Chapter 1

INTRODUCTION AND MOTIVATION

This monograph is concerned with certain problems in estimation and control. Broadly speaking, in estimation theory one is confronted with the following problem: given the values of an observable signal (often called the measurement signal) one would like to estimate (or to predict) the values of another signal (often called the desired signal) that is not directly observable. In control theory, one is given a certain dynamical system (the so-called plant), and the goal is to suitably influence the behavior of the plant through a control signal, so that the plant yields some desired performance. In many applications, these two problems are coupled. In fact, when all that is available from the plant is observations of its output, it turns out that, under certain conditions, we can solve the control problem via a two-step procedure: one first uses the observed output of the plant to estimate the value of certain unobservable signals (that are internal to the plant), and then uses these estimates to construct the required control signals.

The solution to the problem of estimating an unobservable signal given an observable one depends on the relationship between the two signals (*i.e.*, on the model describing them) and on the optimality criterion that one uses to determine the desired estimates. Similar remarks apply to the control problem: the solutions depend on the relationship between the control signals, the exogenous signals, and the plant (*i.e.*, on the model) and on the criterion used to measure the plant's desired performance. Of course, these issues are interrelated: one would like to choose a criterion that is compatible with the model, and vice-versa. However, what influence the choice of model and criterion most significantly are the underlying problem that we are actually trying to solve (be it estimation or control), and the possibility of actually obtaining a solution to the formulated problem that is easily implementable. In other words, physical significance and mathematical tractability.

The subject of estimation and control theory, as covered by all possible choices of signal models and optimality criteria, is a vast, and still developing, one; moreover, its close links with several other fields such as adaptive filtering, signal detection, matrix computation, etc., keep deepening. Here, with some minor exceptions, our attention will be devoted to linear models (where the measurement and desired signals are linearly related) and to quadratic (or quadratically-induced) deterministic and stochastic criteria. The linear models that we shall mostly, but not exclusively, be concerned with are those that are more common in system theory, viz., finite-dimensional linear state-space models and rational transfer matrices. The quadratic criteria that we shall consider include the (now classical) deterministic least-squares and stochastic least-mean-squares criteria, now often called H^2 criteria, as well as the relatively more recent criteria used in H^∞ theory, dynamic game theory, and risk-sensitive estimation and control.

Indeed the major premise of this book is that these apparently different estimation and control problems, with different deterministic and stochastic criteria, can be solved in a unified geometric framework by formulating them in certain indefinite inner-product (or indefinite metric) spaces. These so-called Krein spaces are extensions of Hilbert spaces where the self inner-product of any vector can be positive, negative, or zero. Although Hilbert spaces and Krein spaces share many characteristics, they differ in special ways that turn out to mark the differences between the standard least-mean-squares (LQG or H^2) theories and the more recent H^∞ and game theories.

Apart from rather more transparent derivations of existing results, a major bonus of this unified approach is that it enables us to fairly readily extend to the H^∞ setting much of the huge body of results and insights developed over the last three decades in the now classical H^2 theories of Kalman filtering and LQG control. For example, we shall show how to extend the now favored square-root array algorithms and the fast (Chandrasekhar-Kailath-Morf) algorithms of Kalman filter theory to these new settings. We shall also be guided to new results on the asymptotic behavior of H^∞ filters and controllers, and on the existence and properties of solutions to Riccati equations with (possibly) indefinite coefficient matrices. Our new framework will be also used to study the implications of robust estimation for the (vast and currently highly active) field of adaptive signal processing.

In this introductory chapter we, attempt to provide some motivations for our introduction of indefinite metric spaces by describing and comparing the solutions to several H^2 and H^∞ estimation problems . We shall also overview the scope and contributions of this book. However, before doing so, it will be useful to present some very brief historical remarks.

1.1 Some Historical Remarks

The problem of least-mean-squares estimation of stochastic processes was first investigated by Kolmogorov [Kol39], [Kol41] and Wiener [Wie49]. Wiener, in particular, must be credited with introducing the use of stochastic models and optimization criteria in estimation and control. The assumption that the underlying processes are stationary (and implicitly Gaussian) is crucial to the Wiener and Kolmogorov theory and it was not until the late 1950's and early 1960's that a logjam in effectively extending the theory to nonstationary processes was broken by Kalman's explicit introduction of state-space structure into the problem [Kal60b], [KB61] and [Kal63b]; we should also mention Stratonovich's closely related contemporary and independent emphasis on the use of Markov models [Str60]. The new theory soon acquired the name *Kalman filter theory*, and since then a vast literature on the topic has developed, see *e.g.*, the IEEE reprint volume [Sor85].

Concurrent with the development of Kalman filter theory a closely related theory of optimal control was being developed [Kal60a], [Kal64], [Pon61], [Yak62], [Pop64] and [Won68b]. As in the Kalman filter theory, the underlying assumptions of this theory were that the plant has a known linear (and possibly time-variant) description, and that the exogenous signals (the noises and disturbances) impinging on the feedback system are stochastic in nature, but have known statistical properties. These assumptions turned out to be very well suited to the problems of guidance and control of space vehicles to which the theory was first applied. This theory is now known as linear-quadratic-Gaussian (LQG) control to reflect the fact that the model and optimal controller are linear, that the cost function is quadratic, and that the disturbances are assumed to be stochastic processes with jointly Gaussian distribution.

However, in many applications of estimation and control one is faced with modeling errors and lack of statistical information. In these cases, the aforementioned methods are not directly applicable since the statistics and distributions of the stochastic processes are not known. Moreover, it is not obvious what the behavior of such estimation schemes will be once the assumptions on the statistics and distributions are not met.

Robust control theory itself grew out of the need for designing controllers that were less sensitive to plant modeling errors and to the lack of statistical information on the exogenous signals: by the late 1970's it had been observed that LQG controllers could be highly non-robust with respect to such modeling errors. The H^∞ approach to robust control, inspired by the pioneering work of Zames [Zam81], was extensively studied in the 1980's and has since been approached by numerous authors using various interpolation-theoretic, operator-theoretic, game-theoretic, circuit-theoretic, and system-theoretic techniques [ZF83],[FZ84], [Doy84], [VJ84], [Fra87], [BC87], [Kim87], [Pet87], [Hel87], [FT88], [DGKF89], [Bas89], [Tad90], [ST90], [GGLD90], [KLK91], [PAJ91], [LAKG92], [Sch92], [GA94], [IS94], [DHI94] and [Fei98]. Some recent papers on H^∞ estimation (rather than H^∞ control) include [KN91], [Bas91], [ST92] and [Gri93].

The objective in H^∞ estimation is to come up with estimators that minimize (or in the suboptimal case, bound) the maximum energy gain from the disturbances to the estimation errors. This will guarantee that if the disturbances are small (in energy) then (no matter what the disturbances are) the estimation errors will be as small as possible (in energy). Likewise, in H^∞ control the main idea is to construct controllers that minimize (or in the suboptimal case, bound) the maximum gain from the exogenous signal energy to the performance cost (typically the energy of the regulated and control signals). In either case, the robustness of H^∞ estimators and controllers, with respect to disturbance variation, follows from the fact that they safeguard against the estimator and controller's *worst-case* performance and make no assumptions on the statistics or distributions of the disturbance signals. Of course, since they make no such assumptions about the disturbances, they have to accommodate for all conceivable disturbances, and thus may be over-conservative.

We may begin with the remark that, despite their fundamentally different objectives, the controllers and estimators obtained in H^∞ theory bear a striking resemblance to those obtained in LQG and Kalman filter theory. Nevertheless, there are enough significant differences that various ingenious methods, as mentioned above, were devised to obtain these results; so mastering them can be quite challenging. In the words of Kimura (1997):

"*It is remarkable that H^∞ control allows such a multitude of approaches. It looks entirely different from different viewpoints. This fact certainly implies that H^∞ control is quite rich in logical structure and is versatile as an engineering tool. However, the original question of what is the theoretical core of H^∞ control remains unanswered. Indeed every fundamental notion mentioned has a method of solving the H^∞ control problem associated with it. Unfortunately, however, lengthy chains of reasoning and highly technical manipulations are their common characteristic features* [Kim97] pp. v-vi."

The theme of this monograph is that such very challenging and very different solution methods need not be necessary; the basic LQG and Kalman filtering arguments can still be used, provided we set up appropriate control and estimation problems with elements not in a Hilbert space, but in an indefinite metric (so-called) Krein space. This observation has several different ramifications that we will also explore. First, however, some motivation for introducing indefinite metric spaces.

1.2 Some Motivation for Introducing Indefinite Metric Spaces

In this section we shall motivate the introduction of indefinite metric spaces by reviewing some major results from finite and infinite horizon H^2 and H^∞ state-space estimation theory.

[In Chapters 10 and 11, we shall pose and solve more general problems formulated in an operator-theoretic input-output (rather than state-space) framework.]

1.2.1 State-Space Estimation

Consider the time-variant state-space model

$$\begin{cases} x_{i+1} = F_i x_i + G_i u_i, & x_0 \\ y_i = H_i x_i + v_i, & \end{cases} \quad i \geq 0 \tag{1.2.1}$$

where $F_i \in \mathcal{C}^{n\times n}$, $G_i \in \mathcal{C}^{n\times m}$, and $H_i \in \mathcal{C}^{p\times n}$ are known matrices, x_0, u_i, and v_i are *unknown* quantities and y_i is the measured output. We can regard v_i as a measurement noise and u_i as a process noise or driving disturbance. The problem is to estimate some arbitrary linear combination of the states, say

$$s_i = L_i x_i,$$

where $L_i \in \mathcal{C}^{q\times n}$ is known, using the observations $\{y_0, \ldots y_i\}$. Let

$$\hat{s}_{i|i} = \mathcal{F}_f(y_0, y_1, \ldots, y_i),$$

denote an estimate of s_i given observations $\{y_j\}$ from time 0 up to and including time i, and let

$$\tilde{s}_{i|i} = s_i - \hat{s}_{i|i}, \tag{1.2.2}$$

be the corresponding so-called *filtered* error.

The goal in estimation is to construct the estimates $\hat{s}_{i|i} = \mathcal{F}_f(y_0, y_1, \ldots, y_i)$ such that the filtered error sequence $\{\tilde{s}_{i|i}\}$ be "small". A popular measure for smallness is the *energy*,

$$\sum_{j=0}^{i} \tilde{s}_{j|j}^* \tilde{s}_{j|j}. \tag{1.2.3}$$

The H^2 Approach: Stochastic Disturbance Processes

In the H^2 approach it is assumed that the unknown disturbances $\{x_0, \{u_i\}, \{v_i\}\}$ are zero-mean random variables with known second-order statistics. Here we may assume that the second-order statistics, or variances and covariances, are given by

$$E \begin{bmatrix} x_0 \\ u_i \\ v_i \end{bmatrix} \begin{bmatrix} x_0^* & u_j^* & v_j^* \end{bmatrix} = \begin{bmatrix} \Pi_0 & 0 & 0 \\ 0 & Q_i \delta_{ij} & 0 \\ 0 & 0 & R_i \delta_{ij} \end{bmatrix}. \tag{1.2.4}$$

Once the initial state and disturbances are zero-mean random variables, the same will be true of all the other quantities of interest in (1.2.1), *i.e.*, the state variables $\{x_i\}$, the observations $\{y_i\}$, and the desired signals $\{s_i\}$.

In particular, the filtered error energy (1.2.3) will also be a random variable. Therefore the goal in H^2 estimation is to find a *linear* estimate $\hat{s}_{i|i}$ that minimizes the *expected* filtered error energy, *i.e.*, the average value of (1.2.3). The precise statement follows.

Problem 1.2.1 (H^2 Estimation Problem) *Consider the state-space model (1.2.1) along with the stochastic assumptions (1.2.4). Find a* linear *estimation strategy* $\hat{s}_{j|j} = \mathcal{F}_f(y_0, \ldots, y_j)$ *that minimizes the expected filtered error energy,* i.e.,

$$\min_{\mathcal{F}_f} E \sum_{j=0}^{i} (s_j - \hat{s}_{j|j})^* (s_j - \tilde{s}_{j|j}). \tag{1.2.5}$$

The solution to the above problem is given by the celebrated Kalman filter [Kal60b], [AM79], [KSH98], which we quote below.

