1 IN *SYSTEM THEORY: MODELING, ANALYSIS AND CONTROL*,

T. DJAFERIS AND I. SCHICK, EDS., PP. 419–432, KLUWER, MA, 2000.

DISPLACEMENT STRUCTURE AND $\mathcal{H}_\infty$ PROBLEMS*

T. Constantinescu[1], A. H. Sayed[2], and T. Kailath[3]

[1]Programs in Mathematical Sciences
University of Texas
Richardson, TX 75083 USA
email: tiberiu@utdallas.edu

[2]Department of Electrical Engineering
University of California
Los Angeles, CA 90024 USA
email: sayed@ee.ucla.edu

[3]Department of Electrical Engineering
Stanford University
Stanford, CA 94305 USA
email: tk@rascals.stanford.edu

*The work of A. H. Sayed was partially supported by the National Science Foundation under award CCR-9732376, and that of T. Kailath by the National Science Foundation under award CCR-9628117 and the Army Research Office under grant DAAH-04-96-1-0176.

Abstract:

The solution of $\mathcal{H}_\infty$ problems requires the determination of contractive operators that map certain input signals to certain output signals. Such operators, and tests for their contractiveness, arise naturally in a scattering formulation of the generalized Schur algorithm, which is an efficient procedure for the triangular factorization of matrices with displacement structure. In this paper we explain this connection and show how to reformulate $\mathcal{H}_\infty$ problems, both for the finite and the infinite horizon cases, in terms of equivalent factorization problems for positive-definite matrices with structure.

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1.1 INTRODUCTION

The generalized Schur algorithm is a fast procedure for the factorization of matrices with displacement structure. A major feature of this algorithm is that it admits a powerful physical interpretation as a natural inverse scattering algorithm for determining the parameters of a cascade of elementary sections that combine together to form a layered medium structure; moreover, such scattering cascades map contractive loads at their outputs to contractive functions at their inputs. By studying the flow of signals through these layered media, several important (both old and new) results can be insightfully obtained. Among some of these we may mention the use of energy conservation ideas to obtain various matrix factorization and inversion formulas, layer-peeling and layer-adjoining algorithms for inverse scattering problems, and local blocking (or transmission zero) properties that combine together to yield cascades that satisfy global interpolation conditions. These and other applications are cited in the survey [1].

In this paper we shall demonstrate the usefulness of the contractive mapping property for $\mathcal{H}_\infty$ problems. It will be seen that by properly defining a convenient matrix structure, both conditions for the existence of, and a recursive construction for, $\mathcal{H}_\infty$ solutions are quite directly obtained. The key is that, as just noted, the generalized Schur algorithm constructs a contractive mapping that relates two so-called wave operators (consisting of the input and output signals of the scattering cascade). This fact is exploited to solve $\mathcal{H}_\infty$ problems since these problems are essentially concerned with studying such contractive mappings.

In the next section we review the generalized Schur algorithm and its scattering formulation. Then in the remaining sections we formulate the $\mathcal{H}_\infty$ problem, in both the finite and infinite-horizon cases, and then show how it can be reduced to the equivalent problem of factoring a structured matrix. State-space structure is not assumed, but as is now well-known, such structure can be combined with displacement structure (see, e.g., Sec. 9 of [1]) to reduce the computational burden; state-space methods are studied in [2]–[4].

1.2 THE GENERALIZED SCHUR ALGORITHM

We describe here the generalized Schur algorithm for a special class of structured matrices (more general descriptions can be found in [1, 5]).

Thus let F be an $n \times n$ *strictly* lower triangular matrix. Then we say that an $n \times n$ positive-definite Hermitian matrix R has *displacement structure* with respect to F if it satisfies a displacement equation of the form

$$R - FRF^* = GJG^* \ , \quad J = (I_p \oplus -I_q) \ , \tag{1.1}$$

where J is a signature matrix that specifies the displacement inertia of R, and G is an $n \times r$ so-called generator matrix with $r \ll n$ and $r = (p+q)$. We say that R has structure when the difference $R - FRF^*$ is low rank; its rank r is called the displacement rank of R. Since F is strictly lower triangular, the equation (1.1) has a unique solution R and, therefore, the triple $\{F, G, J\}$ fully characterizes R.

