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SQUARE-ROOT ALGORITHMS FOR STRUCTURED MATRICES, INTERPOLATION, AND COMPLETION PROBLEMS*

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Abstract. We derive square-root based algorithms for structured matrices and discuss potential applications to interpolation and matrix completion problems. The mathematical machinery used here is based on a standard Gaussian elimination technique and on simple results from matrix and linear system theory. We show how to exploit the inherent displacement structure in order to construct a convenient transmission-line cascade that makes evident the required interpolation and completion conditions. We also introduce the concept of time-variant structured matrices and discuss its applications to matrix completions and time-variant interpolation problems.

1. Introduction. Interpolation problems of various types have had many applications in circuit and system theory. A classical paper is that of Youla and Saito [1], which was followed up and significantly extended by Helton [2] and others. We refer to the works of Sarason [3], Adamjan, Arov, and Krein [4], Foias and Frazho [5], Fedvcina [6], Delsarte, Genin, and Kamp [7], Ball and Helton [8], Alpay, Dewilde, and Dym [9, 10, 11], Kimura [12], Ball, Gohberg, and Rodman [13], Limebeer, Anderson, and Green [14, 15], and others, for extensive discussion and references.

The successful application of interpolation problems in control and circuit theory has inspired the study of generalizations to the time-variant setting [16, 17, 18, 19, 20, 21]. We describe here a computationally oriented solution for interpolation problems, in both the time-variant and time-invariant cases, based on a fast algorithm for the recursive triangular factorization of structured matrices. We use the interpolation data to construct a convenient so-called generator for the factorization algorithm, which then leads to a transmission-line cascade of first-order sections that makes evident the interpolation property. This is due to the fact that transmission lines have "transmission zeros": certain inputs at certain frequencies yield zero outputs. In the time-invariant case for example, each section of the cascade can be characterized by a $p \times q$ rational transfer matrix $\Theta_i(z)$ say, that has a left zero-direction vector g_i at a frequency f_i, viz.,

$$g_i\Theta_i(f_i) \equiv \begin{bmatrix} a_i & b_i \end{bmatrix} \begin{bmatrix} \Theta_{i,11} & \Theta_{i,12} \\ \Theta_{i,21} & \Theta_{i,22} \end{bmatrix}(f_i) = \mathbf{0},$$

which makes evident (with the proper partitioning of the row vector g_i and the matrix function $\Theta_i(z)$) the following interpolation property: $a_i\Theta_{i,12}\Theta_{i,22}^{-1}(f_i) = -b_i$.

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We shall begin, after some preliminaries, with a new presentation of earlier results on the factorization of a special class of structured matrices. Sections 4–7 give a survey of new results and applications.

1.1. Some Notation. Let $RH^{\infty}_{p\times q}$ denote the space of $p \times q$ rational matrix-valued functions $K(z)$ that are analytic and bounded inside the unit disc ($|z| < 1$). A matrix valued function $S(z) \in RH^{\infty}_{p\times q}$ that is (strictly) bounded by unity in $|z| < 1$ ($\|S\|_{\infty} < 1$) will be referred to as a *Schur function.* We also use the notation $\mathcal{H}^k_A(z)$ to refer to the following block-Toeplitz upper-triangular matrix

$$\mathcal{H}^k_A(z) = \begin{bmatrix} A(z) & \frac{1}{1!}A^{(1)}(z) & \frac{1}{2!}A^{(2)}(z) & \cdots & \frac{1}{(k-1)!}A^{(k-1)}(z) \\ & A(z) & \frac{1}{1!}A^{(1)}(z) & \cdots & \frac{1}{(k-2)!}A^{(k-2)}(z) \\ & & A(z) & \cdots & \frac{1}{(k-3)!}A^{(k-3)}(z) \\ & & & & \vdots \\ & O & & \ddots & \frac{1}{1!}A^{(1)}(z) \\ & & & & A(z) \end{bmatrix},$$

where $A(z)$ is a rational matrix function analytic at z, $k \geq 1$ is a positive integer, and $A^{(i)}(z)$ denotes the i^{th} derivative at z. We denote by $e_i = \begin{bmatrix} \mathbf{0}_{1\times i} & 1 & \mathbf{0} \end{bmatrix}$ the i^{th} basis vector of the $n-$dimensional space of complex numbers $\mathcal{C}^{\infty\times\backslash}$. The symbol $*$ stands for Hermitian conjugation (complex conjugation for scalars).

1.2. Basic Tools. The mathematical machinery used throughout this work is not much more than elementary matrix and linear systems theory. A key property in our analysis is the simple Gaussian elimination procedure presented now; it will be combined with displacement structure to get a fast factorization algorithm.

We restrict ourselves to Hermitian positive-definite matrices R, even though the results can be extended to more general cases [22, 23, 24]. A classical algorithm for the triangular factorization of $R = [r_{mj}]^{n-1}_{m,j=0}$ is the so-called Schur reduction procedure. The assumption of positive-definiteness guarantees the existence of a triangular factorization of the form $R = LD^{-1}L^*$, where L is lower-triangular and D is a diagonal matrix with positive entries. The columns of L and the diagonal entries of D can be recursively computed as follows: let l_0 and d_0 denote the first column and the $(0,0)$ entry of R respectively,

$$d_0 = r_{00}\ , \quad l_0 = \begin{bmatrix} r_{00} & r_{10} & \cdots & r_{n-1,0} \end{bmatrix}^{\mathbf{T}}.$$

If we subtract from R the outer product $l_0 d_0^{-1} l_0^*$ then we obtain a new matrix with one zero row and column. That is,

(1) $$R - l_0 d_0^{-1} l_0^* = \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & R_1 \end{bmatrix} \equiv \tilde{R}_1\ ,$$

where $R_1 = \left[r^{(1)}_{mj}\right]^{n-2}_{m,j=0}$ is called the Schur complement of r_{00} in R. Expression (1) represents one Schur reduction step and it can be repeated in order to compute the Schur complement $R_2 = \left[r^{(2)}_{mj}\right]^{n-3}_{m,j=0}$ of $r^{(1)}_{00}$ in R_1, and so on. Each further step corresponds to a recursion of the form

(2) $$\begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & R_{i+1} \end{bmatrix} = R_i - l_i d_i^{-1} l_i^*\ ,$$

where $d_i = r_{00}^{(i)}$ (the $(0,0)$ entry of the i^{th} Schur complement R_i), and l_i denotes the first column of R_i. It follows from (2) that R can be expressed as the sum of n rank 1 terms (since $R_n = 0$),

$$R = \sum_{i=0}^{n-1} \begin{bmatrix} \mathbf{0}_i \\ l_i \end{bmatrix} d_i^{-1} \begin{bmatrix} \mathbf{0}_i \\ l_i \end{bmatrix}^* .$$

Therefore, $D =$ diagonal $\{d_0, d_1, \ldots, d_{n-1}\}$ and the nonzero parts of the columns of the lower triangular factor L are given by $\{l_i\}_{i=0}^{n-1}$.

In summary, the triangular factors L and D can be constructed from the first columns of the successive Schur complements R_i. Observe however, that the Schur reduction procedure (2) is a recursive algorithm that operates directly on the entries of R_i. This requires $O(n^3)$ operations (additions and multiplications). The computational complexity can be reduced to $O(rn^2)$ when R exhibits displacement structure (with displacement rank $r \ll n$), since for structured matrices we can replace (2) with an alternative more efficient so-called *generator recursion* to be derived in the next section.

We further state a simple result in matrix theory that plays an important role in the derivation of all so-called square-root algorithms (see, *e.g.*, [25]).

LEMMA 1.2.1 *Consider two $n \times m$ $(n \leq m)$ matrices A and B. If $AJA^* = BJB^*$ is of full rank, for some $m \times m$ signature matrix J, then there exists a $J-$unitary $m \times m$ matrix Θ $(\Theta J \Theta^* = J)$ such that $A = B\Theta$.*

■

2. Array Algorithms. We now consider a positive-definite Hermitian matrix R that has low displacement rank, say r, with respect to the displacement operation $R - FRF^*$. That is, we can write

(3) $$R - FRF^* = GJG^* \ ,$$

for some $n \times r$ so-called generator matrix G and a signature matrix $J = (I_p \oplus -I_q)$. The diagonal entries of the (stable) lower triangular matrix F will be denoted by $\{f_i\}_{i=0}^{n-1}$ $(|f_i| < 1)$. The positive-definiteness of R guarantees the existence of a unique (lower triangular) Cholesky factor $\bar{L} = LD^{-1/2}$ such that $R = \bar{L}\bar{L}^*$, and we shall denote the *nonzero* parts of the columns of $\bar{L}$ by $\{\bar{l}_i\}_{i=0}^{n-1}$ $(\bar{l}_i = l_i d_i^{-1/2})$.

Let g_0 denote the first row of G. It follows from the displacement equation (3) and from the positive-definiteness of R that

$$d_0 = \frac{g_0 J g_0^*}{1 - |f_0|^2} > 0.$$

Consequently, g_0 has positive $J-$norm $(g_0 J g_0^* > 0)$ and we can always choose a $J-$unitary matrix Θ_0 that reduces g_0 to the form (recall Lemma 1.2.1)

(4) $$g_0 \Theta_0 = \begin{bmatrix} \delta_0 & 0 & \ldots & 0 \end{bmatrix} ,$$

where δ_0 is a positive scalar. By comparing the $J-$norm on both sides of (4) we conclude that the value of δ_0 is given by $\delta_0 = \sqrt{d_0(1 - |f_0|^2)}$. Hence, the action of Θ_0 is to reduce the original generator G to the following form

(5) $$G\Theta_0 = \begin{bmatrix} \delta_0 & 0 & 0 \\ x & x & x \\ x & x & x \\ x & x & x \end{bmatrix} \equiv \bar{G} \ ,$$

where the first row of $\bar{G}$ lies along the direction of the basis vector $\begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}$. Clearly, $\bar{G}$ is also a generator of R since $GJG^* = G\Theta_0 J \Theta_0^* G^* = \bar{G}J\bar{G}^*$. We say that $\bar{G}$ is a *proper* generator of R. Moreover, the rotation Θ_0 can be implemented in a variety of ways: by using a sequence of elementary Givens and hyperbolic rotations [26], Householder transformations [27, 28], etc..

It further follows from (3) that we can write

$$\begin{bmatrix} \bar{L} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \bar{L}^* \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} F\bar{L} & G \end{bmatrix} \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & J \end{bmatrix} \begin{bmatrix} \bar{L}^* F^* \\ G^* \end{bmatrix}.$$

This last expression fits into the statement of Lemma 1.2.1. Hence, there exists an $(I \oplus J)-$unitary matrix $\mathbf{\Gamma}$ such that

$$\begin{bmatrix} \bar{L} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} F\bar{L} & G \end{bmatrix} \mathbf{\Gamma}. \tag{6}$$

By examining this identity more closely, we shall derive a fast recursive algorithm for computing the Cholesky factor $\bar{L}$ from knowledge of F and G alone [1]. We first note that the $(I \oplus J)-$unitary transformation $\mathbf{\Gamma}$ can be achieved through a sequence of elementary transformations, say $\mathbf{\Gamma}_0, \mathbf{\Gamma}_1, \mathbf{\Gamma}_2, ..,$ that produce the block zero in the postarray by introducing *one zero row at a time*. We first implement $\mathbf{\Gamma}_0$ as a sequence of two rotations Θ_0 and Γ_0. The first rotation Θ_0 reduces the generator G to proper form, and the second rotation Γ_0 annihilates the remaining nonzero entry δ_0. The overall effect is to annihilate the first row of the G matrix. We then proceed to implement $\mathbf{\Gamma}_1$, $\mathbf{\Gamma}_2, ..$ in a similar fashion.

