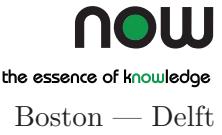


# **Adaptation, Learning, and Optimization over Networks**

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**Ali H. Sayed**  
University of California at Los Angeles



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University of California at Los Angeles

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