Theorem 1.2.1 (Kalman Filter) *The solution to Prob. 1.2.1 is given by*

$$\hat{s}_{j|j} = L_j \hat{x}_{j|j}, \tag{1.2.6}$$

where $\hat{x}_{j|j}$ satisfies the filtered form of the Kalman filter recursions,

$$\hat{x}_{j+1|j+1} = F_j \hat{x}_{j|j} + K_{f,j+1}(y_{j+1} - H_{j+1} F_j \hat{x}_{j|j}), \quad \hat{x}_{-1|-1} = 0 \tag{1.2.7}$$

where

$$K_{f,j} = P_j H_j^* R_{e,j}^{-1} \quad , \quad R_{e,j} = R_j + H_j P_j H_j^* \tag{1.2.8}$$

and where P_j satisfies the Riccati recursion,

$$P_{j+1} = F_j P_j F_j^* + G_j Q_j G_j^* - F_j P_j H_j^* R_{e,j}^{-1} H_j P_j F_j^*, \quad P_0 = \Pi_0. \tag{1.2.9}$$

The H$^\infty$ Approach: Deterministic Disturbance Processes

In the H$^\infty$ approach it is assumed that the disturbances $\{x_0, \{u_i\}, \{v_i\}\}$ are unknown, but *nonrandom*. One cannot therefore speak of expected values, or attempt to minimize the average filtered error energy. Instead one can look at the normalized cost (assuming one considers the filtered error energy as a cost we intend to minimize),

$$\frac{\sum_{j=0}^{i} \tilde{s}_{j|j}^* \tilde{s}_{j|j}}{x_0^* \Pi_0^{-1} x_0 + \sum_{j=0}^{i} u_j^* u_j + \sum_{j=0}^{i} v_j^* v_j} , \tag{1.2.10}$$

which can also be interpreted as the energy gain from the unknown disturbances $\Pi_0^{-1/2} x_0$ and $\{u_j, v_j\}_{j=0}^{i}$ to the filtered error $\{\tilde{s}_{j|j}\}_{j=0}^{i}$. [Here Π_0 is a positive definite weighting matrix.] It is quite clear that if the ratio in (1.2.10) is small then the estimator performs well, and vice-versa. However, the problem with this ratio is that it depends on the disturbances $\{x_0, \{u_i\}, \{v_i\}\}$, which are not known. To overcome this problem, we can consider the worst-case, *i.e.*,

$$\sup_{x_0, \{u_j\}, \{v_j\}} \frac{\sum_{j=0}^{i} \tilde{s}_{j|j}^* \tilde{s}_{j|j}}{x_0^* \Pi_0^{-1} x_0 + \sum_{j=0}^{i} u_j^* u_j + \sum_{j=0}^{i} v_j^* v_j}, \tag{1.2.11}$$

which represents the worst-case energy gain from the disturbances to the filtered errors. The goal in H$^\infty$ estimation is to *minimize this worst-case energy gain*, and the claim is that the resulting estimators will be robust with respect to disturbance variations, since no statistical assumptions are being made about the disturbances, and since we are safeguarding against the worst-case scenario. However, the resulting estimators may of course be over-conservative.

It turns out that there are only a few cases (some of which will be described later in Chapters 7 and 15) where the worst-case energy gain can be explicitly minimized over $\mathcal{F}$. Therefore one normally relaxes the minimization condition and considers the following suboptimal problem.

Problem 1.2.2 (Suboptimal H^∞ Estimation Problem) *Consider the standard state-space model (1.2.1), and a given scalar $\gamma_f > 0$. Determine whether it is possible to bound the worst-case energy gain in (1.2.11) by γ_f^2,* i.e., *whether it is possible to find an estimation strategy $\hat{s}_{j|j} = \mathcal{F}_f(y_0, \ldots, y_j)$ that achieves*

$$\sup_{x_0,\{u_j\},\{v_j\}} \frac{\sum_{j=0}^{i} \tilde{s}_{j|j}^* \tilde{s}_{j|j}}{x_0^* {\Pi_0}^{-1} x_0 + \sum_{j=0}^{i} u_j^* u_j + \sum_{j=0}^{i} v_j^* v_j} < \gamma_f^2. \tag{1.2.12}$$

If this is the case, then find one such H^∞ filter of level γ_f.

Before stating the solution to the *suboptimal* H^∞ problem it will be useful to examine the structure of the problem in slightly more detail. Note that (1.2.12) implies that for all nonzero $\{x_0, \{u_j\}_{j=0}^{i}, \{v_j\}_{j=0}^{i}\}$ we must have

$$\frac{\sum_{j=0}^{i} (\hat{s}_{j|j} - L_j x_j)^* (\hat{s}_{j|j} - L_j x_j)}{x_0^* \Pi_0^{-1} x_0 + \sum_{j=0}^{i} u_j^* u_j + \sum_{j=0}^{i} (y_j - H_j x_j)^* (y_j - H_j x_j)} < \gamma_f^2. \tag{1.2.13}$$

Note, moreover, that (1.2.13) implies that for *all* $k \le i$, we must have

$$\frac{\sum_{j=0}^{k} (\hat{s}_{j|j} - L_j x_j)^* (\hat{s}_{j|j} - L_j x_j)}{x_0^* \Pi_0^{-1} x_0 + \sum_{j=0}^{k} u_j^* u_j + \sum_{j=0}^{k} (y_j - H_j x_j)^* (y_j - H_j x_j)} < \gamma_f^2. \tag{1.2.14}$$

If the $\{y_j\}_{j=0}^{i}$ are all zero it is easy to see that the $\{\hat{s}_{j|j}\}$ must all be zero as well.[1] Therefore we need only consider the case where the $\{y_j\}_{j=0}^{i}$ are a nonzero sequence, and can thus restate Prob. 1.2.2 as follows.

Restatement of Problem 1.2.2: *Given a scalar $\gamma_f > 0$, then (1.2.12) is satisfied if, and only if, there exists a sequence $\{\hat{s}_{j|j} = \mathcal{F}_f(y_0, \ldots, y_j)\}_{j=0}^{i}$ such that for all complex vectors x_0, for all causal sequences $\{u_j\}_{j=0}^{i}$, and for all nonzero causal sequences $\{y_j\}_{j=0}^{i}$, the scalar quadratic form defined by*

$$\begin{aligned} J_{f,k} \; &= \; x_0^* \Pi_0^{-1} x_0 + \sum_{j=0}^{k} u_j^* u_j \\ &\quad + \sum_{j=0}^{k} (y_j - H_j x_j)^* (y_j - H_j x_j) - \gamma_f^{-2} \sum_{j=0}^{k} (\hat{s}_{j|j} - L_j x_j)^* (\hat{s}_{j|j} - L_j x_j) \end{aligned}$$

satisfies

$$J_{f,k} > 0 \quad \textit{for all} \quad 0 \le k \le i. \tag{1.2.15}$$

Note that the *indefinite* quadratic form $J_{f,k}$ can be rewritten as follows:

$$J_{f,k} \; = \; x_0^* \Pi_0^{-1} x_0 + \sum_{j=0}^{k} u_j^* u_j \tag{1.2.16}$$

[1] To see why, suppose that $\{y_j\}_{j=0}^{i}$ is the zero sequence. Then if the disturbances x_0 and $\{u_j\}_{j=0}^{i}$ are also taken to be zero, the denominator in (1.2.13) becomes zero, and hence if $\{\hat{s}_{j|j}\}$ is taken as any nonzero sequence the ratio in (1.2.13) becomes infinite. Since we would like to find an estimation strategy that bounds the energy gain by γ_f^2 for *all* possible disturbances, we conclude that the $\{\hat{s}_{j|j}\}$ must be zero whenever the $\{y_j\}_{j=0}^{i}$ are so.

$$+\sum_{j=0}^{k}\left(\begin{bmatrix} y_j \\ \hat{s}_{j|j} \end{bmatrix} - \begin{bmatrix} H_j \\ L_j \end{bmatrix} x_j\right)^* \begin{bmatrix} I & 0 \\ 0 & -\gamma_f^{-2} I \end{bmatrix}\left(\begin{bmatrix} y_j \\ \hat{s}_{j|j} \end{bmatrix} - \begin{bmatrix} H_j \\ L_j \end{bmatrix} x_j\right).$$

We see that the solution to the H^∞ filtering problem is intimately related to guaranteeing the positivity of an indefinite quadratic form. This suggests that there may be benefit in introducing some sort of indefinite metric space into the problem. In fact, this will become even more evident after we quote the solution to the above H^∞ filtering problem (see [KN91], [ST92], [HSK93b], [GL95], [HSK96b]).

Theorem 1.2.2 (Finite Horizon H^∞ Filter) *An H^∞ filter of level γ_f exists if, and only if, the matrices*

$$R_j = \begin{bmatrix} I_p & 0 \\ 0 & -\gamma_f^2 I_q \end{bmatrix} \quad \text{and} \quad R_{e,j} = \begin{bmatrix} I_p & 0 \\ 0 & -\gamma_f^2 I_q \end{bmatrix} + \begin{bmatrix} H_j \\ L_j \end{bmatrix} P_j \begin{bmatrix} H_j^* & L_j^* \end{bmatrix} \tag{1.2.17}$$

have the same inertia[2] *for all $0 \le j \le i$, where $P_0 = \Pi_0$ and P_j satisfies the Riccati recursion*

$$P_{j+1} = F_j P_j F_j^* + G_j G_j^* - F_j P_j \begin{bmatrix} H_j^* & L_j^* \end{bmatrix} R_{e,j}^{-1} \begin{bmatrix} H_j \\ L_j \end{bmatrix} P_j F_j^*. \tag{1.2.18}$$

If this is the case, then one possible H^∞ estimator is given by

$$\hat{s}_{j|j} = L_j \hat{x}_{j|j}, \tag{1.2.19}$$

where $\hat{x}_{j|j}$ satisfies the recursion,

$$\hat{x}_{j+1|j+1} = F_j \hat{x}_{j|j} + K_{s,j+1}(y_{j+1} - H_{j+1} F_j \hat{x}_{j|j}), \quad \hat{x}_{-1|-1} = 0 \tag{1.2.20}$$

and

$$K_{s,j} = P_j H_j^* (I + H_j P_j H_j^*)^{-1}. \tag{1.2.21}$$

Comparison of the H^2 and H^∞ Solutions

The solution of Theorem 1.2.2 looks very much like the Kalman filter solution of Theorem 1.2.1, except that the Riccati recursion (1.2.18) differs from that of the Kalman filter (1.2.9), since:

- We have indefinite "covariance" matrices, $\begin{bmatrix} I & 0 \\ 0 & -\gamma_f^2 I \end{bmatrix}$.
- The L_i (of the quantity to be estimated) enters the Riccati recursion, (1.2.18).
- We have an additional condition, (1.2.17), that must be satisfied for the filter to exist; in the Kalman filtering problem the L_i do not appear, and the P_i are positive semidefinite, so that (1.2.17) is immediate.

The appearance of the L_i means that the H^∞ estimate of say, the first component of the state vector x_{i+1}, is not the first component of the H^∞ estimate of the whole state vector (because in the first case $L_i = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}$, and in the second case $L_i = I$). This is very different from the situation in the H^2 case, where the estimate of any linear combination of the state is simply given by that linear combination of the state estimate.

[2] By the inertia of a Hermitian matrix, we mean the number of its positive, negative and zero eigenvalues.

The facts that we have the indefinite "covariances", and that the matrix L_i enters the solution of the H^∞ estimation problem, are not suprising once the quadratic form is written as in (1.2.16), where both $I \oplus (-\gamma_f^2 I)$ and L_i appear. However, what is surprising is the fact that the solution to the deterministic H^∞ problem has the same Kalman-filter-like structure of the solution (*cf.* Theorem 1.2.1) of the stochastic H^2 problem; the only difference is that, in the deterministic H^∞ problem, an additional (inertia) condition (1.2.17) is required for a solution to exist.

The Krein Space Approach to the H^∞ Problem

The explanation of these more surprising facts will be given in Chapters 2 and 3. We briefly outline the approach here.

Our method of ensuring that the quadratic form (1.2.16) is strictly positive is the following: first, we need to guarantee that (1.2.16) has a *minimum* over the free variables $\{x_0, \{u_j\}_{j=0}^i\}$ (otherwise the $\{x_0, \{u_j\}_{j=0}^i\}$ can be chosen to make (1.2.16) arbitrarily negative), and, second, we need to choose the estimates $\{\hat{s}_{j|j}\}_{j=0}^i$ such that the value of (1.2.16) at its minimum is strictly positive.