So let Θ_0 be a $J-$unitary matrix that reduces G to proper form, viz.,

$$\begin{bmatrix} F\bar{L} & G \end{bmatrix} \begin{bmatrix} I & \\ & \Theta_0 \end{bmatrix} = \begin{bmatrix} F\bar{L} & \bar{G} \end{bmatrix} \equiv \begin{bmatrix} F\bar{L} & \begin{matrix} \delta_0 & 0 & 0 \\ x & x & x \\ x & x & x \end{matrix} \end{bmatrix}.$$

In order to annihilate the first row of $\bar{G}$ we still need an elementary unitary (Givens) rotation, say Γ_0, that eliminates the nonzero entry δ_0. This can be done by "pivoting" the first column of $\bar{G}$ against the first column of $F\bar{L}$, while keeping all other columns unchanged. This operation produces the first column of $\bar{L}$ (because of (6)), and a matrix G_1 whose significance we shall verify very soon:

$$\begin{bmatrix} \boxed{f_0 d_0^{1/2}} & \mathbf{0} & \boxed{\delta_0} & 0 & 0 \\ x & \multirow{2}{*}{$F_1\bar{L}_1$} & x & x & x \\ x & & x & x & x \end{bmatrix} \begin{bmatrix} c & & s & \\ & I & & \\ s^* & & -c & \\ & & & I \end{bmatrix} = \begin{bmatrix} \bar{l}_0 & \begin{matrix} \mathbf{0} \\ F_1\bar{L}_1 \end{matrix} & \begin{matrix} \boxed{0} & 0 & 0 \\ & G_1 & \end{matrix} \end{bmatrix}, \tag{7}$$

where F_1 and $\bar{L}_1$ are the submatrices obtained after deleting the first row and column of F and $\bar{L}$, respectively. The letters c and s denote the (cosine and sine) parameters of the rotation matrix. Let $\bar{x}_0$ and x_1 denote the first columns of $\bar{G}$ and G_1, respectively. From expression (7) we see that, ignoring the columns that remain unchanged and are thus common to the pre- and post-arrays,

$$\begin{bmatrix} F\bar{l}_0 & \bar{x}_0 \end{bmatrix} \begin{bmatrix} c & s \\ s^* & -c \end{bmatrix} = \begin{bmatrix} \bar{l}_0 & \begin{matrix} 0 \\ x_1 \end{matrix} \end{bmatrix}, \tag{8}$$

where the top entry of $\bar{x}_0$ is δ_0. The rotation parameters are clearly given by

$$c = \frac{1}{\sqrt{1+|\rho_0|^2}} \quad , \quad s = \frac{\rho_0}{\sqrt{1+|\rho_0|^2}} \quad , \quad \rho_0 = \frac{\delta_0}{f_0 d_0^{1/2}} .$$

That is, we can rewrite (8) more explicitly as follows:

$$\begin{bmatrix} F\bar{l}_0 & \bar{x}_0 \end{bmatrix} \begin{bmatrix} f_0^* & \frac{\delta_0}{d_0^{1/2}} \\ \frac{\delta_0}{d_0^{1/2}} & -f_0 \end{bmatrix} = \begin{bmatrix} \bar{l}_0 & \begin{matrix} 0 \\ x_1 \end{matrix} \end{bmatrix},$$

[1] This particular approach, one of several possible ones, is a variation of one suggested by H. Lev-Ari (see [31, Chapter 2] and also Section 3 ahead).

which leads to

$$\begin{bmatrix} 0 \\ x_1 \end{bmatrix} = \Phi_0 \bar{x}_0 = \Phi_0 G \Theta_0 \begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix} ,$$

where we defined the "Blaschke" matrix $\Phi_0 = (I - f_0^* F)^{-1}(F - f_0 I)$. We still need to verify the significance of G_1. Comparing the $(I \oplus J)$-norm on both sides of (7) we obtain

$$F\bar{L}\bar{L}^* F^* + \bar{G}J\bar{G}^* = \bar{l}_0 \bar{l}_0^* + \begin{bmatrix} \mathbf{0} \\ F_1 \bar{L}_1 \end{bmatrix} \begin{bmatrix} \mathbf{0} & \bar{L}_1^* F_1^* \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ G_1 \end{bmatrix} J \begin{bmatrix} \mathbf{0} & G_1^* \end{bmatrix}.$$

But the Cholesky factor of the first Schur complement R_1 is $\bar{L}_1$ itself. Hence, using (3) we get

$$R - \bar{l}_0 \bar{l}_0^* = \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & F_1 R_1 F_1^* \end{bmatrix} + \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & G_1 J G_1^* \end{bmatrix}.$$

Consequently (recall (1)), $R_1 - F_1 R_1 F_1^* = G_1 J G_1^*$, which shows that G_1 is a generator matrix of the Schur complement R_1 with respect to the displacement operation $R_1 - F_1 R_1 F_1^*$. Hence, G_1 is obtained as follows: choose a $J-$unitary rotation Θ_0 that converts the first row of G to the form $\begin{bmatrix} \delta_0 & \mathbf{0} \end{bmatrix}^{\mathbf{T}}$ and apply Θ_0 to G as in (5); keep the last $(r-1)$ columns of $G\Theta_0$ unchanged and multiply the first column by Φ_0; this results in G_1. We can write this transformation in the following compact (array) form:

$$\begin{bmatrix} \mathbf{0} \\ G_1 \end{bmatrix} = G\Theta_0 \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & I_{r-1} \end{bmatrix} + \Phi_0 G \Theta_0 \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Therefore, the effect of the transformation $\mathbf{\Gamma}_0$ (which we implemented as a sequence of two rotations Θ_0 and Γ_0) is to annihilate the first row of G,

$$\begin{bmatrix} F\bar{L} & G \end{bmatrix} \mathbf{\Gamma}_0 = \begin{bmatrix} \bar{l}_0 & \begin{matrix} \mathbf{0} \\ F_1 \bar{L}_1 \end{matrix} & \begin{matrix} \mathbf{0} \\ G_1 \end{matrix} \end{bmatrix}.$$

We can now proceed by annihilating the first row of G_1,

$$\begin{bmatrix} F_1\bar{L}_1 & G_1 \end{bmatrix} \mathbf{\Gamma}_1 = \begin{bmatrix} \bar{l}_1 & \begin{matrix} \mathbf{0} \\ F_2 \bar{L}_2 \end{matrix} & \begin{matrix} \mathbf{0} \\ G_2 \end{matrix} \end{bmatrix} ,$$

where F_2 and $\bar{L}_2$ are the submatrices obtained after deleting the first row and column of F_1 and $\bar{L}_1$, respectively, and so on.

Moreover, it follows from the displacement equation (3) that the first column of R satisfies the relation $l_0 = F l_0 f_0^* + G J g_0^*$. Hence, using the properness of $G\Theta_0$ (see (5)) we have

$$l_0 = (I - f_0^* F)^{-1} G \Theta_0 J \Theta_0^* g_0^* = (I - f_0^* F)^{-1} \bar{x}_0 \delta_0.$$

Using the fact that $\bar{l}_0 = l_0 d_0^{-1/2}$, we get

(9) $$\bar{l}_0 = \sqrt{1 - |f_0|^2}\,(I - f_0^* F)^{-1} \bar{x}_0 , \qquad \text{since} \quad d_0 = \frac{\delta_0^2}{1 - |f_0|^2}.$$

Similar expressions are valid for the other column vectors $\bar{l}_i$, $i \geq 1$. In summary, we are led to the following recursive procedure (see [29, 32] for earlier and different derivations).

ALGORITHM 2.1 *The Cholesky factorization of a positive-definite Hermitian matrix R with displacement structure of the form $R - FRF^* = GJG^*$, can be computed by the recursive procedure*

(10) $$\begin{bmatrix} \mathbf{0} \\ G_{i+1} \end{bmatrix} = G_i \Theta_i \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & I_{r-1} \end{bmatrix} + \Phi_i G_i \Theta_i \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad F_0 = F, \;\; G_0 = G \; ,$$

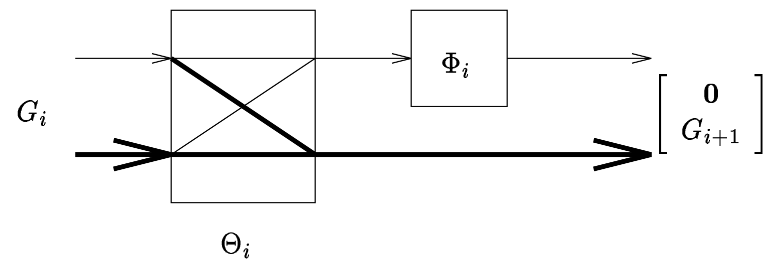

Figure 1: One step of the generator recursion.

$$\Phi_i = (I_{n-i} - f_i^* F_i)^{-1}(F_i - f_i I_{n-i}) \ ,$$

where F_i is the submatrix obtained after deleting the first row and column of F_{i-1}, and Θ_i is an arbitrary $J-$unitary matrix that reduces the first row of G_i (denoted by g_i) to the form $g_i\Theta_i = \begin{bmatrix} \delta_i & 0 & \dots & 0 \end{bmatrix}$. The columns of the Cholesky factor $\bar{L}$ are then given by

$$\bar{l}_i = \sqrt{1 - |f_i|^2}\ (I_{n-i} - f_i^* F_i)^{-1} G_i \Theta_i \begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix}.$$

■

Pictorially, we have the following simple array picture as depicted in Figure 1.

$$G_i = \begin{bmatrix} x & x & x \\ x & x & x \\ x & x & x \\ \vdots & \vdots & \vdots \end{bmatrix} \xrightarrow{\Theta_i} \begin{bmatrix} \delta_i & 0 & 0 \\ x & x & x \\ x & x & x \\ \vdots & \vdots & \vdots \end{bmatrix} \xrightarrow{\Phi_i} \begin{bmatrix} 0 & 0 & 0 \\ x & x & x \\ x & x & x \\ \vdots & \vdots & \vdots \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ G_{i+1} \end{bmatrix}.$$

2.1. First-Order $J-$Lossless Sections. It follows from the square-root argument (using (7)) that the expressions for l_i and G_i can be grouped together into the following revealing expression:

(11)
$$\begin{bmatrix} l_i & \mathbf{0} \\ & G_{i+1} \end{bmatrix} = \begin{bmatrix} F_i l_i & G_i \end{bmatrix} \begin{bmatrix} f_i^* & \frac{\delta_i}{d_i}\begin{bmatrix} 1 & \mathbf{0} \end{bmatrix} \\ \Theta_i \begin{bmatrix} \delta_i \\ \mathbf{0} \end{bmatrix} & \Theta_i \begin{bmatrix} -f_i & \mathbf{0} \\ \mathbf{0} & I_{r-1} \end{bmatrix} \end{bmatrix},$$

which clearly shows that each step of the generator recursion involves a first-order state-space system that appears on the right-hand-side of the above expression. Let $\Theta_i(z)$ denote its $r \times r$ transfer matrix (with inputs from the left), viz.,

$$\Theta_i(z) = \Theta_i \begin{bmatrix} -f_i & \mathbf{0} \\ \mathbf{0} & I_{r-1} \end{bmatrix} + \Theta_i \begin{bmatrix} \delta_i \\ \mathbf{0} \end{bmatrix} (z^{-1} - f_i^*)^{-1} \frac{\delta_i}{d_i} \begin{bmatrix} 1 & \mathbf{0} \end{bmatrix}.$$

It then readily follows, upon simplification, that

(12)
$$\Theta_i(z) = \Theta_i \begin{bmatrix} \frac{z - f_i}{1 - z f_i^*} & \mathbf{0} \\ \mathbf{0} & I_{r-1} \end{bmatrix}.$$

Each such section is clearly $J-$lossless. This follows from the fact that $\Theta_i(z)$ is analytic in $|z| < 1$ due to $|f_i| < 1$, and that $\Theta_i(z) J \Theta_i^*(z) = J$ on $|z| = 1$ since $(z - f_i)/(1 - z f_i^*)$ is a Blaschke factor and Θ_i is $J-$unitary. Furthermore, each $\Theta_i(z)$ also has an important "blocking" property that will be very relevant in the solution of interpolation problems.