A major result concerning such structured matrices R is that the successive Schur complements of R, denoted by R_i, inherit a similar structure. That is, if R_i is the Schur complement of the leading $i \times i$ submatrix of R, then R_i also exhibits displacement structure of the form

$$R_i - F_i R_i F_i^* = G_i J G_i^* ,$$

where F_i is the submatrix obtained after deleting the first i rows and columns of F, and the generator G_i satisfies a recursive construction that we explain below.

Algorithm 1.1 (A generalized Schur algorithm) *Generator matrices G_i for the successive Schur complements R_i of a positive-definite structured matrix R, as in (1.1), can be recursively constructed as follows. Start with $G_0 = G, F_0 = F$, and repeat for $i \geq 0$:*

1. *At step i we have F_i and G_i. Let g_i denote the top row of G_i.*

2. *Choose any $J-$unitary rotation Θ_i that reduces g_i to the form*

$$g_i \Theta_i = \begin{bmatrix} \delta_i & 0 & \ldots & 0 \end{bmatrix} \tag{1.2}$$

Such a rotation always exists in view of the positive-definiteness of R and it can be implemented in many different ways, e.g., as a sequence of elementary unitary and hyperbolic rotations. [If such a transformation cannot be performed, then the given matrix R is not positive-definite; in other words, the generalized Schur algorithm can also be used as a test for positivity.]

3. *Apply Θ_i to G_i leading to the next generator as follows:*

$$\begin{bmatrix} 0 \\ G_{i+1} \end{bmatrix} = G_i \Theta_i \begin{bmatrix} 0 & 0 \\ 0 & I_{r-1} \end{bmatrix} + F_i G_i \Theta_i \begin{bmatrix} 1 & 0 \\ \mathbf{0} & 0 \end{bmatrix} . \tag{1.3}$$

4. *The columns of the Cholesky factor of R, viz., $R = \bar{L}\bar{L}^*$, are given by*

$$\bar{l}_i = G_i \Theta_i \begin{bmatrix} 1 \\ 0 \end{bmatrix} . \tag{1.4}$$

$\diamondsuit$

Pictorially, we have the following (see Fig. 1.1):

$$G_i = \begin{bmatrix} \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \\ \vdots & & \vdots \end{bmatrix} \overset{\Theta_i}{\rightarrow} \begin{bmatrix} \delta_i & 0 & 0 \\ \times' & \times' & \times' \\ \times' & \times' & \times' \\ \vdots & & \vdots \end{bmatrix} \overset{F_i}{\rightarrow} \begin{bmatrix} 0 & 0 & 0 \\ \times'' & \times' & \times' \\ \times'' & \times' & \times' \\ \vdots & & \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ G_{i+1} \end{bmatrix}$$

In words:

- Choose an $r \times r$ $J-$unitary rotation Θ_i that reduces the top row of G_i as in (1.2). We say that G_i is reduced to proper form.
- Apply Θ_i to G_i.
- Multiply the first column of $G_i\Theta_i$ by F_i and keep all other columns unchanged.

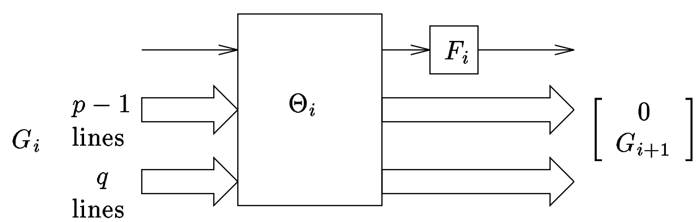

Figure 1.1 Pictorial representation of the generalized Schur algorithm.