In principle, both steps can be attempted algebraically, *e.g.*, using dynamic programming to obtain a recursive solution. However, the algebra becomes quite involved, which is probably why present solutions to the H^∞ problem generally proceed in various other ways, as noted in Sec. 1.1. The situation is quite different in the *stochastic* H^2 problem, where the recursive Kalman filter solution can be obtained by using geometric arguments: projections in the Hilbert space of random variables minimize the relevant quadratic form (1.2.5), and recursive solutions are readily obtained by introducing the innovations process (see Sec. 3.3.1, or most textbooks on Kalman filtering, such as [KSH98]).

The appearance of the Kalman filter recursions in the H^∞ solution suggests that similar methods could be used — the problem is that in the deterministic H^∞ problem there is no Hilbert space (of random variables, or other abstract vectors). However, we shall show in Sec. 2.4.2 that we can define a (partially) equivalent vector-space problem, where now the space is not a Hilbert space but a special kind of indefinite metric space called a Krein space, that allows us to solve our original deterministic problem. Krein spaces differ from Hilbert spaces in certain basic ways — in particular, they can contain non-zero vectors of zero-length (such vectors are called neutral), as well as subspaces that contain non-zero vectors orthogonal to all vectors in the subspace (such subspaces are called degenerate, and such vectors isotropic). A consequence of these facts is that in Krein spaces, projections need not always exist, and even when they do, they only stationarize (and do not necessarily minimize) a quadratic form. The projection, when it exists, can still be recursively computed using (generalized) Kalman filter recursions, but an additional calculation (of the Hessian matrix of the quadratic form) is required to check if the stationary point is actually a minimum.[3] The inertia condition of Theorem 1.2.2 is precisely this additional condition.

Apart from the fact that Theorem 1.2.2 now has a familiar derivation, with a clear explanation of the similarities to, and differences from, the H^2 solution, the link with the Kalman filter derivations allows us to attempt to extend to the H^∞ case several of the many results on Kalman filters derived in the last four decades. Among these, we mention the (square-root) array algorithms, and the fast (Chandrasekhar-Kailath-Morf) algorithms for constant-parameter systems. We shall derive the H^∞ versions of these results in Chapter 5, but we note the results below in order to show the value of the Kalman filter connection.

[3]This is the reason for saying the Krein space problem is only partially equivalent to the deterministic problem. In Hilbert space, there is a complete equivalence, as will be shown in Corollary 2.4.3.

1.2.2 Array Algorithms

$\mathbf{H}^2$ Square-Root Arrays

Recall the Kalman filter recursions of Theorem 1.2.1,

$$\hat{x}_{j+1|j+1} = F_j\hat{x}_{j|j} + K_{f,j+1}(y_{j+1} - H_{j+1}F_j\hat{x}_{j|j}), \quad \hat{x}_{-1|-1} = 0. \tag{1.2.22}$$

The above recursion shows that the key to propagating $\hat{x}_{j|j}$, and thereby the desired estimate, is to have the so-called gain matrices $K_{f,j}$. In Theorem 1.2.1, $K_{f,j}$ is given in terms of a certain variable, P_j, that satisfies the Riccati recursion (1.2.9).

The matrix P_j appearing in this Riccati recursion has the physical meaning of being the variance of the state prediction error, $\tilde{x}_j = x_j - \hat{x}_j$, where $\hat{x}_j = \hat{x}_{j|j-1}$ is the estimate of x_j given $\{y_k, k < j\}$, and therefore has to be positive (semi-)definite. When actually implementing (1.2.9), round-off errors can cause a loss of positive-definiteness, thus throwing all the obtained results into doubt. For this, and other reasons (reduced dynamic range, better conditioning, more stable algorithms, etc.) attention has moved in the Kalman filtering community to so-called array algorithms that propagate square-root factors of P_j, *i.e.*, a matrix, $P_j^{1/2}$ say, such that

$$P_j = P_j^{1/2}(P_j^{1/2})^* \triangleq P_j^{1/2}P_j^{*/2}.$$

The following is one such now well known result (see [MK75], [KSH98]) which we shall derive in Sec. 5.2.

Algorithm 1.2.1 ($\mathbf{H}^2$ Array Algorithm - Filtered Form) *The matrix, $K_{f,j}$, necessary to obtain the state estimates in the filtered form of the conventional Kalman filter*

$$\hat{x}_{j+1|j+1} = F_j\hat{x}_{j|j} + K_{f,j+1}(y_{j+1} - H_{j+1}F_j\hat{x}_{j|j}), \qquad \hat{x}_{-1|-1} = 0,$$

can be updated as follows

$$\begin{bmatrix} R_j^{1/2} & H_jP_j^{1/2} \\ 0 & P_j^{1/2} \end{bmatrix} \Theta_j^{(1)} = \begin{bmatrix} R_{e,j}^{1/2} & 0 \\ K_{f,j}R_{e,j}^{1/2} & P_{j|j}^{1/2} \end{bmatrix}, \tag{1.2.23}$$

$$\begin{bmatrix} F_jP_{j|j}^{1/2} & G_jQ_j^{1/2} \end{bmatrix} \Theta_j^{(2)} = \begin{bmatrix} P_{j+1}^{1/2} & 0 \end{bmatrix} \tag{1.2.24}$$

where $\Theta_j^{(1)}$ and $\Theta_j^{(2)}$ are any unitary matrices that triangularize the above pre-arrays. The algorithm is initialized with $P_0^{1/2} = \Pi_0^{1/2}$.

Note that in the square-root array algorithm, all one needs to do is form the pre-array in (1.2.23) and to triangularize it via a unitary transformation, *i.e.*, a matrix Θ such that $\Theta\Theta^* = \Theta^*\Theta = I$. [An example of a unitary matrix is a rotation.] Once this is done, the quantities necessary to update the array, and to calculate the state estimates, can all be found from the triangularized post-array.

$\mathbf{H}^\infty$ Square-Root Arrays

As shown in Theorem 1.2.2, the desired estimates $\hat{s}_{j|j} = L_i\hat{x}_{j|j}$ are found from the following recursion for $\hat{x}_{j|j}$,

$$\hat{x}_{j+1|j+1} = F_j\hat{x}_{j|j} + K_{s,j+1}(y_{j+1} - H_{j+1}F_j\hat{x}_{j|j}), \quad \hat{x}_{-1|-1} = 0. \tag{1.2.25}$$

Once more, the key to computing the state estimates is to have the gain matrices $K_{s,j}$. These now can be expressed in terms of another variable P_j that satisfies the Riccati recursion (1.2.18).

Using our observation that H$^\infty$ filtering is essentially Kalman filtering in an indefinite space, it is possible to develop square-root array algorithms for the propagation of $K_{s,j}$, as well. However, since the underlying space in H$^\infty$ estimation is indefinite, we need to make certain adjustments. The first is that in an indefinite metric space the "covariance" (or more precisely, Gramian) matrices, such as R_i, are no longer necessarily positive definite. Therefore we need to introduce *indefinite* square-roots $R_i^{1/2}$, of R_i, defined as follows,

$$R_i = R_i^{1/2} J R_i^{*/2},$$

where J is an signature matrix (a diagonal matrix with $+1$ and -1 on the main diagonal) that represents the number of positive and negative eigenvalues of R_i. Moreover, since the counterpart of a rotation in a definite metric space is a *hyperbolic* rotation in an indefinite metric space, we need to replace unitary transformations with J-unitary transformations (*i.e.*, matrices Θ such that $\Theta J \Theta^* = \Theta^* J \Theta = J$).

With these adjustments, we shall show the following result in Chapter 5.

Algorithm 1.2.2 (Central H$^\infty$ Array Algorithm - Filtered Form) *The H$^\infty$ filtering problem with level γ_f has a solution if, and only if, for all $j = 0, \ldots, i$ there exist* J-*unitary (with $J = I_p \oplus (-I_q) \oplus I_n$) matrices, $\Theta_j^{(1)}$, such that*

$$\begin{bmatrix} \begin{bmatrix} I_p & 0 \\ 0 & \gamma_f I_q \end{bmatrix} & \begin{bmatrix} H_j \\ L_j \end{bmatrix} P_j^{1/2} \\ 0 & P_j^{1/2} \end{bmatrix} \Theta_j^{(1)} = \begin{bmatrix} R_{e,j}^{1/2} & 0 \\ K_{f,j} R_{e,j}^{1/2} & P_{j|j}^{1/2} \end{bmatrix}, \tag{1.2.26}$$

$$\begin{bmatrix} F_j P_{j|j}^{1/2} & G_j \end{bmatrix} \Theta_j^{(2)} = \begin{bmatrix} P_{j+1}^{1/2} & 0 \end{bmatrix} \tag{1.2.27}$$

with $R_{e,j}^{1/2}$ lower block triangular, and with $\Theta_j^{(2)}$ unitary. The gain matrix $K_{s,j}$ needed to update the estimates in the central filter recursions

$$\hat{x}_{j|j} = F_{j-1}\hat{x}_{j-1|j-1} + K_{s,j}(y_j - H_j F_{j-1}\hat{x}_{j-1|j-1}), \qquad \hat{x}_{-1|-1} = 0,$$

is equal to

$$K_{s,j} = \bar{K}_{s,j}(I + H_j P_j H_j^*)^{-1/2},$$

where $\bar{K}_{s,j}$ is given by the first block column of $\bar{K}_{f,j} = K_{f,j} R_{e,j}^{1/2}$, and $(I + H_j P_j H_j^*)^{1/2}$ is given by the $(1,1)$ block entry of $R_{e,j}^{1/2}$. The algorithm is initialized with $P_0^{1/2} = \Pi_0^{1/2}$.

An interesting aspect of the above algorithm is that we do not need to explicitly check for the existence condition (1.2.17) required of H$^\infty$ filters. These conditions are built into the array algorithms themselves: if the algorithms can be performed (*i.e.*, if the pre-arrays can be triangularized by a J-unitary transformation) then an H$^\infty$ estimator of the desired level exists, and if they cannot be performed such an estimator does not exist. [More details will be given in Chapter 5.]

1.2.3 Fast Array Algorithms

H^2 Chandrasekhar-Kailath-Morf Recursions

The conventional Kalman filter and square-root array recursions both require $O(n^3)$ operations per iteration (where n is the number of states in the state-space model). However, when the state-space model is *time-invariant* the Chandrasekhar-Kailath-Morf recursions offer an algorithm that requires $O(n^2)$ operations per iteration [Kai73], [KMS73], [MSK74], [MK75], [SK94a].

In what follows we shall assume a time-invariant state-space model of the form

$$\begin{cases} x_{j+1} &= Fx_j + Gu_j, \quad x_0 \\ y_j &= Hx_j + v_j \end{cases} \tag{1.2.28}$$

and we shall assume that the driving and measurement disturbance covariances are constant, *i.e.*, $Q_j = Q \geq 0$ and $R_j = R > 0$, for all j. When this is the case, it turns out that we can write

$$P_{j+1} - P_j = M_j S M_j, \quad \text{for all } j, \tag{1.2.29}$$

where M_j is an $n \times d$ matrix and S is a $d \times d$ signature matrix. Thus, for time-invariant state-space models, $P_{j+1} - P_j$ has rank d for all j and in addition has constant inertia. In several important cases d can be much less than n. When this is true, propagating the smaller matrices M_j, which is equivalent to propagating the P_j, can offer computational reductions. The result is given below and will be proven in Chapter 5.

Algorithm 1.2.3 (Fast H^2 Array Algorithm - Filtered Form) *If F is invertible, the gain matrix $K_{f,j} = \bar{K}_{f,j} R_{e,j}^{-1/2}$ necessary to obtain the state estimates in the filtered form of the conventional Kalman filter*

$$\hat{x}_{j|j} = F\hat{x}_{j-1|j-1} + K_{f,j}(y_j - HF\hat{x}_{j-1|j-1}), \qquad \hat{x}_{-1|-1} = 0,$$

can be computed using

$$\begin{bmatrix} R_{e,j}^{1/2} & HFN_j \\ \bar{K}_{f,j} & FN_j \end{bmatrix} \Theta_j = \begin{bmatrix} R_{e,j+1}^{1/2} & 0 \\ \bar{K}_{f,j+1} & N_{j+1} \end{bmatrix}, \tag{1.2.30}$$

where Θ_j is any J*-unitary matrix (with $J = I_p \oplus S$) that triangularizes the above pre-array. The algorithm is initialized with*

$$R_{e,0} = R + H\Pi_0 H^*, \qquad \bar{K}_{f,0} = \Pi_0 H^* R_{e,0}^{1/2},$$

and $\{N_0, S\}$ are obtained from the factorization

$$F^{-1}(P_1 - \Pi_0)F^{-*} = \Pi_0 + F^{-1}GQG^*F^{-*} - K_{f,0}R_{e,0}K_{f,0}^* - F^{-1}\Pi_0 F^{-*} = N_0 S N_0^*.$$

Note that compared to the earlier (square-root) array formulas, the size of the pre-array in the fast array recursions has been reduced from $(p+n) \times (p+n+m)$ to $(p+n) \times (p+d)$ where m and p are the dimensions of the driving disturbance and output, respectively, and where n is the number of the states. Thus the number of operations for each iteration has been reduced from $O(n^3)$ to $O(n^2 d)$, with d typically much less than n.