LEMMA 2.1.1 *Each first-order section $\Theta_i(z)$ has a transmission zero at f_i and along the direction defined by g_i, viz., $g_i\Theta_i(f_i) = \mathbf{0}$.*

Proof: This is evident from the relation

$$g_i\Theta_i(f_i) \;=\; g_i\Theta_i \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & I_{r-1} \end{bmatrix} = \begin{bmatrix} \delta_i & \mathbf{0} \end{bmatrix} \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & I_{r-1} \end{bmatrix} = \mathbf{0}$$

■

3. General Algorithm. The algorithm derived in Section 2 is in a convenient array form. We verify here that it is a special case of a more general recursion, which under suitable manipulations reduces to the array form discussed above.

THEOREM 3.1 *The Schur complements R_i are also structured with generator matrices G_i, viz., $R_i - F_iR_iF_i^* = G_iJG_i^*$, where G_i is an $(n-i)\times r$ generator matrix that satisfies, along with l_i, the following recursion*

$$\begin{bmatrix} l_i & \begin{matrix} \mathbf{0} \\ G_{i+1} \end{matrix} \end{bmatrix} = \begin{bmatrix} F_il_i & G_i \end{bmatrix} \begin{bmatrix} f_i^* & h_i^*J \\ Jg_i^* & Jk_i^*J \end{bmatrix} , \tag{13}$$

where g_i is the first row of G_i, and h_i and k_i are arbitrary $r\times 1$ and $r\times r$ matrices, respectively, chosen so as to satisfy the embedding relation

$$\begin{bmatrix} f_i & g_i \\ h_i & k_i \end{bmatrix} \begin{bmatrix} d_i & \mathbf{0} \\ \mathbf{0} & J \end{bmatrix} \begin{bmatrix} f_i & g_i \\ h_i & k_i \end{bmatrix}^* = \begin{bmatrix} d_i & \mathbf{0} \\ \mathbf{0} & J \end{bmatrix} , \tag{14}$$

where

$$d_i = \frac{g_iJg_i^*}{1-|f_i|^2} , \tag{15}$$

and F_i is the $(n-i)\times(n-i)$ submatrix obtained after deleting the first row and column of F_{i-1}.

Proof: We prove the result for $i=0$. The same argument holds for $i\geq 1$. Using (1) we write

$$\begin{aligned} \tilde{R}_1 - F\tilde{R}_1F^* \;&=\; -\frac{1}{d_0}\left[f_0^*Fl_0g_0JG^* + GJg_0^*l_0^*F^*f_0 - \frac{1}{d_0}Fl_0g_0Jg_0^*l_0^*F^*\right] + \\ & \quad GJ\left\{J - \frac{g_0^*g_0}{d_0}\right\}JG^*. \end{aligned} \tag{16}$$

We now verify that the right-hand side of the above expression can be put into the form of a *perfect square* by introducing some auxiliary quantities. Consider an $r\times 1$ column vector h_0 and an $r\times r$ matrix k_0 that are defined to satisfy the following relations (in terms of the quantities that appear on the right-hand side of the above expression. We shall see very soon that this is always possible):

$$h_0^*Jh_0 = \frac{g_0Jg_0^*}{d_0^2} , \quad k_0^*Jk_0 = J - \frac{g_0^*g_0}{d_0} , \quad k_0^*Jh_0 = -\frac{f_0g_0^*}{d_0} . \tag{17}$$

Using $\{h_0, k_0\}$, we can rewrite the right-hand side of (16) in the form

$$GJk_0^*Jk_0JG^* + GJk_0^*Jh_0l_0^*F^* + Fl_0h_0^*Jk_0JG^* + Fl_0h_0^*Jh_0l_0^*F^*,$$

which can clearly be factored as $\tilde{G}_1J\tilde{G}_1^*$, where $\tilde{G}_1 = Fl_0h_0^*J + GJk_0^*J$. But the first row and column of $\tilde{R}_1$ are zero. Hence, the first row of $\tilde{G}_1$ is zero, $\tilde{G}_1 = \begin{bmatrix} \mathbf{0} & G_1^{\mathbf{T}} \end{bmatrix}^{\mathbf{T}}$. Moreover, it follows from (17) (and the expression for d_0) that $\{f_0, g_0, h_0, k_0\}$ satisfy the relation

$$\begin{bmatrix} f_0 & g_0 \\ h_0 & k_0 \end{bmatrix}^* \begin{bmatrix} d_0^{-1} & \mathbf{0} \\ \mathbf{0} & J \end{bmatrix} \begin{bmatrix} f_0 & g_0 \\ h_0 & k_0 \end{bmatrix} = \begin{bmatrix} d_0^{-1} & \mathbf{0} \\ \mathbf{0} & J \end{bmatrix} ,$$

which is equivalent to (14) for $i = 0$.

■

It is worth noting that the generator recursion (13) has the same form as the array equation (11) that we wrote earlier. In fact, the matrix defined by

$$\begin{bmatrix} f_i^* & h_i^* J \\ J g_i^* & J k_i^* J \end{bmatrix}$$

is the general form of an elementary transformation that produces the desired zero row on the left-hand side of (13). Moreover, if we consider the transfer matrix $\Theta_i(z)$ associated with the above discrete-time system, viz.,

$$\Theta_i(z) \quad = \quad J k_i^* J + J g_i^* \left[z^{-1} - f_i^* \right]^{-1} h_i^* J. \tag{18}$$

Then, using the embedding relation (14) (or the expressions similar to (17) for h_i and k_i), we readily conclude that

$$\Theta_i(z) J \Theta_i^*(z) = J + \frac{J g_i^* g_i J}{d_i} \, \frac{z z^* - 1}{(1 - z f_i^*)(1 - z^* f_i)} , \tag{19}$$

which confirms that each first-order section $\Theta_i(z)$ is $J-$lossless. Furthermore, the blocking property of $\Theta_i(z)$ is also evident here since

$$g_i \Theta_i(f_i) \quad = \quad g_i J k_i^* J + g_i J g_i^* \frac{f_i}{1 - |f_i|^2} h_i^* J = g_i J k_i^* J + f_i d_i h_i^* J \overset{(14)}{=} \mathbf{0}.$$

Using the embedding relation (14) we can further show [29, 30, 31], following an argument similar to that in [29], that all choices of h_i and k_i are completely specified by $\{f_i, g_i, d_i\}$.

LEMMA 3.1 *All possible choices of h_i and k_i are given by*

$$h_i = \Theta_i^{-1} \left\{ \frac{1}{d_i} \frac{\tau_i - f_i}{1 - \tau_i f_i^*} J g_i^* \right\} \quad \text{and} \quad k_i = \Theta_i^{-1} \left\{ I_r - \frac{1}{d_i} \frac{J g_i^* g_i}{1 - \tau_i f_i^*} \right\} , \tag{20}$$

for an arbitrary $J-$unitary matrix Θ_i and an arbitrary scalar τ_i on the unit circle ($|\tau_i| = 1$).

■

Using expression (20) for h_i and k_i we can rewrite the generator recursion (13) and the transfer matrix (18) in a more convenient form that depends (up to $J-$unitary rotations) only on known parameters (see also [29, 32] for earlier and alternative derivations).

THEOREM 3.2 *The generator recursion (13) and the transfer matrix (18) reduce to*

$$\begin{bmatrix} \mathbf{0} \\ G_{i+1} \end{bmatrix} = \left\{ G_i + (\Phi_i - I_{n-i}) G_i \frac{J g_i^* g_i}{g_i J g_i^*} \right\} \Theta_i \ , \tag{21}$$

$$\Theta_i(z) = \left\{ I_r + [B_i(z) - 1] \frac{J g_i^* g_i}{g_i J g_i^*} \right\} \Theta_i \ , \tag{22}$$

where $B_i(z)$ is a Blaschke factor of the form

$$B_i(z) = \frac{z - f_i}{1 - z f_i^*} \frac{1 - \tau_i f_i^*}{\tau_i - f_i} \ ,$$

and Φ_i is a "Blaschke" matrix given by,

$$\Phi_i = \frac{1-\tau_i f_i^*}{\tau_i - f_i}(F_i - f_i I_{n-i})(I_{n-i} - f_i^* F_i)^{-1}.$$

■

We remark that $(F_i - f_i I_{n-i})$ and $(I_{n-i} - f_i^* F_i)^{-1}$ commute, and hence the expression for Φ_i given above will be the same as the expression given earlier in Algorithm 2.1 if we choose

$$\tau_i = \frac{1+f_i}{1+f_i^*}\ . \tag{23}$$

Notice also that the blocking property of each Section $\Theta_i(z)$ is again evident from expression (22) since $B_i(f_i) = 0$. That is,

$$g_i \Theta_i(f_i) = \left\{ g_i - \frac{g_i J g_i^*}{g_i J g_i^*}\ g_i \right\} \Theta_i = \mathbf{0}.$$

The generator recursion of Theorem 3.2 is the general form of the factorization algorithm and it includes, as special cases, the array algorithm derived in section 2. Observe for instance, that (21) has two parameters that we are free to choose: Θ_i and τ_i. Choosing τ_i as in (23) and Θ_i such that $g_i\Theta_i$ is reduced to the form in (4), we can easily check that Theorem 3.2 reduces to the array algorithm of Section 2.

4. The Tangential Hermite-Fejér Problem. We now show that the algorithm derived in the previous sections also solves interpolation problems. We first state a general Hermite-Fejér interpolation problem that includes many of the classical problems as special cases. We consider m points $\{\alpha_i\}_{i=0}^{m-1}$ inside the open unit disc $\mathcal{D}$ and we associate with each point α_i a positive integer $r_i \geq 1$ and two row vectors $\mathbf{a}_i$ and $\mathbf{b}_i$ partitioned as follows:

$$\mathbf{a}_i = \left[\begin{array}{cccc} u_1^{(i)} & u_2^{(i)} & \ldots & u_{r_i}^{(i)} \end{array}\right], \quad \mathbf{b}_i = \left[\begin{array}{cccc} v_1^{(i)} & v_2^{(i)} & \ldots & v_{r_i}^{(i)} \end{array}\right] \ ,$$

where $u_j^{(i)}$ and $v_j^{(i)}$ $(j = 1, \ldots, r_i)$ are $1 \times p$ and $1 \times q$ row vectors, respectively. That is, $\mathbf{a}_i$ and $\mathbf{b}_i$ are partitioned into r_i row vectors each. If an interpolating point α_i is repeated (say, $\alpha_i = \alpha_{i+1} = \ldots = \alpha_{i+j}$), then we shall further assume that the following condition is satisfied (which rules out degenerate cases [30]):

$$\{u_1^{(i)}, u_1^{(i+1)}, \ldots, u_1^{(i+j)}\} \quad \textit{are linearly independent.} \tag{24}$$

The tangential Hermite-Fejér problem then reads as follows (see, *e.g.*, [5]).

PROBLEM 4.1 *Describe all Schur-type functions $S(z) \in RH^\infty_{p\times q}$ that satisfy*

$$\mathbf{b}_i = \mathbf{a}_i \mathcal{H}_S^{r_i}(\alpha_i) \quad \textit{for} \quad 0 \leq i \leq m-1. \tag{25}$$

■

This statement clearly includes, as special cases, the problems of Carathéodory-Fejér [33, 34, 35], Nevanlinna-Pick [33, 36, 37], and the corresponding tangential (matrix) versions.

