OF THE LMS ALGORITHM</b>	<b>152</b>
7.1	Introduction	152
7.2	Adaptive Filtering	154
7.2.1	Least-Squares Methods	154
7.2.2	Gradient-Based Methods	155
7.2.3	The Question of Robustness	155
7.3	The $H^\infty$ Approach	156
7.3.1	Formulation of the $H^\infty$ Adaptive Filtering Problem	156
7.4	Main Result	157
7.4.1	The Normalized LMS Algorithm	157
7.4.2	The LMS Algorithm	159
7.5	An Illustrative Example	160
7.5.1	Discussion	162
7.6	All $H^\infty$ Adaptive Filters	164
7.7	Risk-Sensitive Optimality	166
7.8	Further Remarks	167
7.9	Conclusion	168
7.10	Notes and References	168
7.A	A First Principles Proof of the $H^\infty$ Optimality of LMS	169
7.A.1	The Normalized LMS Algorithm	169
7.A.2	The LMS Algorithm	170
7.B	Robustness of Least-Squares Estimators	171
7.B.1	Introduction	171
7.B.2	A General $H^\infty$ Bound	172
7.B.3	Proof of the Upper Bounds	174
7.B.4	Proof of the Lower Bounds	177

7.B.5 RLS Adaptive Filtering	179
<b>8 DUALITY</b>	<b>182</b>
8.1 Introduction	182
8.2 An Alternative Scalar Quadratic Form	183
8.3 Dual Bases	184
8.3.1 Algebraic Specification	185
8.3.2 Geometric Specification	186
8.3.3 Linear Models	188
8.4 A Pair of Duality and Equivalence Relationships	190
8.4.1 General Equivalence and Duality Relationships	190
8.5 Dual State-Space Models	192
8.5.1 The Backwards Dual Model	192
8.5.2 The Forwards Dual Model	196
8.5.3 The Mixed Dual Model	199
8.6 Application to Smoothing	201
8.6.1 Two-Filter Formulae	202
8.7 Conclusion	204
8.8 Notes and References	204
<b>9 FINITE-HORIZON CONTROL PROBLEMS</b>	<b>205</b>
9.1 Introduction	205
9.2 The LQR Control Problem	206
9.2.1 Problem Formulation	206
9.2.2 Solution Based on Duality	207
9.3 Full Information $H^2$ Control	211
9.3.1 Problem Formulation	212
9.3.2 Solution	213
9.4 Measurement Feedback $H^2$ Control	216
9.4.1 Problem Formulation	217
9.4.2 The Separation Principle	218
9.5 Full Information $H^\infty$ Control	222
9.5.1 Problem Formulation	222
9.5.2 Solution	223
9.6 Measurement Feedback $H^\infty$ Control	228
9.6.1 Problem Formulation	228
9.6.2 Solution	229
9.7 Towards Infinite Horizon Results	236
9.8 Conclusion	236
9.9 Notes and References	236
<b>10 INPUT-OUTPUT APPROACH TO <math>H^2</math> AND <math>H^\infty</math> ESTIMATION</b>	<b>238</b>
10.1 Preliminaries from Linear Systems	238

---

10.2 A Basic Estimation Problem	241
10.2.1 Special Cases	243
10.3 $H^2$ Estimation	244
10.3.1 Stochastic Interpretation	245
10.3.2 The Noncausal Solution	247
10.3.3 The Causal Solution	248
10.3.4 Special Cases	252
10.3.5 The Question of Robustness	255
10.4 $H^\infty$ Estimation	256
10.4.1 Relations to Risk-Sensitivity	258
10.4.2 The Noncausal Solution	261
10.4.3 The Causal Solution	263
10.5 Suboptimal $H^\infty$ Estimation	265
10.5.1 Main Result	266
10.5.2 Solution via Stationarizing Indefinite Quadratic Forms	273
10.5.3 Special Cases	276
10.6 A Simple Example	280
10.7 Final Remarks	285
10.8 Notes and References	286
<b>11 INPUT-OUTPUT APPROACH TO <math>H^2</math> AND <math>H^\infty</math> CONTROL</b>	<b>288</b>
11.1 Control Problems	288
11.1.1 Full Information Control	288
11.1.2 Measurement Feedback Control	290
11.1.3 Special Cases	292
11.2 $H^2$ Control	293
11.2.1 Connections to LQR and LQG Control	293
11.2.2 The Noncausal Solution	295
11.2.3 The Full Information Causal Solution	295
11.2.4 The Measurement Feedback Solution	298
11.2.5 Special Cases	302
11.2.6 The Question of Robustness	306
11.3 $H^\infty$ Control	309
11.3.1 Relations to Risk-Sensitivity	310
11.3.2 The Noncausal Solution	312
11.3.3 The Causal Solution	313
11.3.4 The Full Information Solution	315
11.3.5 The Measurement Feedback Solution	315
11.3.6 Special Cases	319
11.4 Notes and References	324
<b>12 THE DISCRETE-TIME ALGEBRAIC RICCATI EQUATION</b>	<b>325</b>
12.1 Introduction	325