H^∞ Chandrasekhar-Kailath-Morf Recursions

The fact that H^∞ filtering coincides with H^2 filtering in an indefinite metric space allows us to generalize the conventional Chandrasekhar-Kailath-Morf recursions to the H^∞ setting. The result is given below and will be proven in Chapter 5.

Algorithm 1.2.4 (Central H$^\infty$ Fast Array Algorithm - Filtered Form) *Let F be invertible. Then the H^∞ filtering problem with level γ_f has a solution if, and only if,*

(i) All leading submatrices of

$$R = \begin{bmatrix} I_p & 0 \\ 0 & -\gamma_f^2 I_q \end{bmatrix} \quad \text{and} \quad R_{e,0} = \begin{bmatrix} I_p & 0 \\ 0 & -\gamma_f^2 I_q \end{bmatrix} + \begin{bmatrix} H \\ L \end{bmatrix} \Pi_0 \begin{bmatrix} H^* & L^* \end{bmatrix}$$

have the same inertia.

(ii) For all $j = 0, \ldots, i$, there exist J*-unitary matrices, Θ_j, (where $J = I_p \oplus (-I_q) \oplus S$) such that*

$$\begin{bmatrix} R_{e,j}^{1/2} & \begin{bmatrix} H \\ L \end{bmatrix} F N_j \\ K_{f,j} R_{e,j}^{1/2} & F N_j \end{bmatrix} \Theta_j = \begin{bmatrix} R_{e,j+1}^{1/2} & 0 \\ K_{f,j} R_{e,j+1}^{1/2} & N_{j+1} \end{bmatrix} \tag{1.2.31}$$

with $R_{e,j}^{1/2}$ and $R_{e,j+1}^{1/2}$ block lower triangular.

The algorithm is initialized with, $R_{e,0}$, $K_{f,0} = \Pi_0 \begin{bmatrix} H^* & L^* \end{bmatrix} R_{e,0}^{-1}$, and $\{N_0, S\}$ are obtained from the factorization

$$F^{-1}(P_1 - \Pi_0)F^{-*} = \Pi_0 + F^{-1} G Q G^* F^{-*} - K_{f,0} R_{e,0} K_{f,0}^* - F^{-1} \Pi_0 F^{-*} = N_0 S N_0^*.$$

The gain matrix $K_{s,j}$ needed to update the estimates in the central filter recursions

$$\hat{x}_{j|j} = F_{j-1} \hat{x}_{j-1|j-1} + K_{s,j}(y_j - H_j F_{j-1} \hat{x}_{j-1|j-1}), \qquad \hat{x}_{-1|-1} = 0,$$

is equal to

$$K_{s,j} = \bar{K}_{s,j}(I + H_j P_j H_j^*)^{-1/2},$$

where $\bar{K}_{s,j}$ is given by the first block column of $\bar{K}_{f,j} = K_{f,j} R_{e,j}^{1/2}$, and $(I + H_j P_j H_j^*)^{1/2}$ is given by the $(1,1)$ block entry of $R_{e,j}^{1/2}$.

Note that compared to the H$^\infty$ square-root array formulas, the size of the pre-array in the H$^\infty$ fast array recursions has been reduced from $(p+q+n) \times (p+q+n+m)$ to $(p+q+n) \times (p+q+d)$ where m, p and q are the dimensions of the driving disturbance, output and states to be estimated, respectively, and where n is the number of the states. Thus the number of operations for each iteration has been reduced from $O(n^3)$ to $O(n^2 d)$ with d typically much less than n.

As in the array case, the fast array recursions do not require explicitly checking the inertia conditions required of H$^\infty$ filters — if the recursions can be carried out then an H$^\infty$ estimator of the desired level exists, and if not, such an estimator does not exist.

1.2.4 Infinite Horizon Problems

The H^2 and H$^\infty$ filters of Sec. 1.2.1 are finite-horizon filters, in the sense that they guarantee H^2-optimality, or H$^\infty$-suboptimality, over the finite time-interval, $j = 0, 1, \ldots i$, for some fixed i. It is also possible to consider infinite horizon problems where the time interval is either $[0, +\infty)$ (semi-infinite) or $(-\infty, +\infty)$ (doubly-infinite). In this case, it is reasonable to assume that the state-space model is time-invariant, *i.e.*,

$$\begin{cases} x_{i+1} = F x_i + G u_i \\ y_i = H x_i + v_i \end{cases}, \tag{1.2.32}$$

and that, in the H^2 case, the disturbance covariances are constant, *i.e.*, $Q_i = Q \geq 0$ and $R_i = R > 0$, for all i.

One approach for solving infinite-horizon time-invariant problems is to take the limit, as $i \to \infty$, of the finite-horizon solutions of Theorems 1.2.1 and 1.2.2. Further examination of the statements of these two theorems shows that this essentially amounts to studying the limit of P_i, *i.e.*, the solution to the Riccati recursions (1.2.9) and (1.2.18), as time progresses to infinity. Therefore, in this approach we need to demonstrate the following two items:

(i) That P_i, the solution to the Riccati recursions (1.2.9) and (1.2.18) converges to a limit, say P.

(ii) That the estimator corresponding to the Riccati variable P is indeed a solution to the infinite-horizon H^2 and H^∞ estimation problems.

This approach can indeed be used to solve the infinite-horizon H^2 estimation problem. For example, in Chapter 14, among other results, we show that under some mild (stabilizability and detectability) conditions, the solution of the Riccati recursion (1.2.9) converges exponentially to the unique non-negative definite solution of the discrete-time algebraic Riccati equation (DARE),

$$P = FPF^* + GQG^* - K_p R_e K_p^*, \tag{1.2.33}$$

with $K_p = FPH^* R_e^{-1}$ and $R_e = R + HPH^*$, for all non-negative definite, and even some indefinite, initial conditions, P_0. Once the convergence of P_i is established, one can further show that the estimator corresponding to the limiting value P is H^2-optimal, thereby establishing the following result.

Theorem 1.2.3 (Infinite-Horizon Kalman Filter) *Consider the time-invariant state-space model (1.2.32), and assume that $\{F, GQ^{1/2}\}$ is stabilizable and $\{F, H\}$ is detectable.*[4] *Then the solution to the infinite-horizon H^2 problem is given by*

$$\hat{s}_{j|j} = L\hat{x}_{j|j}, \tag{1.2.34}$$

where $\hat{x}_{j|j}$ satisfies the recursions,

$$\hat{x}_{j+1|j+1} = F\hat{x}_{j|j} + K_f(y_{j+1} - HF\hat{x}_{j|j}), \tag{1.2.35}$$

with

$$K_f = PH^* R_e^{-1} \quad , \quad R_e = R + HPH^* \tag{1.2.36}$$

and where P is the unique non-negative solution to the DARE,

$$P = FPF^* + GQG^* - K_p R_e K_p^*. \tag{1.2.37}$$

However, in the H^∞ case, the issue is much more difficult because R_e (as defined in (1.2.36)) is generally indefinite rather than non-negative definite. In fact, the aforementioned approach of taking the limit of the finite-horizon solution apparently cannot be used since

[4]The condition that $\{F, GQ^{1/2}\}$ be stabilizable (or, equivalently, that it be controllable on and outside the unit circle) can be replaced by the weaker assumption that $\{F, GQ^{1/2}\}$ be controllable on the unit circle. To show this, however, we have to go to a direct derivation (see Sec. 13.2.1), and not use a limiting argument.

the convergence of P_i (item (i) above) is difficult to establish without assuming beforehand that a solution to the infinite-horizon problem exists!

For this reason, we address infinite-horizon H^∞ problems directly, applying the approach discussed at the end of Sec. 1.2.1 to the quadratic form (1.2.16), but with $k = \infty$. However, for several reasons, we first present a different and initially more general alternative approach in which we formulate a general operator-theoretic version of both the H^2 and H^∞ problems. By appropriately specifying the operators involved, we shall see that this formulation applies equally to finite- and infinite-horizon, and discrete-time and continuous-time problems. Then we show that these operator-theoretic problems can be solved by a Wiener-Hopf type of technique. In particular, the key step is a "canonical" factorization of a generalized covariance/spectral function. This approach is developed in Chapter 13 and then specialized to the state-space case where, among other results, we show the following.

Theorem 1.2.4 (Infinite-Horizon H^∞ Filter) *Consider the time-invariant state-space model (1.2.32), and assume that $\{F, G\}$ is controllable on the unit circle and $\{F, H\}$ is detectable. Then an infinite-horizon H^∞ filter of level γ_f exists if, and only if, there exists a solution to the* DARE

$$P = FPF^* + GG^* - K_p R_e K_p^*, \tag{1.2.38}$$

with

$$K_p = FP \begin{bmatrix} H^* & L^* \end{bmatrix} R_e^{-1} \quad \text{and} \quad R_e = \begin{bmatrix} I_p & 0 \\ 0 & -\gamma_f^2 I_q \end{bmatrix} + \begin{bmatrix} H \\ L \end{bmatrix} P \begin{bmatrix} H^* & L^* \end{bmatrix} \tag{1.2.39}$$

such that

(i) $F_p \triangleq F - K_p \begin{bmatrix} H \\ L \end{bmatrix}$ *is stable.*

(ii) $P \geq 0$.

(iii) R_e *and* $\begin{bmatrix} I_p & 0 \\ 0 & -\gamma_f^2 I_q \end{bmatrix}$ *have the same inertia.*

If this is the case, then one possible H^∞ estimator is given by

$$\hat{s}_{j|j} = L\hat{x}_{j|j}, \tag{1.2.40}$$

where $\hat{x}_{j|j}$ satisfies the recursion,

$$\hat{x}_{j+1|j+1} = F\hat{x}_{j|j} + K_s(y_{j+1} - HF\hat{x}_{j|j}), \tag{1.2.41}$$

where

$$K_s = PH^*(I + HPH^*)^{-1}. \tag{1.2.42}$$

Note that, compared to Theorem 1.2.3, for the H^2 case, there are three additional requirements for the infinite-horizon H^∞ filter of Theorem 1.2.4. In the H^2 case, however, all three are automatically satisfied. Indeed, under stabilizability and detectability it can be shown that the DARE (1.2.37) always has a unique non-negative definite solution, thereby establishing condition (ii). Moreover, for this solution it can be shown that $F_p = F - K_p H$ is stable (note that L does not appear in the DARE (1.2.37)), and that $R_e = R + HPH^*$ is positive and so that it has the same inertia as I_p (note here that $q = 0$), thereby establishing conditions (i) and (iii).

It is also interesting to compare the three conditions of Theorem 1.2.4 with the condition of Theorem 1.2.2 required for the existence of a finite-horizon H^∞ filter. The inertia condition (1.2.17) is clearly the finite-horizon counterpart of condition (iii). Moreover, when (1.2.17) is satisfied, it can be shown that $P_i \geq 0$, which is the finite-horizon counterpart of condition (ii). However, condition (i) is a purely infinite-horizon condition and has no finite-horizon counterpart.

Finally, we should also mention that the issue of convergence of the Riccati variable P_i will be taken up in Chapter 14 where conditions on the initial condition P_0 are given such that the finite-horizon H^2 and H^∞ filters converge to infinite-horizon ones (assuming that the corresponding infinite-horizon solutions exist).

1.2.5 Factorization Approach to Infinite Horizon Problems

As mentioned earlier, one direct approach to infinite-horizon H^∞ problems is via canonical factorization; the objects being factorized are certain so-called *Popov* functions. In infinite-horizon H^2 problems, they are just power-spectral-density functions. However, it is interesting that Krein space arguments are useful for studying the problem of canonical spectral factorization even in this much studied case, as we shall demonstrate in this section. Moreover, although we shall begin with the Popov function in the H^2 case, our end result on canonical factorization is equally applicable to H^∞ problems where the Popov function under consideration is indefinite.