4.1. Solvability Condition. The first step in the recursive solution consists in constructing three matrices F, G, and J directly from the interpolation data: F contains the information relative to the points $\{\alpha_i\}$ and the dimensions $\{r_i\}$, G contains the information relative to the direction vectors $\{\mathbf{a}_i\}$ and $\{\mathbf{b}_i\}$, and $J = (I_p \oplus -I_q)$ is a signature matrix. The matrices F and G are constructed as follows: we associate with each α_i a Jordan block $\bar{F}_i$ of size $r_i \times r_i$,

$$\bar{F}_i = \begin{bmatrix} \alpha_i & & & \\ 1 & \alpha_i & & \\ & \ddots & \ddots & \\ & & 1 & \alpha_i \end{bmatrix},$$

and two $r_i \times p$ and $r_i \times q$ matrices U_i and V_i, respectively, which are composed of the row vectors associated with α_i, viz.,

$$U_i = \begin{bmatrix} u_1^{(i)} \\ u_2^{(i)} \\ \vdots \\ u_{r_i}^{(i)} \end{bmatrix} \quad \text{and} \quad V_i = \begin{bmatrix} v_1^{(i)} \\ v_2^{(i)} \\ \vdots \\ v_{r_i}^{(i)} \end{bmatrix}.$$

Then $F =$ diagonal $\{\bar{F}_0, \bar{F}_1, \ldots, \bar{F}_{m-1}\}$ and

$$G = \begin{bmatrix} U_0 & V_0 \\ U_1 & V_1 \\ \vdots & \vdots \\ U_{m-1} & V_{m-1} \end{bmatrix} \equiv \begin{bmatrix} \mathbf{U} & \mathbf{V} \end{bmatrix}. \tag{26}$$

Let $n = \sum_{i=0}^{m-1} r_i$ and $r = p+q$, then F and G are $n \times n$ and $n \times r$ matrices respectively. We shall denote the diagonal entries of F by $\{f_i\}_{i=0}^{n-1}$ (for example, $f_0 = f_1 = \ldots = f_{r_0-1} = \alpha_0$). We also associate with the interpolation Problem 4.1 the following displacement equation

$$R - FRF^* = GJG^*. \tag{27}$$

R is clearly unique since F is a stable matrix ($|f_i| < 1, \ \forall\, i$). We shall prove in the next section that by applying the array algorithm to F and G we obtain a transmission-line cascade $\mathbf{\Theta}(z)$ that parametrizes all solutions of the Hermite-Fejér problem. Meanwhile, we verify that the above construction of F, G, and R allows us to prove the necessary and sufficient conditions for the existence of solutions (see also [38, 39] for related discussion).

THEOREM 4.1.1 *The tangential Hermite-Fejér problem is solvable if, and only if, R is positive-definite.*

Proof: If R is positive-definite then the recursive procedure described later finds a solution $S(z)$. Conversely, assume there exists a solution $S(z)$ satisfying the interpolation conditions (25), and let $\{S_i\}_{i=0}^{\infty}$ be the Taylor series coefficients of $S(z)$ around the origin, viz.,

$$S(z) = S_0 + zS_1 + z^2S_2 + z^3S_3 + \ldots$$

Define the (semi-infinite) block lower-triangular Toeplitz matrix

$$\mathcal{S} = \begin{bmatrix} S_0 & & & \\ S_1 & S_0 & & O \\ S_2 & S_1 & S_0 & \\ \vdots & & & \ddots \end{bmatrix},$$

as well as the (semi-infinite) matrices

$$\mathbf{U} = \begin{bmatrix} \mathbf{U} & F\mathbf{U} & F^2\mathbf{U} & \ldots \end{bmatrix} \quad \text{and} \quad \mathbf{V} = \begin{bmatrix} \mathbf{V} & F\mathbf{V} & F^2\mathbf{V} & \ldots \end{bmatrix}.$$

We can easily check that because of (25) we get $\mathbf{V} = \mathbf{U}\mathcal{S}$. But R in (27) is given by

$$R = \mathbf{U}\mathbf{U}^* - \mathbf{V}\mathbf{V}^* \ = \mathbf{U}\left(I - \mathbf{S}\mathbf{S}^*\right)\mathbf{U}^*$$

Moreover, $\mathcal{S}$ is a strict contraction (since $S(z)$ is a Schur-type function with $\|S\|_\infty < 1$) and it follows from (24) that $\mathbf{U}\mathbf{U}^* > 0$ (see [30, 31]). Hence, $R > 0$.

■

4.2. Interpolation Properties. We already know how to construct a convenient structure (27) from the interpolation data. We remark that we only know F, G, and J, whereas the matrix R itself *is not known* a priori. In fact, the recursive procedure described here *does not* require R. It only uses the matrices F, G, and J that are constructed *directly* from the interpolation data.

We now verify that if we apply the array algorithm to G in (27), we then obtain a cascade $\mathbf{\Theta}(z)$,

$$\mathbf{\Theta}(z) = \Theta_0(z)\Theta_1(z)\ldots\Theta_{n-1}(z),$$

of first order $J-$lossless sections that parametrizes all solutions of the Hermite-Fejér interpolation problem. This follows from the fact that the first-order sections have local blocking properties, $g_i\Theta_i(f_i) = \mathbf{0}$, which reflect into a global blocking property for the entire cascade, as we readily verify.

Consider the first-order section $\Theta_0(z)$. It follows from its local blocking property that

$$e_0 G\Theta_0(f_0) = g_0\Theta_0(f_0) = \mathbf{0}$$

But the Jordan structure of $\bar{F}_0$ (with eigenvalue $\alpha_0 = f_0 = f_1 = \ldots = f_{r_0-1}$) imposes a stronger condition on $\Theta_0(z)$. Note for example, that the following relation follows immediately from the array form (10) (g_0 and g_1 are the first rows of G and G_1, respectively)

$$g_1 = g_0\Theta_0^{(1)}(f_0) + e_1 G\Theta_0(f_0)\ .$$

More precisely, by comparing the second row on both sides of (10) for $i = 0$, we conclude that

$$\begin{aligned} g_1 &= e_1 G\Theta_0 \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & I_{r-1} \end{bmatrix} + e_1\Phi_0 G\Theta_0 \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \\ &= e_1 G\Theta_0(f_0) + \frac{1}{1-|f_0|^2} e_0 G\Theta_0 \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \\ &= e_1 G\Theta_0(f_0) + g_0\Theta_0^{(1)}(f_0). \end{aligned}$$

Therefore, the first row of G_1 is obtained as a linear combination of the first two rows of G,

$$g_1 \ \equiv \ e_0 G_1 = \begin{bmatrix} e_0 G & e_1 G \end{bmatrix} \begin{bmatrix} \Theta_0^{(1)}(f_0) \\ \Theta_0(f_0) \end{bmatrix}.$$

This result can be extended to show that the k^{th} row of G_1 ($k < r_0$) is obtained as a linear combination of the first $(k+1)$ rows of G, and so on. Putting these remarks together leads to

(28) $$\begin{bmatrix} e_0 G & e_1 G & \ldots & e_{r_0-1} G \end{bmatrix} \mathcal{H}_{\Theta_0}^{r_0}(\alpha_0) = \begin{bmatrix} \mathbf{0} & e_0 G_1 & e_1 G_1 & \ldots & e_{r_0-2} G_1 \end{bmatrix}.$$

Therefore, when the first r_0 rows of G propagate through $\Theta_0(z)$ we obtain the first $r_0 - 1$ rows of G_1 at $z = \alpha_0$. This argument can be continued [30, 31] to conclude the following result: let s_i denote the total size of the Jordan blocks prior to $\bar{F}_i$: $s_i = \sum_{p=0}^{i-1} r_p\ ,\ \ s_0 = 0$.

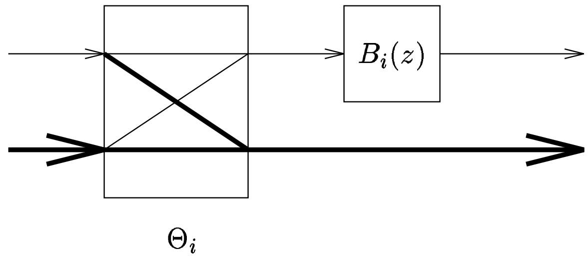

Figure 2: A $J-$lossless first-order section $\Theta_i(z)$.

THEOREM 4.2.1 *The transfer matrix* $\mathbf{\Theta}(z)$ *satisfies the global blocking property*

(29) $$\begin{bmatrix} e_{s_i}G & e_{s_i+1}G & \dots & e_{s_i+r_i-1}G \end{bmatrix} \mathcal{H}_{\mathbf{\Theta}}^{r_i}(\alpha_i) = \mathbf{0}.$$

■

The row vector on the left hand-side of (29) is composed of the r_i row vectors in $\begin{bmatrix} U_i & V_i \end{bmatrix}$ associated with α_i, viz., $\begin{bmatrix} u_1^{(i)} & v_1^{(i)} & u_2^{(i)} & v_2^{(i)} & \dots & u_{r_i}^{(i)} & v_{r_i}^{(i)} \end{bmatrix}$. If we now partition $\mathbf{\Theta}(z)$ accordingly with $J = (I_p \oplus -I_q)$,

$$\mathbf{\Theta}(z) = \begin{bmatrix} \mathbf{\Theta}_{11}(z) & \mathbf{\Theta}_{12}(z) \\ \mathbf{\Theta}_{21}(z) & \mathbf{\Theta}_{22}(z) \end{bmatrix},$$

it is then a standard result that $S(z) = -\mathbf{\Theta}_{12}(z)\mathbf{\Theta}_{22}^{-1}(z)$ is a Schur-type function due to the $J-$losslessness of $\mathbf{\Theta}(z)$, and we conclude from (29) that it satisfies the required interpolation conditions. Moreover, all solutions $S(z)$ are parametrized in terms of a linear fractional transformation based on $\mathbf{\Theta}(z)$ (see [11, 13, 21, 30] for details and related discussion).

LEMMA 4.2.1 *All solutions* $S(z)$ *of the tangential Hermite-Fejér problem are given by a linear fractional transformation of a Schur matrix function* $K(z)$ ($\|K\|_\infty < 1$)

(30) $$S(z) = -\left[\mathbf{\Theta}_{11}(z)K(z) + \mathbf{\Theta}_{12}(z)\right]\left[\mathbf{\Theta}_{21}(z)K(z) + \mathbf{\Theta}_{22}(z)\right]^{-1}.$$

■

4.3. Transmission-Line Structure. Each section $\Theta_i(z)$ can be schematically represented as shown in Figure 2. Figure 3 shows a scattering interpretation of the cascade $\mathbf{\Theta}(z)$, where $\mathbf{\Sigma}(z)$ is the scattering matrix defined by

$$\mathbf{\Sigma}(z) = \begin{bmatrix} \mathbf{\Theta}_{11} - \mathbf{\Theta}_{12}\mathbf{\Theta}_{22}^{-1}\mathbf{\Theta}_{21} & -\mathbf{\Theta}_{12}\mathbf{\Theta}_{22}^{-1} \\ \mathbf{\Theta}_{22}^{-1}\mathbf{\Theta}_{21} & \mathbf{\Theta}_{22}^{-1} \end{bmatrix}(z).$$

The solution $S(z)$ is the transfer matrix from the top left $(1 \times p)$ input to the bottom left $(1 \times q)$ output, with a Schur-type load $(-K(z))$ at the right end. Therefore, we are led to the following $O(rn^2)$ recursive algorithm for the solution of the Hermite-Fejér problem.