Another useful conclusion follows by combining the generator recursion (1.3) with the expression (1.4) for the $i-$th column of the Cholesky factor. Indeed, define

$$l_i \triangleq \sqrt{d_i}\, \bar{l}_i \,,$$

where $1/\sqrt{d_i}$ is the top entry of $\bar{l}_i$ (and equal to $|\delta_i|^2$). That is, the top entry of l_i is normalized to unity. Then it can be verified that (1.3) and (1.4) lead to the following expression

$$\begin{bmatrix} l_i & \begin{matrix} 0 \\ G_{i+1} \end{matrix} \end{bmatrix} = \begin{bmatrix} F_i l_i & G_i \end{bmatrix} \begin{bmatrix} 0 & \frac{\delta_i}{d_i} \begin{bmatrix} 1 & 0 \end{bmatrix} \\ \Theta_i \begin{bmatrix} \delta_i \\ 0 \end{bmatrix} & \Theta_i \begin{bmatrix} 0 & 0 \\ 0 & I_{r-1} \end{bmatrix} \end{bmatrix} . \qquad (1.5)$$

We can therefore regard the transformation that appears on the right-hand side as the system matrix of a first-order linear state-space system; the rows of $\{G_i\}$ and $\{G_{i+1}\}$ can be regarded as inputs and outputs of this system, respectively, and the entries of $\{l_i, F_i l_i\}$ can be regarded as the corresponding current and future states. If we let $\Theta_i(z)$ denote the transfer function of the linear system (with inputs from the left), viz.,

$$\Theta_i(z) = \Theta_i \begin{bmatrix} 0 & 0 \\ 0 & I_{r-1} \end{bmatrix} + \Theta_i \begin{bmatrix} \delta_i \\ 0 \end{bmatrix} (z-0)^{-1} \frac{\delta_i}{d_i} \begin{bmatrix} 1 & 0 \end{bmatrix},$$

simple algebra will show that the above expression collapses to

$$\Theta_i(z) = \Theta_i \begin{bmatrix} z^{-1} & 0 \\ 0 & I_{r-1} \end{bmatrix}. \tag{1.6}$$

We therefore see that each step of the generalized Schur recursions can be regarded as giving rise to a first-order section $\Theta_i(z)$ - see Fig. 1.2.

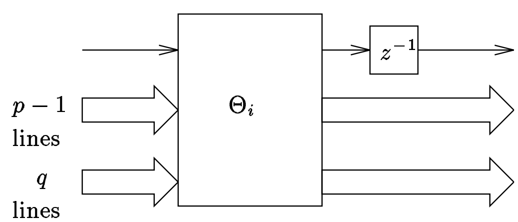

Figure 1.2 Elementary section $\Theta_i(z)$.

A succession of steps of the generalized Schur algorithm would therefore lead to a feedforward cascade of sections, say for $(n+1)$ steps,

$$\Theta(z) = \Theta_0(z)\Theta_1(z)\ldots\Theta_n(z) = \begin{bmatrix} \Theta_{11}(z) & \Theta_{12}(z) \\ \Theta_{21}(z) & \Theta_{22}(z) \end{bmatrix}, \tag{1.7}$$

which we partition accordingly with $J = (I_p \oplus -I_q)$. That is, $\Theta_{11}(z)$ is $p \times p$, $\Theta_{12}(z)$ is $p \times q$, $\Theta_{21}(z)$ is $q \times p$, and $\Theta_{22}(z)$ is $q \times q$. The transformation implied by $\Theta(z)$ is depicted in Fig. 1.3, with the input terminals denoted by $\{i_1, i_2\}$ and the output terminals denoted by $\{o_1, o_2\}$. It is also a $J-$lossless transformation.

The associated scattering or transmission line cascade would then be (see Fig. 1.4):

$$\Sigma(z) = \begin{bmatrix} \Theta_{11}(z) - \Theta_{12}(z)\Theta_{22}^{-1}(z)\Theta_{21}(z) & -\Theta_{12}(z)\Theta_{22}^{-1}(z) \\ \Theta_{22}^{-1}(z)\Theta_{21}(z) & \Theta_{22}^{-1}(z) \end{bmatrix}. \tag{1.8}$$

Such cascades map any strictly contractive load (or Schur function) that connects o_1 to o_2 to a contractive (transfer) function at the left-hand terminals of the cascade. The key fact is that the flow on the last q lines is reversed (without affecting the values of the signals inside the cascade).