To this end, consider the time-invariant state-space model

$$\begin{cases} x_{i+1} &= Fx_i + u_i \\ y_i &= Hx_i + v_i \end{cases} \tag{1.2.43}$$

where F is stable, $\{F, H\}$ is observable[5] and the disturbances are zero-mean *stationary* random processes with

$$E\begin{bmatrix} u_i \\ v_i \end{bmatrix}\begin{bmatrix} u_j^* & v_j^* & 1 \end{bmatrix} = \begin{bmatrix} Q\delta_{ij} & S\delta_{ij} & 0 \\ S^*\delta_{ij} & R\delta_{ij} & 0 \end{bmatrix}.$$

[*Note that, for notational convenience, but without loss of generality, we are assuming* $G = I$.] Taking z-transforms, we can rewrite (1.2.43) as

$$y(z) = H(zI-F)^{-1}u(z) + v(z) = \begin{bmatrix} H(zI-F)^{-1} & I \end{bmatrix}\begin{bmatrix} u(z) \\ v(z) \end{bmatrix}. \tag{1.2.44}$$

Recall that if an $m \times 1$ stationary process $\{r_i\}$ is applied to a $p \times m$ stable linear system with transfer matrix $H(z)$ to yield an output process $\{s_i\}$, the so-called output z-spectrum, defined as the z-transform of the autocorrelation sequence,

$$S_s(z) = \mathcal{Z}\left\{Es_j s_{j-i}^*\right\} = \sum_{i=-\infty}^{\infty} Es_j s_{j-i}^* z^{-i}, \tag{1.2.45}$$

is given by

$$S_s(z) = H(z)S_r(z)H^*(z^{-*}), \tag{1.2.46}$$

where $S_r(z) = \mathcal{Z}\left\{Er_j r_{j-i}^*\right\}$ is the z-spectrum of $\{r_i\}$. Therefore, in our case, the output z-spectrum of $\{y_i\}$ is given by

$$S_y(z) = \begin{bmatrix} H(zI-F)^{-1} & I \end{bmatrix}\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix}\begin{bmatrix} (z^{-1}I-F^*)^{-1}H^* \\ I \end{bmatrix}. \tag{1.2.47}$$

[5] In Chapter 13 we shall see that both these conditions can be replaced with the less restrictive condition that $\{F, H\}$ is detectable.

Note that the matrix appearing in the center of (1.2.47) is the covariance of the disturbances $\{u_i, v_i\}$, so that we have

$$\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \geq 0. \tag{1.2.48}$$

This implies that $S_y(e^{j\omega}) \geq 0$, which is a defining property of a power spectral density matrix generated by a true stochastic process.

However, let us calculate the output spectrum in an alternative fashion. The steady-state covariance of the state x_i, defined by $\bar{\Pi} = \lim_{i\to\infty} Ex_i x_i^*$, satisfies the (discrete-time) Lyapunov equation

$$\bar{\Pi} = F\bar{\Pi}F^* + Q. \tag{1.2.49}$$

we note that the solution $\bar{\Pi}$ to (1.2.49) exists and is unique, since F is stable. Thus, in steady-state, the autocorrelation function of the output process is given by

$$R_{y,i} = Ey_j y_{j-i}^* = \begin{cases} HF^i\bar{\Pi}H^* + HF^{i-1}S & i > 0 \\ R + H\bar{\Pi}H^* & i = 0 \\ H\bar{\Pi}F^{*i}H^* + S^*F^{*(i-1)}H^* & i < 0 \end{cases}$$

Taking the z-transform of $R_{y,i}$ in the above expression, the output z-spectrum can be written as

$$S_y(z) = \begin{bmatrix} H(zI-F)^{-1} & I \end{bmatrix} \begin{bmatrix} 0 & F\bar{\Pi}H^* + S \\ H\bar{\Pi}F^* + S^* & R + H\bar{\Pi}H^* \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{bmatrix}. \tag{1.2.50}$$

Comparing (1.2.47) with (1.2.50) we see that the only difference between these two *representations* of the output z-spectrum is the matrix appearing in the center of these equations. In the case of (1.2.47) we saw that this matrix was the covariance of the disturbances $\{u_i, v_i\}$. However, in (1.2.50) the center matrix

$$\begin{bmatrix} 0 & F\bar{\Pi}H^* + S \\ H\bar{\Pi}F^* + S^* & R + H\bar{\Pi}H^* \end{bmatrix}, \tag{1.2.51}$$

is *indefinite*. Note that $S_y(e^{j\omega}) \geq 0$, of course, even though the center matrix (1.2.51) is not non-negative definite and cannot be thought of as the covariance of some random variables, say $\{u_i^{(1)}, v_i^{(1)}\}$. (Indeed $u_i^{(1)}$ would need to have zero variance but nonzero cross-variance with $v_i^{(1)}$!) However, conditioned by the discussions of the previous sections, we can be emboldened to *broaden our domain of discourse*, and instead of random variables, consider disturbances $\{u_i, v_i\}$ that belong to an abstract *indefinite* (so-called Krein) space. Then the matrix (1.2.51) can be considered as the covariance (more precisely, Gramian) of such an abstract disturbance $\{u_i^{(1)}, v_i^{(1)}\}$. To show that such a step has some value, we shall use this broadening of possibilities to explore the question of describing all *center matrices* that can give rise to the same output z-spectrum.

An Equivalence Class for Input Gramians

To this end, consider the state-space model (1.2.43) but now suppose that the inputs $\{u_i, v_i\}$ are such that

$$\left\langle \begin{bmatrix} u_i \\ v_i \end{bmatrix}, \begin{bmatrix} u_j \\ v_j \end{bmatrix} \right\rangle = \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \delta_{ij}, \tag{1.2.52}$$

where $Q = Q^*$, $R = R^*$ are Hermitian, but otherwise arbitrary, as is S. Note that we have replaced the notation $Eu_i v_j^*$ with $\langle u_i, v_j \rangle$ since we are now considering the $\{u_i, v_i\}$ to lie in an indefinite space so that the matrix appearing in (1.2.52) may be *indefinite*. Now associated with the state-space model (1.2.43) and the inputs (1.2.52), we may define the *Popov function*

$$\Sigma(z) \triangleq S_y(z) = \begin{bmatrix} H(zI-F)^{-1} & I \end{bmatrix} \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} (z^{-1}I-F^*)^{-1}H^* \\ I \end{bmatrix}. \tag{1.2.53}$$

We can readily see that the Popov function is a generalization of the z-spectrum, since

$$S_y(z) = \mathcal{Z}\left\{\langle y_j, y_{j-i}\rangle\right\}, \tag{1.2.54}$$

and the only difference is that now $S_y(z)$ need not be non-negative definite on the unit circle, $|z| = 1$.

Now suppose that we intend to add white and stationary disturbances $\{\bar{u}_i, \bar{v}_i\}$ (orthogonal to the original $\{u_i, v_i\}$) to the state-space model (1.2.43) such that the output z-spectrum $S_y(z)$ remains unchanged. In other words, the output of the state-space model

$$\left\{ \begin{array}{rcl} x_{i+1} + \bar{x}_{i+1} & = & F(x_i + \bar{x}_i) + u_i + \bar{u}_i \\ y_i + \bar{y}_i & = & H(x_i + \bar{x}_i) + v_i + \bar{v}_i \end{array} \right. \tag{1.2.55}$$

should still have Popov function equal to $S_y(z)$, given in (1.2.53).

The Gramian matrix of the new disturbances $\{u_i + \bar{u}_i, v_i + \bar{v}_i\}$ is given by

$$\begin{bmatrix} Q+\bar{Q} & S+\bar{S} \\ S^*+\bar{S}^* & R+\bar{R} \end{bmatrix},$$

and the new output z-spectrum by

$$S_{y+\bar{y}}(z) = \begin{bmatrix} H(zI-F)^{-1} & I \end{bmatrix} \begin{bmatrix} Q+\bar{Q} & S+\bar{S} \\ S^*+\bar{S}^* & R+\bar{R} \end{bmatrix} \begin{bmatrix} (z^{-1}I-F^*)^{-1}H^* \\ I \end{bmatrix}.$$

Now by linearity, $S_{y+\bar{y}}(z) = S_y(z) + S_{\bar{y}}(z)$. Therefore if $S_y(z)$ is to be unchanged, this implies that $S_{\bar{y}}(z)$, the z-spectrum of the process $\{\bar{y}_i\}$ defined by

$$\left\{ \begin{array}{rcl} \bar{x}_{i+1} & = & F\bar{x}_i + \bar{u}_i \\ \bar{y}_i & = & H\bar{x}_i + \bar{v}_i \end{array} \right. , \tag{1.2.56}$$

must be *zero*. Now a simple calculation shows that

$$\langle \bar{y}_i, \bar{y}_i\rangle = \bar{R} + H\langle \bar{x}_i, \bar{x}_i\rangle H^*, \tag{1.2.57}$$

so that if we define the Hermitian matrix $Z = -\langle \bar{x}_i, \bar{x}_i\rangle$, (note that since the variables in (1.2.56) belong to an indefinite metric space, Z is in general indefinite) we may write

$$\langle \bar{y}_i, \bar{y}_i\rangle = \bar{R} - HZH^* = 0, \tag{1.2.58}$$

or $\bar{R} = HZH^*$. Likewise, a similar computation for $i > j$, shows that

$$\langle \bar{y}_i, \bar{y}_j\rangle = HF^{i-j-1}(F\langle \bar{x}_i, \bar{x}_i\rangle H^* + \bar{S}) = HF^{i-j-1}(-FZH^* + \bar{S}). \tag{1.2.59}$$

Thus choosing

$$\bar{S} = FZH^* \tag{1.2.60}$$

we see that

$$\langle \bar{y}_i, \bar{y}_j\rangle = 0. \tag{1.2.61}$$

Finally, using the state equation in (1.2.56) we may write

$$-Z = -FZF^* + \bar{Q}. \tag{1.2.62}$$

Combining (1.2.58), (1.2.60) and (1.2.62) shows that the indefinite variables $\{\bar{u}_i, \bar{v}_i\}$ can have as Gramian matrix

$$\begin{bmatrix} \bar{Q} & \bar{S} \\ \bar{S}^* & \bar{R} \end{bmatrix} = \begin{bmatrix} -Z + FZF^* & FZH^* \\ HZF^* & HZH^* \end{bmatrix}, \tag{1.2.63}$$

for some Hermitian Z (which is the negative of the state Gramian matrix of the stationary process $\bar{x}_i$).

We summarize the discussion, and somewhat extend it, in the following lemma.

Lemma 1.2.1 (Equivalence Class for Input Gramians) .

(a) For any Hermitian Z, the output z-spectrum of the state-space model (1.2.43)

$$S_y(z) = \begin{bmatrix} H(zI-F)^{-1} & I \end{bmatrix} \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{bmatrix},$$

is invariant under the input Gramian transformation

$$\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \rightarrow \begin{bmatrix} Q - Z + FZF^* & S + FZH^* \\ S^* + HZF^* & R + HZH^* \end{bmatrix}. \tag{1.2.64}$$

(b) If for an observable system $\{F, H\}$, there exist input Gramians

$$\begin{bmatrix} Q_1 & S_1 \\ S_1^* & R_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} Q_2 & S_2 \\ S_2^* & R_2 \end{bmatrix}$$

that yield the same output spectrum, i.e.,

$$\begin{aligned} &\begin{bmatrix} H(zI-F)^{-1} & I \end{bmatrix} \begin{bmatrix} Q_1 & S_1 \\ S_1^* & R_1 \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{bmatrix} \\ &= \begin{bmatrix} H(zI-F)^{-1} & I \end{bmatrix} \begin{bmatrix} Q_2 & S_2 \\ S_2^* & R_2 \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{bmatrix} \end{aligned}$$

then there exists a unique Hermitian Z such that

$$\begin{bmatrix} Q_1 & S_1 \\ S_1^* & R_1 \end{bmatrix} = \begin{bmatrix} Q_2 - Z + FZF^* & S_2 + FZH^* \\ S_2^* + HZF^* & R_2 + HZH^* \end{bmatrix}.$$

Proof of Lemma 1.2.1: We have already proven part (a) by our Krein space arguments above. However, to reassure the reader, we note that one can directly show via a calculation that (1.2.64) is true for any Hermitian matrix Z — just check (after some algebra) that

$$0 = \begin{bmatrix} H(zI-F)^{-1} & I \end{bmatrix} \begin{bmatrix} -Z + FZF^* & FZH^* \\ HZF^* & HZH^* \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{bmatrix},$$

is true for any $Z = Z^*$.