ALGORITHM 4.3.1 *The Hermite-Fejér problem can be recursively solved as follows:*

- *Construct* F, G, *and* J *from the interpolation data as described in Section 4.1.*
- *Start with* $F_0 = F$, $G_0 = G$, *and apply the array form (10) of the generator recursion for* $i = 0, 1, \dots, n-1$.
- *Each step provides a first-order section* $\Theta_i(z)$ *completely specified by* f_i, g_i, *and* Θ_i *as in (12) or (22).*

- *The cascade of sections $\mathbf{\Theta}(z)$ satisfies the relation*

$$\left[\begin{array}{cccccccc} u_1^{(i)} & v_1^{(i)} & u_2^{(i)} & v_2^{(i)} & \ldots & u_{r_i}^{(i)} & v_{r_i}^{(i)} \end{array}\right] \mathcal{H}_{\mathbf{\Theta}}^{r_i}(\alpha_i) = \mathbf{0}, \quad 0 \le i \le m-1.$$

- *Then $S(z) = -\mathbf{\Theta}_{12}(z)\mathbf{\Theta}_{22}^{-1}(z)$ satisfies*

$$\mathbf{b}_i = \mathbf{a}_i \mathcal{H}_S^{r_i}(\alpha_i), \quad 0 \le i \le m-1.$$

- *All solutions $S(z)$ are parametrized by an arbitrary Schur function $K(z)$,*

$$S(z) = -\left[\mathbf{\Theta}_{11}(z)K(z) + \mathbf{\Theta}_{12}(z)\right]\left[\mathbf{\Theta}_{21}(z)K(z) + \mathbf{\Theta}_{22}(z)\right]^{-1}.$$

■

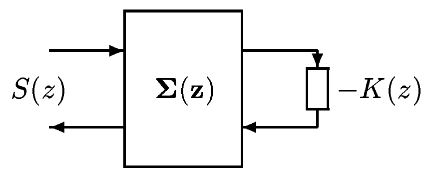

Figure 3: Scattering interpretation.

5. Time-Variant Displacement Structure. We now extend the notion of displacement structure to the time-variant setting and show that we can also study matrices that exhibit structured time-variations, special cases of which often arise in adaptive filtering [40], time-variant interpolation [20], and matrix completion problems [21].

We consider an $n \times n$ time-variant positive-definite Hermitian matrix $R(t) = [r_{mj}(t)]_{m,j=0}^{n-1}$, and we shall say that it has a *time-variant Toeplitz-like* structure if the difference,

$$R(t) - F(t)R(t-1)F^*(t),$$

has low rank, say $r(t)$ (usually $r(t) \ll n$), for some lower triangular $n \times n$ matrix $F(t)$ whose diagonal entries we denote by $\{f_i(t)\}_{i=0}^{n-1}$ ($|f_i(t)| < 1$). It follows from the low rank property that we can write

$$R(t) - F(t)R(t-1)F^*(t) = G(t)J(t)G^*(t) \quad , \tag{31}$$

where $G(t)$ is an $n \times r(t)$ so-called generator matrix, and $J(t) = (I_{p(t)} \oplus -I_{q(t)})$ is an $r(t) \times r(t)$ signature matrix. The main question that we treat in this section is the following: given the Cholesky factor of $R(t-1)$ and knowing that $R(t)$ satisfies a displacement equation of the form (31), how to efficiently and recursively determine the Cholesky factor of $R(t)$?

We may repeat here the same square-root argument as is in the time-invariant case (Section 2) [40, 31]. We shall instead, present the general algorithm and discuss applications in time-variant interpolation and completion problems.

Following the same reasoning as in Section 3 we can prove the following result.

THEOREM 5.1 *The Schur complements $R_i(t)$ are also structured with generator matrices $G_i(t)$, viz.,*

$$R_i(t) - F_i(t)R_i(t-1)F_i^*(t) = G_i(t)J(t)G_i^*(t) \quad ,$$

where $G_i(t)$ is an $(n-i) \times r(t)$ generator matrix that satisfies, along with $l_i(t)$, the following recursion

$$\left[\begin{array}{cc} l_i(t) & \begin{array}{c} \mathbf{0} \\ G_{i+1}(t) \end{array} \end{array}\right] = \left[\begin{array}{cc} F_i(t)l_i(t-1) & G_i(t) \end{array}\right] \left[\begin{array}{cc} f_i^*(t) & h_i^*(t)J(t) \\ J(t)g_i^*(t) & J(t)k_i^*(t)J(t) \end{array}\right], \tag{32}$$

where $g_i(t)$ is the first row of $G_i(t)$, and $h_i(t)$ and $k_i(t)$ are arbitrary $r(t) \times 1$ and $r(t) \times r(t)$ matrices, respectively, chosen so as to satisfy the embedding relation

$$(33) \qquad \begin{bmatrix} f_i(t) & g_i(t) \\ h_i(t) & k_i(t) \end{bmatrix} \begin{bmatrix} d_i(t-1) & \mathbf{0} \\ \mathbf{0} & J(t) \end{bmatrix} \begin{bmatrix} f_i(t) & g_i(t) \\ h_i(t) & k_i(t) \end{bmatrix}^* = \begin{bmatrix} d_i(t) & \mathbf{0} \\ \mathbf{0} & J(t) \end{bmatrix} ,$$

where

$$(34) \qquad d_i(t) = |f_i(t)|^2 d_i(t-1) + g_i(t) J(t) g_i^*(t) ,$$

and $F_i(t)$ is the $(n-i) \times (n-i)$ submatrix obtained after deleting the first row and column of $F_{i-1}(t)$.

■

The generator recursion (32) has a transmission-line picture in terms of a cascade of elementary sections as shown in Figure 4, where each section depends on the parameters $\{f_i(t), g_i(t), h_i(t), k_i(t)\}$. The Δ block represents a storage element where the present value of $l_i(t)$ is stored for the next time instant, and the block with $F_i(t)$ can be implemented as a tapped-delay filter with time-variant coefficients [20, 40, 31].

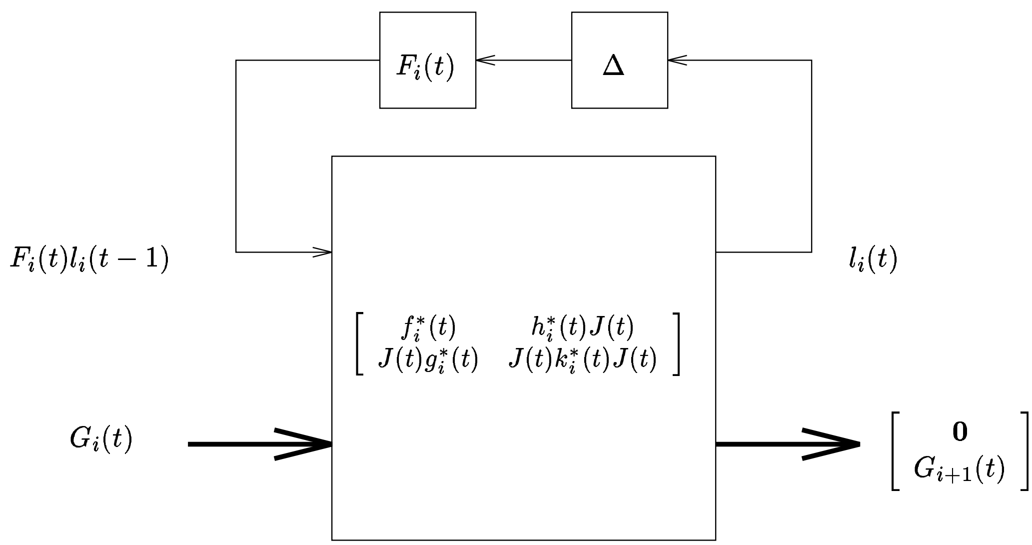

Figure 4: Time-variant transmission-line structure of the recursive algorithm.

LEMMA 5.1 *All possible choices of $h_i(t)$ and $k_i(t)$ are given by*

$$h_i(t) = \Theta_i^{-1}(t) \left\{ \frac{1 - \tau_i^*(t) f_i(t)}{\tau_i^*(t) d_i(t) - d_i(t-1) f_i^*(t)} J(t) g_i^*(t) \right\} ,$$

$$(35) \qquad k_i(t) = \Theta_i^{-1}(t) \left\{ I_{r(t)} - \frac{\tau_i^*(t) J(t) g_i^*(t) g_i(t)}{\tau_i^*(t) d_i(t) - d_i(t-1) f_i^*(t)} \right\} ,$$

where $\Theta_i(t)$ is an arbitrary $J(t)-$unitary matrix, and $\tau_i(t)$ is an arbitrary complex number chosen on the circle $|\tau_i(t)|^2 = d_i(t-1)/d_i(t)$.

■

5.1. Time-Variant Derivatives. Before discussing the application of the above algorithm to time-variant interpolation and matrix completion problems, we first introduce some notation and extend the notion of *"derivatives"* to the time-variant setting [20, 21]. We consider a finite-dimensional linear *time-variant* state-space model with a bounded upper-triangular transfer operator $\mathbf{T}$. The matrix entries of $\mathbf{T}$ are denoted by T_{ij} (of dimensions

$r(i) \times r(j))$ and correspond to the time-variant Markov parameters of the underlying state-space model:

$$\mathbf{T} = \begin{bmatrix} \ddots & \ddots & & & & \\ & T_{-1,-1} & T_{-1,0} & T_{-1,1} & \cdots & \\ & & \boxed{T_{00}} & T_{01} & T_{02} & \cdots \\ & \mathrm{O} & & T_{11} & T_{12} & T_{13} \\ & & & & \ddots & \ddots \end{bmatrix},$$

where $\boxed{T_{00}}$ denotes the $(0,0)$ entry of $\mathbf{T}$. We further consider a *stable* sequence of scalar points $\{f(t)\}_{t\in\mathbb{Z}}$ ($\mathbb{Z}$ is the set of integers), viz., $\exists\ c > 0$ such that $|f(t)| < c < 1$ for all t. We also introduce the symmetric functions $s_k^{(n)}$ of n variables (taken k at a time). That is, $s_0^{(n)} = 1$ and

$$s_k^{(n)}(x_1, x_2, \ldots, x_n) = \sum_{1\le i_1<\ldots<i_k\le n} x_{i_1} x_{i_2} \ldots x_{i_k}.$$

For a uniformly bounded sequence of $1 \times r(t)$ row vectors $\{u(t)\}_{t\in\mathbb{Z}}$, viz., $\exists\ \bar{c} > 0$ such that $\|u(t)\| < \bar{c}$ for all t, we define the $1 \times r(t)$ row vector $u(t) \bullet \mathbf{T}(f(t))$ as follows

$$u(t) \bullet \mathbf{T}(f(t)) = u(t)T_{tt} + f(t)u(t-1)T_{t-1,t} + f(t)f(t-1)u(t-2)T_{t-2,t} + \ldots.$$

This corresponds to a time-variant tangential evaluation along the direction defined by $u(t)$. More generally, we define the $1 \times r(t)$ row vectors (for $p \ge 0$)

$$u(t) \bullet \frac{1}{p!}\mathbf{T}^{(p)}(f(t)) \equiv \sum_{m=0}^{\infty} s_m^{(m+p)} \left[f(t), f(t-1), \ldots, f(t-m-p+1)\right] u(t-m-p)T_{t-m-p,t}\,.$$

We shall also use the compact notation $\begin{bmatrix} u_1(t) & u_2(t) \end{bmatrix} \bullet \mathcal{H}^2_{\mathbf{T}}(f(t))$ to denote the row vector $\begin{bmatrix} u_1(t) \bullet \mathbf{T}(f(t)) & u_1(t) \bullet \frac{1}{1!}\mathbf{T}^{(1)}(f(t)) + u_2(t) \bullet \mathbf{T}(f(t)) \end{bmatrix}$, which we also write as

$$\begin{bmatrix} u_1(t) & u_2(t) \end{bmatrix} \bullet \begin{bmatrix} \mathbf{T}(f(t)) & \frac{1}{1!}\mathbf{T}^{(1)}(f(t)) \\ & \mathbf{T}(f(t)) \end{bmatrix}.$$

More generally, we write $\begin{bmatrix} u_1(t) & u_2(t) & \ldots & u_r(t) \end{bmatrix} \bullet \mathcal{H}^r_{\mathbf{T}}(f(t)) =$

$$\begin{bmatrix} u_1(t) & u_2(t) & \ldots & u_r(t) \end{bmatrix} \bullet \begin{bmatrix} \mathbf{T}(f(t)) & \frac{1}{1!}\mathbf{T}^{(1)}(f(t)) & \frac{1}{2!}\mathbf{T}^{(2)}(f(t)) & \ldots & \frac{1}{(r-1)!}\mathbf{T}^{(r-1)}(f(t)) \\ & \mathbf{T}(f(t)) & \frac{1}{1!}\mathbf{T}^{(1)}(f(t)) & \ldots & \frac{1}{(r-2)!}\mathbf{T}^{(r-2)}(f(t)) \\ & & \ddots & & \vdots \\ & \mathrm{O} & & \mathbf{T}(f(t)) & \frac{1}{1!}\mathbf{T}^{(1)}(f(t)) \\ & & & & \mathbf{T}(f(t)) \end{bmatrix}.$$

We remark that the above expressions for time-variant derivatives and tangential evaluation reduce to the standard definitions in the time-invariant case, where $\mathcal{T}$ is a Toeplitz operator.

