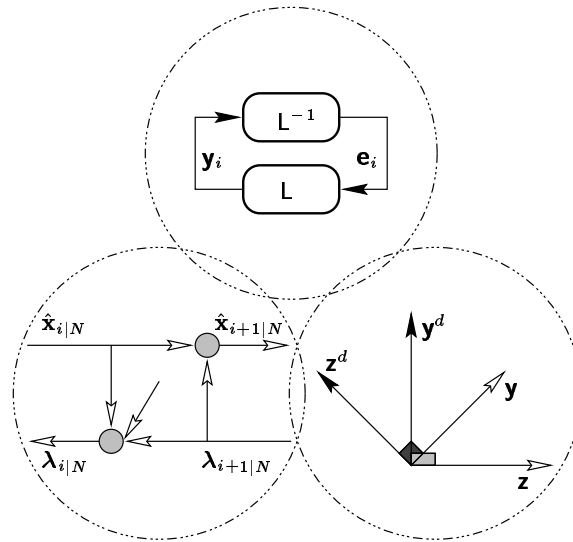


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# LINEAR ESTIMATION

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To our parents and our families.

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