For part (b), note that since $\{Q_1, R_1, S_1\}$ and $\{Q_2, R_2, S_2\}$ generate the same output z-spectrum we can write

$$\begin{bmatrix} H(zI-F)^{-1} & I \end{bmatrix} \begin{bmatrix} Q_1 - Q_2 & S_1 - S_2 \\ S_1^* - S_2^* & R_1 - R_2 \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{bmatrix} = 0.$$

Thus, if we define the indefinite variables $\{\bar{u}_i, \bar{v}_i\}$ such that

$$\langle \begin{bmatrix} \bar{u}_i \\ \bar{v}_i \end{bmatrix}, \begin{bmatrix} \bar{u}_j \\ \bar{v}_j \end{bmatrix} \rangle = \begin{bmatrix} \bar{Q} & \bar{S} \\ \bar{S}^* & \bar{R} \end{bmatrix} \delta_{ij} = \begin{bmatrix} Q_1 - Q_2 & S_1 - S_2 \\ S_1^* - S_2^* & R_1 - R_2 \end{bmatrix} \delta_{ij},$$

then the state-space model

$$\begin{cases} \bar{x}_{i+1} &= F\bar{x}_i + \bar{u}_i \\ \bar{y}_i &= H\bar{x}_i + \bar{v}_i \end{cases},$$

must have a null output z-spectrum. Using the arguments presented before the statement of the Lemma, this implies that

$$\begin{aligned} \bar{Q} &= -Z + FZF^* \\ \bar{R} &= HZH^* \\ 0 &= F^{i-j-1}(-FZH^* + \bar{S}) \quad \text{for } i > j \end{aligned}$$

where, as before, we have defined $\langle \bar{x}_i, \bar{x}_i \rangle = -Z$. Note that since F is stable the first of the above equations shows that Z is unique[6].

Moreover, the last equation shows that

$$\mathcal{O}(-FZH^* + \bar{S}) = 0,$$

where

$$\mathcal{O} = \begin{bmatrix} H^* & F^*H^* & F^{2*}H^* & \ldots \end{bmatrix}^*$$

is the observability map. When $\{F, H\}$ is observable, $\mathcal{O}$ is full rank and we conclude that

$$-FZH^* + \bar{S} = 0.$$

We have thus shown that there exists a unique Hermitian Z such that

$$\begin{bmatrix} \bar{Q} & \bar{S} \\ \bar{S}^* & \bar{R} \end{bmatrix} = \begin{bmatrix} Q_1 - Q_2 & S_1 - S_2 \\ S_1^* - S_2^* & R_1 - R_2 \end{bmatrix} = \begin{bmatrix} -Z + FZF^* & FZH^* \\ HZF^* & HZH^* \end{bmatrix}$$

from which the statement of part (b) follows.

■

Lemma 1.2.1 shows the great freedom that is obtained by allowing the disturbances $\{u_i, v_i\}$ to have an indefinite Gramian matrix. We were thus able to parametrize all input Gramians that give rise to the same Popov function in terms of a Hermitian matrix Z. This matrix has the interpretation of being the steady state Gramian of the state vector in a state-space model that generates zero output spectrum. [The reader at this point may want to verify that the choice $Z = \bar{\Pi}$ in (1.2.64), where $\bar{\Pi}$ is as in (1.2.49), relates the input covariances in (1.2.47) and (1.2.50).]

[6] In fact, we only require that F have no two eigenvalues $\{\lambda_i, \lambda_j\}$ such that $1 - \lambda_i\lambda_j^* = 0$ for the solution to the Lyapunov equation $Z = FZF^* + \bar{Q}$ to be unique.

Factorization of the Popov Function and Relations to the DARE

A major application of the degree of freedom available via the Hermitian matrix Z, is to choose Z such that the center matrix in the Popov function drops rank. In particular, let $Z = P$ be a Hermitian choice that makes the $(n+p) \times (n+p)$ center matrix have rank p, *i.e.*,

$$\begin{bmatrix} Q - P + FPF^* & S + FPH^* \\ S^* + HPF^* & R + HPH^* \end{bmatrix} = \begin{bmatrix} K_p \\ I \end{bmatrix} R_e \begin{bmatrix} K_p^* & I \end{bmatrix}, \tag{1.2.65}$$

where K_p and R_e are are $n \times p$ and $p \times p$ matrices, respectively. [Note that, since the Popov function $S_y(z)$ is a $p \times p$ transfer matrix, we cannot, in general, hope to reduce the rank beyond p.]

Now making the rank of the center matrix equal to p is of significance, since it leads to the following factorization of the Popov function

$$S_y(z) = \left[H(zI - F)^{-1}K_p + I\right] R_e \left[H(z^{-*}I - F)^{-1}K_p + I\right]^*. \tag{1.2.66}$$

We refer to the above as a factorization of the Popov function since the three factors $H(zI - F)^{-1}K_p + I$, R_e and $K_p^*(z^{-1}I - F^*)^{-1}H^* + I$ are all $p \times p$ matrices. In particular, when the transfer matrix $H(zI - F)^{-1}K_p + I$ has a causal and bounded inverse, *i.e.*, when

$$\left[I + H(zI - F)^{-1}K_p\right]^{-1} = I - H(zI - F + K_pH)^{-1}K_p, \tag{1.2.67}$$

is analytic for all $|z| \geq 1$, the above factorization is known as the *canonical* factorization of the Popov function. As we shall see in Chapter 10, the canonical factorization is a key element in both H^2 and H^∞ estimation and control and will be the route we take to obtain infinite horizon results. In the H^2 case, $S_y(z)$ is positive definite on the unit circle, which implies that $R_e > 0$. In the H^∞ case, $S_y(z)$ is indefinite, which implies that R_e is indefinite. Thus, assuming that R_e is Hermitian and has the same inertia as the signature matrix J, we may write $R_e = R_e^{1/2} J R_e^{*/2}$, and hence

$$S_y(z) = \left[H(zI - F)^{-1}K_pR_e^{1/2} + R_e^{1/2}\right] J \left[H(z^{-*}I - F)^{-1}K_pR_e^{1/2} + R_e^{1/2}\right]^*. \tag{1.2.68}$$

The above factorization is referred to as a "J-spectral factorization".

At this point it is also possible to point out the connection between factorizations of the Popov function and solutions of discrete-time algebraic Riccati equations. To this end, note that we can rewrite (1.2.65) as

$$\left\{ \begin{array}{rcl} Q - P + FPF^* & = & K_pR_eK_p^* \\ S + FPH^* & = & K_pR_e \\ R + HPH^* & = & R_e \end{array} \right. \tag{1.2.69}$$

When R_e is invertible, we have $K_p = (FPH^* + S)R_e^{-1}$, and that P satisfies the DARE

$$P = FPF^* + Q - K_pR_eK_p^*. \tag{1.2.70}$$

What we have therefore shown is that if P is a Hermitian solution to the above DARE then the center matrix in (1.2.65) drops rank and a factorization of the Popov function is obtained. Moreover, in view of (1.2.67), if the Hermitian solution P is such that $F - K_pH$ is stable (*i.e.*, has all its eigenvalues strictly inside the unit circle), then the corresponding factorization is canonical. In fact, this connection will allow us to treat the positive (semi-)definite and indefinite cases in a unified fashion in Chapter 13, and to obtain general existence results for solutions of Riccati equations in the (possibly) indefinite case.

A Digression: The KYP Lemma

The results of Lemma 1.2.1 are independent of whether or not $\{y_i\}$ is a *true* stochastic process, *i.e.*, whether or not its z-spectrum, $S_y(z)$, is nonnegative on the unit circle. When in fact that is true, we have a further strong result: the KYP Lemma.

Theorem 1.2.5 (KYP **Lemma**) *Consider the observable pair* $\{F, H\}$*. Then the following two statements are equivalent:*

(i) $S_y(z) \geq 0$ *for all* $z = e^{j\omega} \notin \lambda(F)$[7]*, where*

$$S_y(z) = \begin{bmatrix} H(zI-F)^{-1} & I \end{bmatrix} \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{bmatrix}.$$

(ii) There exists a Hermitian Z *such that*

$$\begin{bmatrix} Q - Z + FZF^* & S + FZH^* \\ S^* + HZF^* & R + HZH^* \end{bmatrix} \geq 0. \tag{1.2.71}$$

The point is the following. Recall that in view of Lemma 1.2.1, we may write, for any Hermitian Z,

$$S_y(z) = \begin{bmatrix} H(zI-F)^{-1} & I \end{bmatrix} \begin{bmatrix} Q - Z + FZF^* & S + FZH^* \\ S^* + HZF^* & R + HZH^* \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{bmatrix}.$$

Thus, Theorem 1.2.5 states that $S_y(z)$ is a true z-spectral density function (*i.e.*, it is nonnegative definite on the unit circle) if, and only if, there exists true stochastic inputs with nonnegative definite covariance

$$\begin{bmatrix} Q - Z + FZF^* & S + FZH^* \\ S^* + HZF^* & R + HZH^* \end{bmatrix} \geq 0,$$

that generate it! This has special significance for the problem of stochastic realization since it states that any nonnegative definite rational z-spectral density function can be realized by a finite-dimensional state-space model driven by true stochastic processes. The Theorem also gives a recipe as to how to find this realization via feasible solutions of the *linear matrix inequality* (LMI), (1.2.71).

In Appendix 3.A we shall show how it is possible to use simple Krein space geometry ideas to give a further interpretation, as well as a proof, of a time-variant counterpart of the KYP lemma.

1.2.6 H^2 and H^∞ Control Problems

The methods described so far can also be used to study control problems with quadratic (or quadratically-induced) cost functions. Here we briefly indicate why this is true. Our approach is further elaborated in Chapter 8 and is based on the fact that the H^2 and H^∞ (full-information) control problems are dual, in a precise sense, to H^2 and H^∞ estimation problems. Therefore the control solutions can essentially be written down by inspection.

[7] $\lambda(F)$ denotes the spectrum, or set of eigenvalues, of the matrix F.

Full Information Control

We begin by introducing the full information control problem for the time-variant state-space model

$$x_{i+1} = F_i x_i + G_{1,i} w_i + G_{2,i} u_i, \quad 0 \le i \le N \tag{1.2.72}$$

where $F_i \in \mathcal{C}^{n\times n}$, $G_{1,i} \in \mathcal{C}^{n\times m_1}$, and $G_{2,i} \in \mathcal{C}^{n\times m_2}$ are known matrices, w_i is an exogenous input and u_i is the control input. The $\{w_i\}$ may be interpreted as process noise or driving disturbance. Moreover, suppose we are given a linear combination of the states,

$$s_i = L_i x_i, \tag{1.2.73}$$

where $L_i \in \mathcal{C}^{q\times n}$ is known, that we intend to *regulate* using the control signal u_i. In the full-information control problem it is assumed that the control signal at time i has access to all the exogenous inputs up to time i, *i.e.*, the initial condition x_0 and the inputs $\{w_j, j \le i\}$. In other words, we may write

$$u_i = \mathcal{F}(x_0, w_0, \ldots, w_i). \tag{1.2.74}$$

In control, the goal is to "cost-effectively" keep the regulated signal $\{s_i\}$ small. By this we mean that we intend to keep $\{u_i\}$ and $\{s_i\}$ simultaneously "small". A popular measure for smallness is the *quadratic cost function*,

$$\sum_{j=0}^{N} u_i^* Q_i^c u_i + \sum_{j=0}^{N} s_i^* R_i^c s_i + x_{N+1}^* P_{N+1}^c x_{N+1}, \tag{1.2.75}$$

where $Q^c > 0$, $R^c \ge 0$, and $P_{N+1}^c \ge 0$ are given. Note that we have introduced a penalty term for the final state x_{N+1}, since we may want to keep it small as well.

The H^2 Approach: Stochastic Disturbance Processes

As mentioned earlier, in the H^2 approach it is assumed that the unknown disturbances are zero-mean random variables with known second-order statistics. Here the disturbances are $\{x_0, \{w_i\}\}$, whose second-order statistics, or variances and covariances, are given by

$$E \begin{bmatrix} x_0 \\ w_i \end{bmatrix} \begin{bmatrix} x_0^* & w_j^* \end{bmatrix} = \begin{bmatrix} \Pi_0 & 0 \\ 0 & Q_i \delta_{ij} \end{bmatrix}. \tag{1.2.76}$$

Once the initial state and disturbances are zero-mean random variables, the same will be true of all the other quantities of interest in (1.2.72), *i.e.*, the state variables $\{x_i\}$, the regulated signals $\{s_i\}$, and the control inputs $\{u_i\}$.

In particular, the quadratic cost (1.2.75) will also be a random variable. Therefore the goal in H^2 control is to find a *linear* control strategy $u_i = \mathcal{F}(x_0, w_0, \ldots, w_i)$ that minimizes the *expected* quadratic cost, *i.e.*, the average value of (1.2.75). The precise statement follows.