6. Time-Variant Hermite-Fejér. We now extend the statement of the Hermite-Fejér problem to the time-variant setting. This extension includes as special cases the time-variant versions of the Carathéodory-Fejér and Nevanlinna-Pick problems studied in [16, 18, 19].

We consider m stable points $\{\alpha_i(t)\}_{i=0}^{m-1}$ inside the open unit disc, and we associate with each point $\alpha_i(t)$ a positive integer $r_i \ge 1$ and uniformly bounded row vectors $\mathbf{a}_i(t)$ and $\mathbf{b}_i(t)$ partitioned as follows

$$\mathbf{a}_i(t) = \begin{bmatrix} u_1^{(i)}(t) & u_2^{(i)}(t) & \ldots & u_{r_i}^{(i)}(t) \end{bmatrix}, \quad \mathbf{b}_i(t) = \begin{bmatrix} v_1^{(i)}(t) & v_2^{(i)}(t) & \ldots & v_{r_i}^{(i)}(t) \end{bmatrix},$$

where $u_j^{(i)}(t)$ and $v_j^{(i)}(t)$ $(j = 1, \ldots, r_i)$ are $1 \times p(t)$ and $1 \times q(t)$ row vectors, respectively. The time-variant Hermite-Fejér interpolation problem then reads as follows.

PROBLEM 6.1 *Given m stable points $\{\alpha_i(t)\}$ with the associated data $r_i, \mathbf{a}_i(t)$, and $\mathbf{b}_i(t)$, describe all upper triangular strictly contractive transfer operators $\mathbf{S}$ ($\|\mathbf{S}\|_\infty < 1$) that satisfy*

$$\mathbf{b}_i(t) = \mathbf{a}_i(t) \bullet \mathcal{H}_{\mathbf{S}}^{r_i}(\alpha_i(t)) \quad \text{for} \quad 0 \le i \le m-1. \tag{36}$$

■

The first step in the solution consists in constructing three matrices $F(t), G(t)$, and $J(t)$ directly from the interpolation data as in the time-invariant case: we define $J(t) = (I_{p(t)} \oplus -I_{q(t)})$, and associate with each $\alpha_i(t)$ a Jordan block $\bar{F}_i(t)$ of size $r_i \times r_i$,

$$\bar{F}_i(t) = \begin{bmatrix} \alpha_i(t) & & & \\ 1 & \alpha_i(t) & & \\ & \ddots & \ddots & \\ & & 1 & \alpha_i(t) \end{bmatrix},$$

and two $r_i \times p(t)$ and $r_i \times q(t)$ matrices $U_i(t)$ and $V_i(t)$, respectively, which are composed of the row vectors associated with $\alpha_i(t)$,

$$U_i(t) = \begin{bmatrix} u_1^{(i)}(t) \\ u_2^{(i)}(t) \\ \vdots \\ u_{r_i}^{(i)}(t) \end{bmatrix} \quad \text{and} \quad V_i(t) = \begin{bmatrix} v_1^{(i)}(t) \\ v_2^{(i)}(t) \\ \vdots \\ v_{r_i}^{(i)}(t) \end{bmatrix}.$$

Then $F(t) =$ diagonal $\{\bar{F}_0(t), \bar{F}_1(t), \ldots, \bar{F}_{m-1}(t)\}$ and

$$G(t) = \begin{bmatrix} U_0(t) & V_0(t) \\ U_1(t) & V_1(t) \\ \vdots & \vdots \\ U_{m-1}(t) & V_{m-1}(t) \end{bmatrix} \equiv \begin{bmatrix} \mathbf{U}(t) & \mathbf{V}(t) \end{bmatrix}. \tag{37}$$

Let $n = \sum_{i=0}^{m-1} r_i$ and $r(t) = p(t) + q(t)$, then $F(t)$ and $G(t)$ are $n \times n$ and $n \times r(t)$ matrices respectively. We shall denote the diagonal entries of $F(t)$ by $\{f_i(t)\}_{i=0}^{n-1}$ (for example, $f_0(t) = f_1(t) = \ldots = f_{r_0-1}(t) = \alpha_0(t)$). We also associate with the interpolation problem the time-variant displacement equation

$$R(t) - F(t)R(t-1)F^*(t) = G(t)J(t)G^*(t). \tag{38}$$

We shall further assume that the interpolation data satisfy the following nondegeneracy condition, which is automatically satisfied in many problems,

$$\mathcal{U}(t) \equiv \begin{bmatrix} \ldots & F(t)F(t-1)\mathbf{U}(t-2) & F(t)\mathbf{U}(t-1) & \mathbf{U}(t) \end{bmatrix}$$

has the property,

$$\mathcal{U}(t)\mathcal{U}^*(t) > \mu > 0 \quad \text{for all } t, \tag{39}$$

where μ is a fixed constant. The more general case is treated in [21] and will be briefly discussed later. The proof of the next theorem follows the same lines as that of Theorem 4.1.1 [20].

THEOREM 6.1 *The tangential Hermite-Fejér problem is solvable if, and only if, there exists a real number $\epsilon > 0$, independent of t, such that the solution $R(t)$ of (38) satisfies $R(t) > \epsilon I$ for all t.*

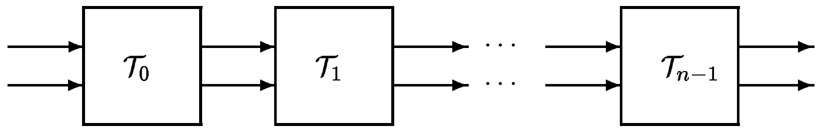

Figure 5: Cascade of first-order time-variant sections.

■

Before proceeding further, we first state [20] the implications of stability and uniform boundedness of the interpolation data $\{f_i(t),\ \mathbf{a}_i(t),\ \mathbf{b}_i(t)\}$ on the boundedness of the quantities $d_i(t)$ and $g_i(t)$ that are needed in the recursive procedure.

LEMMA 6.1 *The sequences* $\{d_i(t)\}_{t\in\mathbb{Z}}$ *and* $\{g_i(t)\}_{t\in\mathbb{Z}}$ *obtained through the recursive Schur reduction procedure are uniformly bounded. More specifically, there exist real numbers* b_d, c_d, *and* c_g *(independent of* t*) such that*

$$0 < b_d < d_i(t) < c_d \quad \text{and} \quad \|g_i(t)\| < c_g \quad \textit{for all } t.$$

■

6.1. Interpolation Properties. Observe again that each generator step as in (32) involves a linear first-order discrete-time system (in state-space form) that appears on the right-hand side of (32), viz.,

$$(40) \qquad \left[\begin{array}{cc} \mathbf{x}_i(t+1) & \mathbf{y}_i(t) \end{array}\right] = \left[\begin{array}{cc} \mathbf{x}_i(t) & \mathbf{w}_i(t) \end{array}\right] \left[\begin{array}{cc} f_i^*(t) & h_i^*(t)J(t) \\ J(t)g_i^*(t) & J(t)k_i^*(t)J(t) \end{array}\right],$$

where $\mathbf{x}_i(t)$ is the state and $\mathbf{w}_i(t)$ is a $1 \times r(t)$ *row* input vector at time t. As in the time-invariant case, the first-order system (40) also has important $J(t)-$losslessness and blocking properties.

Let $\mathbf{T}_i = \left[\begin{array}{c} T_{lj}^{(i)} \end{array}\right]_{l,j=-\infty}^{\infty}$ denote the upper-triangular transfer operator of the i^{th} section (40), where $T_{lj}^{(i)}$ denote the $r(l) \times r(j)$ time-variant Markov parameters of $\mathbf{T}_i$ and are given by

$$\begin{aligned} T_{ll}^{(i)} &= J(l)k_i^*(l)J(l) \ , \\ T_{l,l+1}^{(i)} &= J(l)g_i^*(l)h_i^*(l+1)J(l+1) \ , \\ T_{lj}^{(i)} &= J(l)g_i^*(l)f_i^*(l+1)f_i^*(l+2)\ldots f_i^*(j-1)h_i^*(j)J(j) \quad \text{for } j > l+1. \end{aligned}$$

After n recursive steps (recall that $G(t)$ has n rows) we obtain a cascade of sections $\mathbf{T}$ defined by (Figure 5)

$$(41) \qquad \mathbf{T} = \mathbf{T}_0\mathbf{T}_1\ldots\mathbf{T}_{n-1}.$$

Our purpose is to prove that all solutions $\mathbf{S}$ to the interpolation Problem 6.1 can be parametrized in terms of a linear fractional transformation based on $\mathbf{T}$.

We already know that $\{f_i(t), g_i(t)\}_{t\in\mathbb{Z}}$ are stable and uniformly bounded sequences (recall Lemma 6.1). Moreover, it is always possible [20, 21] to choose the free parameters $\Theta_i(t)$ and $\tau_i(t)$ in (35) so as to guarantee the uniform boundedness of the sequences $\{h_i(t), k_i(t)\}_{t\in\mathbb{Z}}$. This is, for example, the case if we set $\Theta_i(t) = I_r$ and choose $\tau_i(t)$ on the circle of radius

$\sqrt{\frac{d_i(t-1)}{d_i(t)}}$ but in the opposite direction of $f_i(t)$. Other choices are also possible and lead to time-variant lattice structures [20].

It is then a standard result that the boundedness of $\{f_i(t), g_i(t), h_i(t), k_i(t)\}$ assures the boundedness of the corresponding operator $\mathbf{T}_i$ (see, *e.g.*, [41]). Moreover, if we define the direct sum $\mathcal{J} = \bigoplus_{t\in\mathbb{Z}} J(t)$, then it readily follows from the embedding relation (33) that each $\mathbf{T}_i$ also satisfies the following $\mathbf{J}$-losslessness property,

$$\mathcal{T}_i \mathcal{J} \mathcal{T}_i^* = \mathcal{J} \quad \text{and} \quad \mathcal{T}_i^* \mathcal{J} \mathcal{T}_i = \mathcal{J}$$

Furthermore, each section $\mathbf{T}_i$ satisfies an important time-variant blocking property.