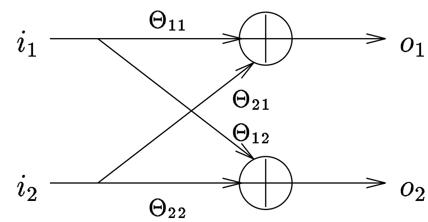

Figure 1.3 The feedforward cascade.

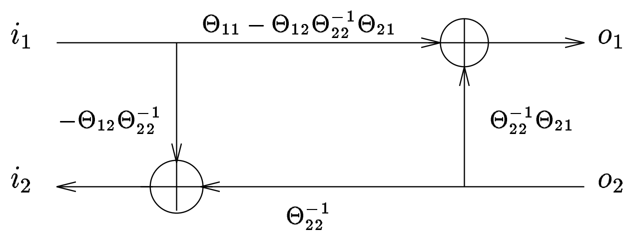

Figure 1.4 The scattering cascade.

1.3 THE FINITE-HORIZON $\mathcal{H}_\infty$ PROBLEM

Consider the estimation problem of Fig. 1.5 where $\{\mathcal{H}, \mathcal{L}\}$ are initially taken as finite upper triangular Toeplitz matrices whose entries are the Markov (or impulse response) parameters of the systems they represent; the Toeplitz assumption means that $\{\mathcal{H}, \mathcal{L}\}$ represent time-invariant systems; we also assume that the systems are initially at rest.

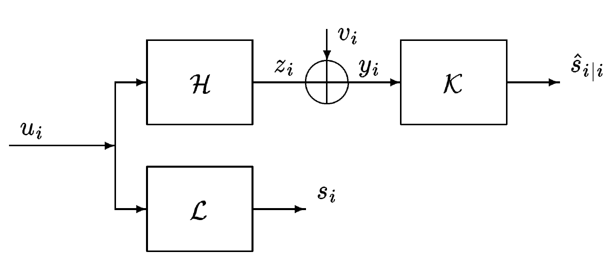

Figure 1.5 A block diagram for a general estimation problem.

For a particular time instant N, the input-output (convolution) map of $\mathcal{H}$ can be described in matrix form as follows:

$$\begin{bmatrix} \boxed{u_0} & \dots & u_N \end{bmatrix} \begin{bmatrix} \boxed{H_0} & H_1 & H_2 & H_3 & \dots & H_N \\ & H_0 & H_1 & H_2 & \dots & H_{N-1} \\ & & H_0 & H_1 & & \vdots \\ & & & \ddots & \ddots & \vdots \\ & & & & \ddots & H_1 \\ & & & & & H_0 \end{bmatrix} = \begin{bmatrix} \boxed{z_0} & \dots & z_N \end{bmatrix}$$

where $\boxed{\cdot}$ denotes values occurring at the initial time instant 0, and the $\{H_k\}$ denote the Markov parameters of $\mathcal{H}$. We assume that the $\{u_k\}$ are $1 \times p$ and the $\{H_k\}$ are $p \times q$. That is, $\mathcal{H}$ maps p dimensional inputs into q dimensional outputs. Likewise, the input-output map of $\mathcal{L}$ can be be described as

$$\begin{bmatrix} \boxed{u_0} & \dots & u_N \end{bmatrix} \begin{bmatrix} \boxed{L_0} & L_1 & L_2 & L_3 & \dots & L_N \\ & L_0 & L_1 & L_2 & \dots & L_{N-1} \\ & & L_0 & L_1 & & \vdots \\ & & & \ddots & \ddots & \vdots \\ & & & & \ddots & L_1 \\ & & & & & L_0 \end{bmatrix} = \begin{bmatrix} \boxed{s_0} & \dots & s_N \end{bmatrix}$$