---

12.2 $H^2$ and $H^\infty$ Problems and Factorization	327
12.2.1 The $H^2$ Case	327
12.2.2 The $H^\infty$ Case	327
12.3 The Popov Function	328
12.3.1 An Equivalence Class for Input Gramians	329
12.4 The Discrete-time Algebraic Riccati Equation	332
12.4.1 Relation to Factorization of the Popov Function	333
12.5 A General Existence Result	334
12.6 The Positive Case	341
12.6.1 The Case of $R > 0$ and $Q - SR^{-1}S^* \geq 0$	346
12.6.2 The KYP Lemma	348
12.6.3 The Positive Real and Bounded Real Lemmas	350
12.7 The Case of Nonsingular $F - GSR^{-1}H$	351
12.8 The Nehari Problem	352
12.8.1 Relationship to Hankel Operators	361
12.8.2 Main Result	362
12.9 The Case of $Q - SR^{-1}S \geq 0$ and $R$ Indefinite	364
12.9.1 Main Result	367
12.10 The Symplectic Matrix	370
12.10.1 The Case of Nonsingular $F$	371
12.10.2 The Case of Singular $F$	374
12.11 Conclusion	375
12.12 Notes and References	375
<b>13 INFINITE HORIZON RESULTS FOR STATE-SPACE MODELS</b>	<b>377</b>
13.1 Introduction	377
13.2 $H^2$ Problems	381
13.2.1 $H^2$ Estimation	381
13.2.2 Full Information $H^2$ Control	384
13.2.3 Measurement Feedback $H^2$ Control	386
13.3 $H^\infty$ Problems	390
13.3.1 $H^\infty$ Estimation	390
13.3.2 Full Information $H^\infty$ Control	393
13.3.3 Measurement Feedback $H^\infty$ Control	396
13.4 Concluding Remarks	402
13.5 Notes and References	402
13.5.1 Approaches to $H^\infty$ Control and Estimation	403
<b>14 ASYMPTOTIC BEHAVIOR</b>	<b>407</b>
14.1 The Riccati Recursion	407
14.2 Overview of Results	408
14.3 Solutions to the Riccati Recursion for Different Initial Conditions	410
14.4 Some General Convergence Results	413

---

14.4.1	The Dual Riccati Equation	414
14.5	Convergence in the $H^2$ Case	417
14.5.1	Convergence with $P_0 = 0$	419
14.5.2	Convergence with Stabilizability	423
14.6	Convergence in the $H^\infty$ Case	425
14.6.1	Convergence with $P_0 = 0$	430
14.7	Concluding Remarks	433
14.8	Notes and References	433
<b>15</b>	<b>OPTIMAL <math>H^\infty</math> SOLUTIONS</b>	<b>434</b>
15.1	Introduction	434
15.1.1	Overview of Results	435
15.2	Some Examples	437
15.2.1	Equalization	437
15.2.2	Full Information Tracking	449
15.2.3	Measurement Feedback Tracking	451
15.2.4	Filtering Signals in Additive Noise	455
15.3	Formulae for $\gamma_{opt}$	458
15.3.1	Some Notation	458
15.3.2	The Time-Invariant Case	461
15.3.3	The Time-Variant Finite-Horizon Case	465
15.4	Applications	468
15.4.1	The Equalization Problem	468
15.4.2	Filtering Signals from Additive Noise	470
15.4.3	One-Step-Ahead Prediction	471
15.5	Worst-Case Non-Estimability and Worst-Case Complete Estimability	477
15.5.1	Worst-Case Non-Estimability	477
15.5.2	Application to Equalization with Delay	481
15.6	Conclusion	482
15.7	Notes and References	482
15.A	Some Useful Lemmas	483
<b>16</b>	<b>CONTINUOUS-TIME RESULTS AND FINAL REMARKS</b>	<b>485</b>
16.1	Continuous-Time Problems	485
16.1.1	State-Space Estimation and Initial Results	485
16.1.2	Canonical Factorization and the Innovations Process	491
16.1.3	State-Space Structure	495
16.1.4	$H^\infty$ Filtering	502
16.1.5	Control Problems	503
16.1.6	Infinite Horizon Results	507
16.2	Closing Comments	515
16.3	Notes and References	515

<b>INDEX</b>	<b>515</b>
<b>BIBLIOGRAPHY</b>	<b>523</b>