Problem 1.2.3 (Full-Information H^2 Control Problem) *Consider the standard state-space model (1.2.72) along with the stochastic assumptions (1.2.76). Find a linear full-information control strategy $u_i = \mathcal{F}(x_0, w_0, \ldots, w_i)$ that minimizes the expected quadratic cost,* i.e.,

$$\min_{\mathcal{F}} \; E \left[\sum_{j=0}^{N} u_i^* Q_i^c u_i + \sum_{j=0}^{N} s_i^* R_i^c s_i + x_{N+1}^* P_{N+1}^c x_{N+1} \right]. \tag{1.2.77}$$

The solution to the above problem is well-known [Kal60a], [AM71], [KS72], and is quoted below.

Theorem 1.2.6 (Full-Information H^2 Controller) *The solution to Prob. 1.2.3 is given by*

$$u_j = -K_j^{c*}(F_j x_j + G_{1,j} w_j), \quad j = 0, 1, \ldots N, \tag{1.2.78}$$

with

$$K_j^c = P_{j+1}^c G_{2,j} R_{e,j}^{-c} \quad \textit{and} \quad R_{e,j}^c = Q_j^c + G_{2,j}^* P_{j+1}^c G_{2,j} \tag{1.2.79}$$

and where P_j^c satisfies the backwards Riccati recursion,

$$P_j^c = F_j^* P_{j+1}^c F_j + L_j^* R_j^c L_j - F_j^* P_j^c G_{2,j} R_{e,j}^{-c} G_{2,j}^* P_j^c F_j, \quad P_{N+1}^c. \tag{1.2.80}$$

The H^∞ Approach: Deterministic Disturbance Processes

As menioned earlier, in the H^∞ approach it is assumed that the disturbances $\{x_0, \{w_i\}\}$ are *nonrandom.* One cannot therefore speak of expected values, or attempt to minimize the average quadratic cost. Instead one can look at the normalized cost,

$$\frac{\sum_{j=0}^N u_j^* Q_j^c u_j + \sum_{j=0}^N s_j^* R_j^c s_j + x_{N+1}^* P_{N+1}^c x_{N+1}}{x_0^* \Pi_0{}^{-1} x_0 + \sum_{j=0}^i w_j^* Q_j^{-1} w_j}\ , \tag{1.2.81}$$

which can also be interpreted as the energy gain from the disturbances, $\Pi_0^{-1/2} x_0$ and $\{Q_j^{-1/2} u_j\}_{j=0}^i$, to the control signal, $\{Q_j^{c*/2} u_j\}_{j=0}^i$, the regulated signal, $\{R_j^{c*/2} s_j\}_{j=0}^i$, and the final cost $P_{N+1}^{c*/2} x_{N+1}$. [Here Π_0 and Q_j are positive definite weighting matrices.] It is quite clear that if the ratio in (1.2.10) is small then the controller performs well, and vice-versa. However, the problem with this ratio is that it depends on the disturbances $\{x_0, \{w_j\}\}$. To overcome this deficiency, we can consider the worst-case, *i.e.*,

$$\sup_{x_0, \{w_j\}} \frac{\sum_{j=0}^N u_j^* Q_j^c u_j + \sum_{j=0}^N s_j^* R_j^c s_j + x_{N+1}^* P_{N+1}^c x_{N+1}}{x_0^* \Pi_0{}^{-1} x_0 + \sum_{j=0}^i w_j^* Q_j^{-1} w_j}\ , \tag{1.2.82}$$

which represents the worst-case gain from the disturbance energy to the quadratic cost. The goal in H^∞ control is to minimize this worst-case gain, and the claim is that the resulting controllers will be robust with respect to disturbance variations, since no statistical assumptions are being made about the disturbances. However, the resulting controllers may of course be over-conservative.

As with H^∞ estimation, there are only a few cases where the worst-case energy gain can be explicitly minimized over $\mathcal{F}$. Therefore one normally relaxes the minimization condition and considers the following suboptimal problem.

Problem 1.2.4 (Suboptimal Full-Information H^∞ Control) *Consider the state-space model (1.2.72), and a given scalar $\gamma > 0$. Determine whether it is possible to bound the worst-case energy gain in (1.2.82) by γ^2,* i.e., *whether it is possible to find a full-information control strategy $u_j = \mathcal{F}(x_0, w_0, \ldots, w_j)$ that achieves*

$$\sup_{x_0, \{w_j\}} \frac{\sum_{j=0}^N u_j^* Q_j^c u_j + \sum_{j=0}^N s_j^* R_j^c s_j + x_{N+1}^* P_{N+1}^c x_{N+1}}{x_0^* \Pi_0{}^{-1} x_0 + \sum_{j=0}^i w_j^* Q_j^{-1} w_j} < \gamma^2. \tag{1.2.83}$$

If this is the case, then find one such full-information H^∞ controller of level γ.

Following an argument similar to what was presented in Sec. 1.2.1, it is possible to show that the solution to the full-information H^∞ control problem is intimately related to guaranteeing the positivity of an indefinite quadratic form. However, we shall not repeat the details here and shall defer this discusion to Chapter 9. Instead let us quote the solution (see also [BB95], [GL95], [ZDG96]).

Theorem 1.2.7 (Suboptimal Full Information H^∞ Controller) *A solution to Problem 1.2.4 exists if, and only if,*

(i) $\Pi_0^{-1} - \gamma^{-2} P_0^c > 0$

(ii) The matrices

$$\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \quad \text{and} \quad R_{e,i}^c = \begin{bmatrix} Q_i^c + G_{2,i}^* P_{i+1}^c G_{2,i} & G_{2,i}^* P_{i+1}^c G_{1,i} \\ G_{1,i}^* P_{i+1}^c G_{2,i} & -\gamma^2 Q_i^w + G_{1,i}^* P_{i+1}^c G_{1,i} \end{bmatrix}$$

have the same inertia for all $i = 0, 1, \ldots, N$, where P_i^c satisfies the backwards Riccati recursion,

$$P_i^c = F_i^* P_{i+1}^c F_i + L_i^* R_i^c L_i - F_i^* P_{i+1}^c \begin{bmatrix} G_{2,i} & G_{1,i} \end{bmatrix} (R_{e,i}^c)^{-1} \begin{bmatrix} G_{2,i}^* \\ G_{1,i}^* \end{bmatrix} P_{i+1}^c F_i, \quad (1.2.84)$$

initialized with P_{N+1}^c. If this is the case, then one possible full-information H^∞ controller is given by

$$\bar{u}_i = -K_j^{c*}(F_i x_i + G_{1,i} w_i). \quad (1.2.85)$$

where

$$K_j^c = P_{i+1}^c G_{2,i} (R_{G^c,i})^{-1} \quad \text{and} \quad R_{G^c,i} = Q_i^c + G_{2,i}^* P_{i+1}^c G_{2,i}. \quad (1.2.86)$$

Duality to Estimation

Comparing the results of Theorems 1.2.1 and 1.2.6 on H^2 estimation and control, and Theorems 1.2.2 and 1.2.7 on H^∞ estimation and control, demonstrates the duality between the solutions to H^2 and H^∞ estimation and control problems. Indeed, if in Theorem 1.2.1, we make the transformations

$$F_i \to F_i^* \ , \ G_i \to L_i^* \ , \ H_i \to G_{2,1}^* \ , \ Q_j \to R_j^c \ , \ R_j \to Q_j^c$$

and reverse forwards-time to backwards-time, then the Riccati recursion (1.2.9) for P_i is transformed to the Riccati recursion (1.2.80) for P_i^c. Moreover, under these transformations, the gain matrix $K_{f,i}$ (of the estimation solution) becomes the negative conjugate transpose of the gain matrix $-K_i^{c^*}$ (of the control solution). Finally, the reader may note that a similar transformation relates the Riccati recursions, the conditions for existence, and the gain matrices, of the solutions of Theorems 1.2.2 and 1.2.7.

In view of the above duality in the solutions, it may be expected that the original problems themselves should be dual in some sense, and, in fact, in Chapters 8 and 11 we shall see that this is the case. In particular, in Chapter 8 we show that the duality between estimation and control problems can be explained in terms of the concepts of dual bases and dual spaces for (definite- and indefinite-metric) linear spaces; once we show this duality, the solution to the control problem can be directly written down knowing the solution to the estimation problem, as will be done in Chapter 9.

Measurement Feedback Control

In the full information control problem it is assumed that the control signal u_i has access to the initial condition x_0 and the past and current values of the exogenous inputs $\{w_j, j \leq i\}$. A more realistic control problem is one for which the control signal u_i has access to past and current values of a certain *observations* (or measurement) signal,

$$y_i = H_i x_i + v_i, \tag{1.2.87}$$

i.e., u_i has access to $\{y_j, j \leq i\}$, where $\{v_j\}$ is an unknown additive disturbance. Such a control problem is referred to as a *measurement feedback* control problem.

If we further assume that the additive disturbance $\{v_j\}$ is a zero-mean random process with known second-order statistics,

$$E \begin{bmatrix} x_0 \\ w_i \\ v_i \end{bmatrix} \begin{bmatrix} x_0^* & w_j^* & v_j^* \end{bmatrix} = \begin{bmatrix} \Pi_0 & 0 & 0 \\ 0 & Q_i\delta_{ij} & 0 \\ 0 & 0 & R_i\delta_{ij} \end{bmatrix}, \tag{1.2.88}$$

we can study the H^2 measurement feedback control problem that attempts to choose $u_i = \mathcal{F}(y_0, \ldots, y_i)$ so as to minimize the expected quadratic cost,

$$\min_{\mathcal{F}} \; E\left[\sum_{j=0}^{N} u_i^* Q_i^c u_i + \sum_{j=0}^{N} s_i^* R_i^c s_i + x_{N+1}^* P_{N+1}^c x_{N+1}\right]. \tag{1.2.89}$$

Likewise, if assume $\{v_j\}$ is nonrandom, we can consider the H^∞ measurement feedback control problem that attempts to choose $u_i = \mathcal{F}(y_0, \ldots, y_i)$ so as to bound the worst-case energy gain,

$$\sup_{x_0, \{w_j, v_j\}} \frac{\sum_{j=0}^{N} u_j^* Q_j^c u_j + \sum_{j=0}^{N} s_j^* R_j^c s_j + x_{N+1}^* P_{N+1}^c x_{N+1}}{x_0^* \Pi_0^{-1} x_0 + \sum_{j=0}^{i} w_j^* Q_j^{-1} w_j + \sum_{j=0}^{i} v_j^* R_j^{-1} v_j} < \gamma^2. \tag{1.2.90}$$

An interesting feature of both the H^2 and H^∞ measurement feedback control problems is that they can be solved via a two-step procedure known as the *separation principle*: we first solve a full information control problem, assuming the states of the system are known, and then use the observations sequence $\{y_j\}$ to estimate the unknown states. The desired control signal is then based on our estimates of the state, rather than on the inaccessible state itself. In this manner the measurement feedback control problem is separated into a full information control problem and an estimation problem. [Further details of the separation principle will be given in Chapters 9 and 11.]

1.3 Outline of the Contents

For ease of reference, we now briefly review the material in the succeeding chapters. We also note that browsing through the opening paragraphs of each chapter will provide additional perspective on the scope and contributions of this monograph.

Chapter 2: Linear Estimation in Krein Spaces

Chapters 2 and 3 present a self-contained theory for linear estimation in Krein spaces. Chapter 2 begins with a brief introduction to Krein spaces, and then proceeds to study Krein spaces and to compare their geometrical properties (such as the existence of projections onto linear subspaces) with the corresponding geometrical properties of Hilbert (or definite metric) spaces. An important consequence is the relation between Krein space projections and the computation of stationary points of certain indefinite quadratic forms. In particular, while Hilbert space projections always minimize definite quadratic forms, it is shown that Krein space projections achieve (stochastic or deterministic) minima if, and only if, certain additional conditions are met. We shall see that it is these special ways in which Hilbert and Krein speces differ that will mark out the differences between the conventional H^2 and more recent H^∞ theories.

Chapters 3: State-Space Models in Krein Space

Here the results of Chapters 2 are specialized to state-space models, or, more specifically, to linear state-space models driven by inputs that lie in a Krein space. This then leads to the centerpiece of the ensuing theory, viz., a Krein space generalization of the classical Kalman filter, obtained using the innovations approach. Then, by noting a certain so-called partial equivalence between stochastic and deterministic problems, we show that this Krein space Kalman filter allows one to recursively compute the stationary point of certain indefinite quadratic forms, or, equivalently, to recursively compute the canonical factorization of certain indefinite Gramians (or transfer operators). This is the reason that Kalman filter theory plays such an important role in the solution of problems in H^2 and H^∞ estimation and control, quadratic game theory, risk-sensitive optimization and adaptive filtering.