We also assume, for simplicity, that the $\{L_k\}$ are $p \times q$. Hence, $\mathcal{L}$ maps p dimensional inputs into q dimensional outputs. We denote the input and output sequences in vector form, for example,

$$u \triangleq \begin{bmatrix} \boxed{u_0} & \dots & u_N \end{bmatrix}, \quad v \triangleq \begin{bmatrix} \boxed{v_0} & \dots & v_N \end{bmatrix},$$

and similarly for $\{y, z, s\}$. We then write more compactly

$$z = u\mathcal{H}\ , \quad s = u\mathcal{L}\ , \quad y = u\mathcal{H} + v\ , \tag{1.9}$$

where $\{\mathcal{H}, \mathcal{L}\}$ denote the above upper triangular matrices of Markov parameters.

The sequences $\{u, v\}$ are assumed to be unknown and the problem that we consider is to design a causal system $\mathcal{K}$ that estimates s_i, the unobservable output of $\mathcal{L}$, using the observations $\{y_j, 0 \le j \le i\}$. These estimates will be denoted by $\hat{s}_{i|i}$ and the estimation error by $\tilde{s}_{i|i} = s_i - \hat{s}_{i|i}$. The causality of $\mathcal{K}$ here means that it is required to be an upper triangular matrix (similar to $\mathcal{H}$ and $\mathcal{L}$).

Let $\mathcal{T}_\mathcal{K}$ denote the matrix that maps the unknown disturbances $\{u, v\}$ to the estimation error sequence $\{\tilde{s}\}$, where

$$\tilde{s} \triangleq \begin{bmatrix} \boxed{\tilde{s}_{0|0}} & \tilde{s}_{1|1} & \dots & \tilde{s}_{N|N} \end{bmatrix}.$$

This map is given by

$$\mathcal{T}_\mathcal{K}(u, v) = u\mathcal{L} - (v + u\mathcal{H})\mathcal{K} \;=\; u(\mathcal{L} - \mathcal{H}\mathcal{K}) - v\mathcal{K}\,. \tag{1.10}$$

In the much studied, generally called an $\mathcal{H}_\infty$ design, the selection of $\mathcal{K}$ is based on the following criterion (see, e.g., [2, 3, 4]).

Problem 1.1 ($\mathcal{H}_\infty$ design criterion) *Given $\gamma > 0$, it is required to describe all causal (upper triangular) matrices $\mathcal{K}$ that satisfy*

$$\|\mathcal{T}_{\mathcal{K}}\|_{2,ind} \triangleq \sup_{u,v\neq 0} \frac{\|\mathcal{T}_{\mathcal{K}}(u,v)\|}{(\|u\|^2+\|v\|^2)^{1/2}} < \gamma \,, \tag{1.11}$$

where $\|\cdot\|$ denotes the Euclidean norm of its argument.

$\diamondsuit$

To solve the above problem using the displacement structure machinery, we construct the following matrices directly from the given Markov parameters:

$$U \triangleq \begin{bmatrix} \gamma I_p & -\gamma H_0 \\ 0 & -\gamma H_1 \\ \vdots & \vdots \\ 0 & -\gamma H_N \end{bmatrix}, \qquad V \triangleq \begin{bmatrix} L_0 \\ L_1 \\ \vdots \\ L_N \end{bmatrix},$$

$$Z \triangleq \begin{bmatrix} 0 & & & \\ I_p & 0 & & \\ & \ddots & \ddots & \\ & & I_p & 0 \end{bmatrix}, \quad G \triangleq [U \quad V], \quad J = \begin{bmatrix} I_p & & \\ & I_q & \\ & & -I_q \end{bmatrix},$$

and introduce the displacement equation

$$R - ZRZ^* = GJG^* \,. \tag{1.12}$$

Theorem 1.1 (Solvability condition) *A solution to Prob. 1.1 exists if, and only if, the solution matrix R of (1.12) is positive-definite. The proof below also provides a constructive procedure for finding one such $\mathcal{K}$.*

Proof: Introduce the matrices (also called wave operators)

$$\mathcal{U} \;=\; [\; Z^{N-1}U \;\; \ldots \;\; ZU \;\; U \;] \,, \;\; \mathcal{V} = [\; Z^{N-1}V \;\; \ldots \;\; ZV \;\; V \;] \,.$$