THEOREM 6.1.1 *Each first-order section* $\mathbf{T}_i$ *satisfies*

$$\begin{bmatrix} \cdots & f_i(t)f_i(t-1)g_i(t-2) & f_i(t)g_i(t-1) & g_i(t) & ? \end{bmatrix} \mathcal{T}_i = \begin{bmatrix} \mathbf{0} & ? \end{bmatrix} ,$$

where $g_i(t)$ *is at the* t^{th} *position of the row vector. Consequently,* $g_i(t) \bullet \mathcal{T}_i(f_i(t)) = \mathbf{0}$.

Proof: This follows directly from the embedding result (33) (as well as from the fact that each step of the generator recursion (32) produces a zero row). The output of $\mathbf{T}_i$ at time t is given by

$$\begin{aligned} \mathbf{y}_i(t) &= \ldots + f_i(t)f_i(t-1)g_i(t-2)T_{t-2,t} + f_i(t)g_i(t-1)T_{t-1,t} + g_i(t)T_{tt} \\ &= \left[-d_i(t-1) + d_i(t-1)\right] f_i(t)h_i^*(t)J(t) = \mathbf{0} \ , \end{aligned}$$

where we substituted the expressions for the Markov parameters and used

$$\begin{aligned} d_i(t) &= g_i(t)J(t)g_i^*(t) + f_i(t)g_i(t-1)J(t-1)g_i^*(t-1)f_i^*(t) + \\ & \quad f_i(t)f_i(t-1)g_i(t-2)J(t-2)g_i^*(t-2)f_i^*(t-1)f_i^*(t) + \ldots \end{aligned}$$

The same argument holds for the previous outputs.

■

The $\mathbf{J}$-losslessness and blocking properties of each section $\mathbf{T}_i$ reflect on the entire cascade $\mathbf{T}$, and it readily follows that $\mathbf{T}$ is a bounded upper-triangular linear operator that satisfies $\mathbf{T}\mathcal{J}\mathbf{T}^* = \mathbf{T}^*\mathcal{J}\mathbf{T} = \mathcal{J}$. It also follows from the last theorem that $\mathbf{T}$ satisfies an important global blocking property.

THEOREM 6.1.2 *The entire cascade* $\mathbf{T}$ *satisfies the global blocking property*

$$\begin{bmatrix} \cdots & F(t)F(t-1)G(t-2) & F(t)G(t-1) & G(t) & \mathbf{0} & \mathbf{0} & \cdots \end{bmatrix} \mathbf{T} = \begin{bmatrix} \mathbf{0} & ? \end{bmatrix} , \tag{42}$$

where $G(t)$ *is in the* t^{th} *position. That is, if we apply to* $\mathbf{T}$ *the block input*

$$\tilde{\mathbf{U}}(t) = \begin{bmatrix} \cdots & F(t)F(t-1)G(t-2) & F(t)G(t-1) & G(t) & \mathbf{0} & \mathbf{0} & \cdots \end{bmatrix}$$

then the output is zero up to and including time t.

Proof: This follows from the generator recursion (32) and from the Jordan structure of $F(t)$. When the first row of $\tilde{\mathbf{U}}(t)$ goes through the first section $\mathbf{T}_0$, it annihilates the output of the entire cascade $\mathbf{T}$ due to the blocking property of $\mathbf{T}_0$. When the second row of $\tilde{\mathbf{U}}(t)$ goes through $\mathbf{T}_0$, we obtain at the output of $\mathbf{T}_0$ (as a consequence of (32) and the Jordan structure of $F(t)$) a zero-direction vector for $\mathbf{T}_1$, which again annihilates the output of the entire cascade $\mathbf{T}$, and so on.

■

Expression (42) is closely related to the interpolation conditions of Problem 6.1. To motivate this, we denote by $s_i = \sum_{p=0}^{i-1} r_p$, $s_0 = 0$, the total size of the Jordan blocks prior to $\bar{F}_i(t)$. By comparing terms on both sides of (42) (and by using the Jordan structure of $F(t)$) we can verify that (42) can be rewritten in the following form

$$\begin{bmatrix} e_{s_i}G(t) & e_{s_i+1}G(t) & \ldots & e_{s_i+r_i-1}G(t) \end{bmatrix} \bullet \mathcal{H}_{\mathbf{T}}^{r_i}(\alpha_i(t)) = \mathbf{0}, \tag{43}$$

where the row vector on the left hand-side of (43) is composed of the r_i row vectors in $\begin{bmatrix} U_i(t) & V_i(t) \end{bmatrix}$ associated with $\alpha_i(t)$, viz.,

$$\begin{bmatrix} u_1^{(i)}(t) & v_1^{(i)}(t) & u_2^{(i)}(t) & v_2^{(i)}(t) & \ldots & u_{r_i}^{(i)}(t) & v_{r_i}^{(i)}(t) \end{bmatrix}.$$

We now show how to parametrize all solutions to the interpolation problem in terms of $\mathcal{T}$. If we partition the matrix entries T_{lj} of the cascade $\mathbf{T}$ accordingly with $J(l)$ and $J(j)$,

$$T_{lj} = \begin{bmatrix} T_{11}^{lj} & T_{12}^{lj} \\ \\ T_{21}^{lj} & T_{22}^{lj} \end{bmatrix},$$

and consider the triangular operators

$$\mathbf{T}_{12} = \left[T_{12}^{lj}\right]_{l,j=-\infty}^{\infty} \quad \text{and} \quad \mathbf{T}_{22}^{(i)} = \left[T_{22}^{lj}\right]_{l,j=-\infty}^{\infty}.$$

Then it can be shown [20, 21] that $\mathbf{S} = -\mathbf{T}_{12}\mathbf{T}_{22}^{-1}$ is an upper-triangular strictly contractive operator. It also follows from Theorem 6.1.2 that $\mathbf{S}$ satisfies the required interpolation conditions. For instance, we conclude from (43) that

$$\begin{bmatrix} \ldots & f_0(t)f_0(t-1)g_0(t-2) & f_0(t)g_0(t-1) & g_0(t) & \mathbf{0} & \mathbf{0} & \ldots \end{bmatrix} \begin{bmatrix} \mathbf{T}_{12} \\ \mathbf{T}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & ? \end{bmatrix},$$

or equivalently,

$$v_1^{(0)}(t) = u_1^{(0)}(t) \bullet \mathbf{S}(f_0(t)).$$

This argument can be continued, as in the time-invariant case, to show that $\mathcal{S}$ satisfies the remaining interpolation conditions. Moreover, we can parametrize all solutions as follows [20, 21].

THEOREM 6.1.3 *All solutions* $\mathbf{S}$ *to the tangential Hermite-Fejér problem are given through a linear fractional transformation of a strictly contractive upper-triangular operator* $\mathbf{K}$,

$$\mathbf{S} = -\left[\mathbf{T}_{11}\mathbf{K} + \mathbf{T}_{12}\right]\left[\mathbf{T}_{21}\mathbf{K} + \mathbf{T}_{22}\right]^{-1}. \tag{44}$$

■

7. Completion Problems. The discussion in the previous sections was restricted to structured matrices R with scalar entries r_{mj}. The results however, are more general. We now state a result proved in [21], which shows that, under a certain positivity condition, it is always possible to associate an abstract interpolation problem with a time-variant structured matrix, and that the existence of a solution to this abstract problem is characterized by a natural embedding property. This result includes as particular cases earlier developments in [9, 42, 43].

We consider a family of block matrices, depending on the parameter $t \in \mathbb{Z}$,

$$R(t) = \left[r_{mj}(t)\right]_{m,j=0}^{n-1}, \quad r_{mj}(t) \in \mathbf{L}(\mathcal{H}_j(t), \mathcal{H}_m(t)),$$

where $\{\mathcal{H}_m(t)\}_{m=0}^{n-1}$ are families of Hilbert spaces depending on the parameter $t \in \mathbb{Z}$, and for two Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$ we use the notation $\mathbf{L}(\mathcal{H}, \mathcal{H}')$ to denote the set of linear bounded operators acting from $\mathcal{H}$ into $\mathcal{H}'$. If we define $\mathcal{H}(t) = \bigoplus_{m=0}^{n-1} \mathcal{H}_m(t)$, then $R(t) \in \mathbf{L}(\mathcal{H}(t))$. We also consider two families $\{\mathbf{F}(t)\}_{t\in\mathbb{Z}}$ and $\{\mathbf{G}(t)\}_{t\in\mathbb{Z}}$ of Hilbert spaces, two families of bounded linear operators,

$$F(t) \in \mathbf{L}(\mathcal{H}(t-1), \mathcal{H}(t)) \quad \text{and} \quad G(t) \in \mathbf{L}(\mathbf{F}(t) \oplus \mathbf{G}(t), \mathcal{H}(t)),$$

and we define the symmetry $J(t) = (I_{\mathbf{F}(t)} \oplus -I_{\mathbf{G}(t)})$, on $\mathbf{F}(t) \bigoplus \mathbf{G}(t)$, where $I_{\mathbf{F}(t)}$ denotes the identity operator on the space $\mathbf{F}(t)$. We also write $G(t) = \begin{bmatrix} \mathbf{U}(t) & \mathbf{V}(t) \end{bmatrix}$, where

$$\mathbf{U}(t) \in \mathbf{L}(\mathbf{F}(t), \mathcal{H}(t)) \quad \text{and} \quad \mathbf{V}(t) \in \mathbf{L}(\mathbf{G}(t), \mathcal{H}(t)).$$

We assume that $\{F(t)\}_{t\in\mathbb{Z}}$ is a uniformly bounded family of lower triangular operators, with stable diagonal entries $\{f_0(t), f_1(t), \ldots, f_{n-1}(t)\}$, viz.,

$$\exists\ c_f > 0 \ \text{ such that } \ \|f_i(t)\| \le c_f < 1 \ \text{ for all } \ t \in \mathbb{Z} \ \text{ and } \ i = 0, 1, \ldots, n-1.$$

We shall say that $\{F(t)\}_{t\in\mathbb{Z}}$ is a *stable* family (these conditions can be relaxed [21]). We also assume that $\{G(t)\}_{t\in\mathbb{Z}}$ is a uniformly bounded family of operators, viz.,

$$\exists\ c_G > 0 \ \text{ such that } \ \|G(t)\| \le c_G \ \text{ for all } \ t \in \mathbb{Z}.$$

We shall also say that $\{R(t)\}_{t\in\mathbb{Z}}$ has a time-variant displacement structure with respect to the family of operators $\{F(t), G(t), J(t)\}_{t\in\mathbb{Z}}$ if $\{R(t)\}_{t\in\mathbb{Z}}$ satisfies the time-variant displacement equation

$$R(t) - F(t)R(t-1)F^*(t) = G(t)J(t)G^*(t) \ , \tag{45}$$

where the symbol $*$ refers to the adjoint operator ($F^*(t) = F(t)^*$). The cardinal number, $r(t) = \dim \mathbf{F}(t) + \dim \mathbf{G}(t)$, is called the displacement rank of $R(t)$ in (45). We say that (45) has a *Pick solution* if, and only if, $R(t)$ is positive-semidefinite for every $t \in \mathbb{Z}$. If we define,

$$\mathbf{U}(t) = \begin{bmatrix} \ldots & F(t)F(t-1)\mathbf{U}(t-2) & F(t)\mathbf{U}(t-1) & \mathbf{U}(t) \end{bmatrix}$$

and

$$\mathbf{V}(t) = \begin{bmatrix} \ldots & F(t)F(t-1)\mathbf{V}(t-2) & F(t)\mathbf{V}(t-1) & \mathbf{V}(t) \end{bmatrix} \ ,$$

then we remark that $\mathbf{U}(t)$ and $\mathbf{V}(t)$ are well defined bounded linear operators,

$$\mathbf{U}(t) \in \mathbf{L}(\bigoplus_{j\le t} \mathbf{F}(j), \mathcal{H}(t)), \quad \mathbf{V}(t) \in \mathbf{L}(\bigoplus_{j\le t} \mathbf{G}(j), \mathcal{H}(t)) \ ,$$

and that $R(t)$ is given by

$$R(t) = \mathbf{U}(t)\mathbf{U}^*(t) - \mathbf{V}(t)\mathbf{V}^*(t). \tag{46}$$

The following result [21] shows that the existence of a Pick solution of (45) is equivalent to the existence of an upper triangular contraction relating $\mathbf{U}(t)$ and $\mathbf{V}(t)$. We have already encountered a special case of this result in the proof of Theorem 4.1.1 (and also Theorem 6.1).