The results presented in Chapters 2 and 3 will be repeatedly used throughout the remaining chapters.

Chapter 4: Finite-Horizon H^∞ Filtering

In this chapter, we apply the results of Chapters 2 and 3 to solve the H^∞ a posteriori and a priori filtering, as well as the smoothing, estimation problems and parametrize all possible solutions. Although most of the results on H^∞ filtering, apart from, say, certain equivalent conditions for the existence of solutions, are not new, the approach and the derivations are. The new approach is also used to solve the l-step ahead H^∞ prediction problem, which seems not to have been considered before.

Chapter 5: Array Algorithms

A major bonus of our approach to quadratic estimation and control problems is that, apart from rather more transparent derivations of existing results, it shows a way to apply to the H^∞ (and various other) settings many of the results developed for Kalman filtering and LQG control over the last three decades. Chapter 5 is the first time that we truly deliver on this claim by developing square-root array and Chandrasekhar-Kailath-Morf array algorithms for H^∞ filtering problems.[8] The H^∞ square-root array algorithms involve propagating the *indefinite* square-roots of the quantities of interest and have the property that the appropriate inertia of these quantities is preserved. For systems that are constant, or whose time-variation is structured in a certain way, the Chandrasekhar-Kailath-Morf array algorithms allow a reduction in the computational effort per iteration from $O(n^3)$ to $O(n^2)$, where n is the number of states. The H^∞ square-root and Chandrasekhar-Kailath-Morf array algorithms both have the interesting feature that one does not need to explicitly check for the inertia conditions required for the existence of H^∞ filters. These conditions are built

[8] In the H^2 case, these algorithms are known to have many useful features (see [KSH98]). They seem to be difficult to obtain via the earlier approaches to the H^∞ problems.

into the algorithms themselves so that an H^∞ estimator of the desired level exists if, and only if, the algorithms can be executed. However, it remains to be studied how the favorable numerical performance of the array algorithms in the H^2 case can be carried over to H^∞ problems.

Chapter 6: Several Related Problems

Here we present applications of the Krein space estimation theory of Chapters 2 and 3 to risk-sensitive filtering, quadratic game-theoretic filtering, and finite-memory adaptive filtering, and some other problems. The point is that all these problems can be cast into the problem of calculating the stationary point of a certain indefinite quadratic form. Therefore, by considering the appropriate state space models and error Gramians, we can use Krein space estimation theory to calculate these stationary points and to study their properties. Although many of the connections between H^∞, risk-sensitive and quadratic game-theoretic estimation and control are known in the literature, the material of this chapter sheds further light on these connections and provides a new perspective on these problems.

Chapter 7: H^∞-Optimality of the LMS Algorithm

Chapter 7 uses the connection between adaptive filtering and state-space estimation to study adaptive filtering with an H^∞ criterion. In particular, it is shown that the celebrated LMS (least-mean-squares) adaptive algorithm is H^∞-optimal. The LMS algorithm has been long regarded as an *approximate* solution to either a stochastic or a deterministic least-squares problem, and it essentially amounts to updating the weight vector estimates along the direction of the instantaneous gradient of a quadratic cost function. In this chapter it is shown that LMS can be regarded as the exact solution to a minimization problem in its own right. Namely, it is established that it is a minimax filter: it minimizes the maximum energy gain from the disturbances to the prediction errors, while the closely related so-called normalized LMS algorithm minimizes the maximum energy gain from the disturbances to the filtered errors. Moreover, the results of Chapter 6 show that they also minimize a certain exponential cost function and are thus also risk-sensitive optimal. The various implications of these results are also discussed, and it is shown how they provide theoretical justification for the widely observed excellent robustness properties of the LMS filter.

In order to compare the robustness of other adaptive filtering algorithms with the (H^∞-optimal) LMS and normalized LMS algorithms, Appendix 7.B examines the robustness of least-squares-based adaptive filters, such as the RLS algorithm, from the H^∞ point of view. The basic result is the derivation of certain upper and lower bounds for the H^∞ norm of the RLS algorithm (and in fact, more generally, of the Kalman filter) with respect to prediction and filtered errors. The main conclusion is that, unlike LMS and normalized LMS which do not allow for any amplification of the disturbances, the RLS algorithm does allow for such amplification. This fact can be especially pronounced in the prediction error case. Moreover, it is also shown that the H^∞ norm for RLS is data-dependent, whereas for LMS and normalized LMS it is not so because the H^∞ norm is simply unity.

Chapter 8: Duality

Kalman (1960) was the first to point out that the solutions of the quadratic regulator control problem and the linear least-mean-squares estimation problem for state-space models can be regarded as being "dual" in a certain sense. However, few textbooks explore the underlying reasons for this duality, or note that we can have dualities between different estimation problems (or different control problems). In this chapter, we introduce the concept of duality through the geometrical notion of dual bases for linear spaces spanned by a set of nonorthogonal basis vectors. We also study the consequences of duality when the underlying spaces have state-space structure, and show that the dual bases inherit a form of state-space structure. In particular, depending on the choice of the original basis, we show that the dual bases can have backwards-time, forwards-time, or mixed, so-called complementary state-space models. [The backwards-time complementary model, for example, is closely related to the adjoint system.] We also discuss some of the applications of duality to smoothing problems. The main application of the dual approach to quadratic problems, however, is in control, which is the subject of the next chapter.

Chapter 9: Finite-Horizon Control Problems

Chapter 9 is devoted to finite-horizon H^2 and H^∞ control problems. The main building block in our study is the *indefinite* LQR (linear-quadratic-regulator) problem, whose solution is most naturally obtained by appealing to the duality of Chapter 8. Using the solution to the indefinite LQR problem we then solve the finite-horizon full-information and measurement-feedback H^2 and H^∞ control problems. The full-information control problem is recognized to be the dual of the estimation problem studied in Chapter 4, whereas the solution to the measurement-feedback problem requires a certain a two-step procedure which effectively *separates* the problem into a full-information control problem and an estimation problem.

Chapter 10: Input-Output Aproach to H^2 and H^∞ Estimation

The various estimation and control problems solved in the previous chapters are all *finite-horizon* problems. To extend these solutions to the infinite-horizon case, it would appear that we must resort to one of the following two strategies: (i), develop a theory for linear estimation in Krein spaces with an infinite number of observations, along the lines of Chapters 2 and 3, and then use the same type of geometric arguments as in Chapters 4-9 to obtain the infinite-horizon solutions, or (ii), study the limiting behavior of the finite-horizon solutions as time progresses to infinity, and thereby obtain the infinite-horizon solutions.

Although the first approach is interesting, it becomes somewhat complicated because of the infinite-dimensional Krein spaces that arise, and the crucial questions of existence, boundedness, and stability that come with them. The second approach, as mentioned earlier in Sec. 1.2.4, runs into problems, since to establish the convergence of the finite-horizon H^∞ solutions we need to first assume the existence of an infinite-horizon solution.

Therefore, to solve the infinite-horizon problem, a fresh outlook seems helpful, and so in Chapters 10 and 11 we take a more fundamental input-output approach to estimation and control. This approach has the benefit that it allows the structure and properties of the H^2 and H^∞ solutions (such as the parametrization of all solutions, the separation principles, the maximum entropy properties, etc.) to be revealed and understood in their most transparent form, and without specifying the finite/infinite-horizon, time-variant/invariant, finite/infinite-dimensional, or even discrete-time/continuous-time nature of the problem. [These properties are at times obscured by the fine details that occur in specific problems.] In this framework the solutions to H^2 and H^∞ problems are obtained from the canonical factorization of certain (possibly indefinite) transfer operators. This serves as yet another illustration of the unification of H^2 and H^∞ problems achieved via considering indefinite metric spaces. It also motivates us to further consider state-space structure which enables the required canonical factorizations to be explicitly found.

Chapter 11: Input-Output Aproach to H^2 and H^∞ Control

While Chapter 10 studies the estimation problem at the general input-output level of transfer operators, Chapter 11 is concerned with control problems. The various H^2 and H^∞, full-information and measurement-feedback, problems, along with their respective stochastic and deterministic interpretations, and noncausal and causal (optimal and and suboptimal) solutions are described. As mentioned earlier, the solutions are described at the level of transfer operators and require certain canonical factorizations.

Chapter 12: The Discrete-Time Algebraic Riccati Equation

Chapter 13 studies the celebrated discrete-time algebraic Riccati equation (DARE) which arises in an impressive range of applications in state-space systems and control theory. The main reason is that solutions to the DARE allow one to perform the canonical factorization of certain Popov functions, which is also a required step in the solution of infinite horizon H^2 and H^∞ solutions.

Although a great deal is known about the Riccati equation when the coefficient matrices are positive semi-definite, much less is known when these coefficients are indefinite matrices. In this chapter the DARE is considered in the full generality of this, so-called, indefinite case and the results are then particularized to some important special cases. In addition to the aforementioned H^2 and H^∞ cases, these include the KYP, bounded-real, and positive real lemmas, as well as the celebrated Nehari problem. The main result is that solutions to the DARE, or more precisely a system of algebraic Riccati equations (SDARE), exists if, and only if, a certain proper factorization of the Popov function exists. Additional conditions are then given under which the solution to the DARE becomes stabilizing, Hermitian, positive semi-definite, etc. We also relate the solutions of the DARE to a so-called symplectic matrix, and show that the wellknown invariant subspace method can be used to compute solutions of the DARE in the indefinite case as well.

Chapter 13: Infinite Horizon Results for State-Space Models

The results of Chapters 10 and 12 are used to study the infinite horizon H^2 and H^∞ estimation and control problems for time-invariant state-space models. This is the case most studied in the literature, and so in this chapter we make a more detailed comparison with earlier results and approaches.

Chapter 14: Asymptotic Behavior

Our solution to the infinite-horizon H^2 and H^∞ estimation and control problems were obtained by studying the infinite-horizon problem directly, rather than by studying the limiting behavior of the finite-horizon solution. Therefore, the question of the asymptotic behavior of the finite-horizon H^2 and H^∞ solutions remains, which is the subject of this chapter.

Assuming a time-invariant state-space structure, the main objective is to find conditions under which, for a given initial condition, the solution to the Riccati recursion converges to a solution of the associated DARE. The main result states that if a stabilizing solution to the DARE exists, and if a certain sequence of matrices is uniformly nonsingular, then the Riccati recursion converges (exponentially) to the unique stabilizing solution. In the general case, the aforementioned nonsingularity conditions need to be recursively checked. However, in some special cases they can be reduced to more simple and more explicit requirements on the initial condition. In particular, when the coefficient matrices of the Riccati recursion are positive semi-definite, as is the case in H^2 problems, we can guarantee the convergence of the Riccati recursion for all positive semi-definite, and even some indefinite, initial conditions.

Moreover, in the case frequently encountered in H^∞ filtering and control, we can guarantee convergence for all positive semi-definite initial conditions that are less than or equal to a certain positive semi-definite matrix.

Chapter 15: Optimal H^∞ Solutions

Most of the results in the earlier chapters are devoted to the study of so-called suboptimal H^∞ problems, on the ground that optimal solutions were known only in a handful of cases. However, this is perhaps too pessimistic a conclusion. In Chapter 7 we presented optimal H^∞ solutions for the adaptive filtering problem. In Chapter 15 we show that optimal H^∞ solutions can also be obtained for several other interesting problems, including equalization, full-information and measurement-feedback tracking, filtering signals in additive noise, and one-step-ahead prediction. For more general H^∞ estimation and control problems, we obtain a formula for the optimal H^∞ norm. While this formula does not always lead to simple frequency-domain characterizations, it does give considerable insight into the "structure" of estimation and control problems, and, in particular, to the questions of "worst-case complete estimability" and "worst-case non-estimability", viz., the questions of whether a given estimation or control problem is "easy" or "difficult" to solve.

Chapter 16: Final Remarks

The results presented in the earlier chapters were almost all exclusively devoted to discrete-time systems. In this chapter we very briefly outline how these results can be generalized to continuous-time systems, bearing in mind that a more detailed analysis requires another occasion. As a general principle, we note that the input-output operator-theoretic approaches of Chapters 10 and 11 do not differ substantially in the discrete-time and continuous-time cases, whereas the state-space approaches (as in Chapters 3, 4 and 9) do exhibit certain differences. Finally, we comment on some possible directions for further work and research.