THEOREM 7.1 *The time-variant displacement equation (45) has a Pick solution $R(t)$ if, and only if, there exists an upper triangular contraction $\mathbf{S}$ ($\|\mathbf{S}\| \le 1$),*

$$\mathbf{S} \in \mathbf{L}(\bigoplus_{t\in\mathbb{Z}} \mathbf{G}(t), \bigoplus_{t\in\mathbb{Z}} \mathbf{F}(t)),$$

such that

(47) $$\mathbf{V}(t) = \mathbf{U}(t) P_{\mathbf{F}}(t) \mathbf{S} / \underset{j \le t}{\oplus} \mathbf{G}(j) \quad \text{for every } t \in \mathbb{Z}$$

where $P_{\mathbf{F}}(t)$ *denotes the orthogonal projection of* $\underset{t \in \mathbb{Z}}{\oplus} \mathbf{F}(t)$ *onto* $\underset{j \le t}{\oplus} \mathbf{F}(j)$.

■

It can be further shown that the contraction $\mathbf{S}$ is the solution of a general interpolation problem [21]. We shall instead show that several completion problems recently considered in connection with moment theory can be solved within the framework of displacement structure theory.

We fix a family $\{\mathbf{E}(n)\}_{n \in \mathbb{Z}}$ of Hilbert spaces and a positive integer p. We now verify that the solution of the following band completion problem [44] follows as a special case of Theorem 7.1.

PROBLEM 7.1 *Given a family* $\{\tilde{Q}_{ij} / \ i, j \in \mathbb{Z}, \ \ |j-i| \le p\}$ *of operators such that* $\tilde{Q}_{ij} = \tilde{Q}^*_{ji}$ *and* $\tilde{Q}_{ij} \in \mathbf{L}(\mathbf{E}(j), \mathbf{E}(i))$, *it is required to find conditions for the existence of a positive definite kernel* $M = [Q_{ij}]_{i,j \in \mathbb{Z}}$ *such that for* $i, j \in \mathbb{Z}$ *and* $|j - i| \le p$, $Q_{ij} = \tilde{Q}_{ij}$.

■

By a positive definite kernel we mean an application $M = [Q_{ij}]_{i,j \in \mathbb{Z}}$ on $\mathbb{Z} \times \mathbb{Z}$ such that for $i, j \in \mathbb{Z}$, we have

$$Q_{ij} \in \mathbf{L}(\mathbf{E}(j), \mathbf{E}(i)) \quad \text{and} \quad \sum_{i,j=-n}^{n} < Q_{ij} h_j, h_i > \ \ge 0 \ ,$$

for every integer $n > 0$ and every set of vectors $\{h_{-n}, h_{-n+1}, \cdots, h_n\}$, $h_k \in \mathbf{E}(k)$, $|k| \le n$. In case $\tilde{Q}_{ij} = \tilde{Q}_{|j-i|}$, the above problem is the well-known truncated trigonometric moment problem. The case $\mathbf{E}(n) = \mathbf{0}$ for $|n|$ large enough, was solved in [44].

Without loss of generality, we can suppose $\tilde{Q}_{ii} = I$ for all $i \in \mathbb{Z}$. Define the spaces,

(48) $$\mathcal{H}(t) = \overset{p}{\underset{k=0}{\oplus}} \mathbf{E}(-t+k), \ \ \mathbf{F}(t) = \mathbf{G}(t) = \mathbf{E}(-t) \ ,$$

and the operators

(49) $$\mathbf{U}(t) = \begin{bmatrix} I \\ \tilde{Q}_{-t+1,-t} \\ \tilde{Q}_{-t+2,-t} \\ \vdots \\ \tilde{Q}_{-t+p,-t} \end{bmatrix} \quad \text{and} \quad \mathbf{V}(t) = \begin{bmatrix} \mathbf{0} \\ \tilde{Q}_{-t+1,-t} \\ \tilde{Q}_{-t+2,-t} \\ \vdots \\ \tilde{Q}_{-t+p,-t} \end{bmatrix} .$$

We also consider the operators $J(t) = (I_{\mathbf{F}(t)} \bigoplus -I_{\mathbf{G}(t)})$,

(50) $$F(t) = \begin{bmatrix} \mathbf{0} & & & \\ I & \mathbf{0} & & \\ & \ddots & \ddots & \\ & & I & \mathbf{0} \end{bmatrix}, \quad G(t) = \begin{bmatrix} \mathbf{U}(t) & \mathbf{V}(t) \end{bmatrix} .$$

These elements specify a displacement structure of the form (45), and the following result follows from Theorem 7.1.

THEOREM 7.2 *Problem 7.1 has solutions if, and only if, the displacement equation associated with the data (48)–(50), has a Pick solution.*

The following so-called tangential Carathéodory-Fejér) problem is a special case of Problem

6.1 and arises, for example, in model validation [45].

PROBLEM 7.2 *Given families of matrices* $\{U_i(t), V_i(t)\}_{t\in\mathbb{Z}}$, $i = 0, 1, \dots, n-1$, *it is required to find conditions for the existence of an upper triangular contraction* $\mathcal{S}$ *such that*

$$\begin{bmatrix} U_0(t-n+1) & \dots & U_{n-1}(t) \end{bmatrix} \begin{bmatrix} \mathbf{S}_{t-n+1,t-n+1} & & \cdots & \mathbf{S}_{t-n+1,t} \\ & \ddots & & \vdots \\ & & \mathbf{S}_{t-1,t-1} & \mathbf{S}_{t-1,t} \\ & O & & \mathbf{S}_{tt} \end{bmatrix} = \begin{bmatrix} V_0(t-n+1) & \dots & V_{n-1}(t) \end{bmatrix}.$$

■

This problem can be stated as imposing linear constraints on the "time-variant derivatives" of $\mathbf{S}$. To reduce the problem to our framework we define for all t,

$$\mathbf{U}(t) = \begin{bmatrix} U_0(t) \\ U_1(t) \\ \vdots \\ U_{n-1}(t) \end{bmatrix}, \quad \mathbf{V}(t) = \begin{bmatrix} V_0(t) \\ V_1(t) \\ \vdots \\ V_{n-1}(t) \end{bmatrix},$$

$$F(t) = \begin{bmatrix} \mathbf{0} & & & \\ I & \mathbf{0} & & \\ & \ddots & \ddots & \\ & & I & \mathbf{0} \end{bmatrix}, \quad G(t) = \begin{bmatrix} \mathbf{U}(t) & \mathbf{V}(t) \end{bmatrix}.$$

THEOREM 7.3 *The tangential Carathéodory-Fejér problem has solutions if, and only if, the displacement equation associated with the above data has a Pick solution. This is equivalent to the condition*

$$\mathbf{U}(t)\mathbf{U}^*(t) \geq \mathbf{V}(t)\mathbf{V}^*(t) \quad \textit{for all } t \in \mathbb{Z},$$

with

$$\mathbf{U}(t) = \begin{bmatrix} & & & U_0(t) \\ & & U_0(t-1) & U_1(t) \\ & \cdot^{\cdot^{\cdot}} & & \vdots \\ U_0(t-n+1) & \dots & U_{n-2}(t-1) & U_{n-1}(t) \end{bmatrix};$$

$$\mathbf{V}(t) = \begin{bmatrix} & & & V_0(t) \\ & & V_0(t-1) & V_1(t) \\ & \cdot^{\cdot^{\cdot}} & & \vdots \\ V_0(t-n+1) & \dots & V_{n-2}(t-1) & V_{n-1}(t) \end{bmatrix}.$$

■

Another completion problem whose solution can be obtained as a special case of Theorem 7.1 is the so-called strong Parrott problem [46, 47].

PROBLEM 7.3 *Given matrices* B_{ij}, $1 \leq j \leq i \leq n$, $S = \begin{bmatrix} S_1 & S_2 & \dots & S_n \end{bmatrix}$ *and* $T = \begin{bmatrix} T_1 & T_2 & \dots & T_n \end{bmatrix}$, *it is required to find conditions for the existence of a contraction* $\mathbf{T}$ *of the form*

$$\mathcal{T} = \begin{bmatrix} B_{11} & & & \\ B_{21} & B_{22} & & ? \\ \vdots & & \ddots & \\ B_{n1} & B_{n2} & \dots & B_{nn} \end{bmatrix},$$

such that $S\mathbf{T} = T$, where ? denotes unspecified entries.

■

To put this problem into our framework, we define

$$\mathbf{U}(t) = \begin{bmatrix} \mathbf{0} \\ I \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}, \quad 1 \le t \le n-1, \qquad \mathbf{U}(t) = \begin{bmatrix} S_{n+t} \\ I \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}, \quad -n+1 \le t \le 0 \ ,$$

$$\mathbf{V}(0) = \begin{bmatrix} T_1 \\ B_{n1} \\ B_{n-1,1} \\ \vdots \\ B_{11} \end{bmatrix}, \ \mathbf{V}(1) = \begin{bmatrix} T_2 \\ \mathbf{0} \\ B_{n2} \\ \vdots \\ B_{22} \end{bmatrix}, \ \mathbf{V}(2) = \begin{bmatrix} T_3 \\ \mathbf{0} \\ \mathbf{0} \\ B_{n3} \\ \vdots \\ B_{33} \end{bmatrix}, \ldots, \ \mathbf{V}(n-1) = \begin{bmatrix} T_n \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \\ B_{nn} \end{bmatrix},$$

$$F(t) = \begin{bmatrix} I & & & & & \\ \mathbf{0} & \mathbf{0} & & & & \\ & I & \mathbf{0} & & & \\ & & I & \mathbf{0} & & \\ & & & \ddots & \ddots & \\ & & & & I & \mathbf{0} \end{bmatrix}, \quad G(t) = \begin{bmatrix} \mathbf{U}(t) & \mathbf{V}(t) \end{bmatrix}, \quad \text{for} \quad -n+1 \le t \le n-1 \ ,$$

and all the elements equal to zero for the other time indices. We then have the following result (using the relaxed version of Theorem 7.1 in [21]).

THEOREM 7.4 *The strong Parrott problem has solutions if, and only if, the time-variant displacement equation associated with the above data has a Pick solution.*

■

8. Concluding Remarks. We discussed several applications of the displacement structure concept to interpolation and matrix completion problems. We emphasized that a transmission-line cascade arises naturally in the study of fast factorization algorithms for structured matrices, and that it has physically meaningful blocking properties or transmission zeros. This simple fact was then exploited to solve interpolation problems in both the time-variant and time-invariant settings. We also showed how several completion or moment problems fit naturally into the framework discussed in this paper.

Our recursive interpolation solution constructs a time-variant transmission-line cascade by *implicitly* considering the triangular factorization of the Pick matrix $R(t)$; unlike several earlier solutions, we do not require explicit knowledge of the matrices $R(t)$ or $R^{-1}(t)$. The whole recursive procedure works only with the matrices $F(t)$ and $G(t)$ that are constructed directly from the interpolation data. The overall computational complexity of the procedure is $O(r(t)n^2)$ operations (additions and multiplications) per time step, where $r(t)$ is a so-called displacement rank (the number of columns of the matrix $G(t)$).

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