Using the expressions for U and V, and the fact that Z is the shift matrix, we can easily verify that $\mathcal{V}$ is a reversed lower triangular matrix and that $\mathcal{U}$ is a reversed block lower triangular matrix, viz.,

$$\mathcal{V} = \begin{bmatrix} & & & & L_0 \\ & & & L_0 & L_1 \\ & & L_0 & L_1 & L_2 \\ & \cdot & \cdot & & \vdots \\ L_0 & L_1 & \ldots & \ldots & L_N \end{bmatrix}, \tag{1.13}$$

$$\mathcal{U} = \begin{bmatrix} & & & & & \gamma I_p & -\gamma H_0 \\ & & & \gamma I_p & -\gamma H_0 & 0 & -\gamma H_1 \\ & & \cdot & 0 & -\gamma H_1 & 0 & -\gamma H_2 \\ & \cdot & & \vdots & \vdots & \vdots & \vdots \\ \gamma I_p & -\gamma H_0 & \ldots & 0 & -\gamma H_{N-1} & 0 & -\gamma H_N \end{bmatrix} . \tag{1.14}$$

Moreover, the solution R of (1.12) is unique and given by $R = \mathcal{U}\mathcal{U}^* - \mathcal{V}\mathcal{V}^*$.

Now assume R is positive-definite, so that $\mathcal{U}\mathcal{U}^* > \mathcal{V}\mathcal{V}^*$. This is equivalent to the existence of a strictly contractive matrix $\mathcal{S}$ such that $\mathcal{V} = \mathcal{U}\mathcal{S}$ [6]. In fact, we can be more specific about $\mathcal{S}$ and show that it has to be a Toeplitz upper triangular (block) matrix. This follows from the special triangular forms of both $\mathcal{U}$ and $\mathcal{V}$. To see this, let $\tilde{I}$ denote the reversed (block) diagonal matrix with entries I_p on the anti-diagonal. Then $\mathcal{V} = \mathcal{U}\mathcal{S}$ implies that

$$\tilde{I}\mathcal{V} = \tilde{I}\mathcal{U}\mathcal{S} . \tag{1.15}$$

Moreover, $\tilde{I}\mathcal{V}$ and $\tilde{I}\mathcal{U}$ will be (block) Toeplitz upper triangular (in fact $\tilde{I}\mathcal{V} = \mathcal{L}$). It then follows that $\mathcal{S}$ must also be (block) Toeplitz upper triangular.

We thus established the existence of a strictly contractive Toeplitz upper triangular matrix $\mathcal{S}$ such that (we partition the entries of $\mathcal{S}$ accordingly with the entries of $\mathcal{U}$):

$$\mathcal{V} = \mathcal{U} \begin{bmatrix} S_0^1 & S_1^1 & S_2^1 & \ldots & S_N^1 \\ S_0^2 & S_1^2 & S_2^2 & \ldots & S_N^2 \\ & S_0^1 & S_1^1 & & \cdot \\ & S_0^2 & S_1^2 & & \cdot \\ & & \ddots & \ddots & \cdot \\ & & & & S_0^1 \\ & & & & S_0^2 \end{bmatrix} . \tag{1.16}$$

From (1.15) we conclude that

$$\tilde{I}\mathcal{V} = \mathcal{L} = \begin{bmatrix} -\gamma\mathcal{H} & \gamma I \end{bmatrix} \begin{bmatrix} \mathcal{S}_2 \\ \mathcal{S}_1 \end{bmatrix} = -\gamma\mathcal{H}\mathcal{S}_2 + \gamma\mathcal{S}_1 , \tag{1.17}$$

where $\mathcal{S}_1$ and $\mathcal{S}_2$ are upper triangular matrices that are obtained from the partitionings of the entries of $\mathcal{S}$,

$$\mathcal{S}_2 = \begin{bmatrix} S_0^2 & S_1^2 & \ldots & S_N^2 \\ & S_0^2 & & \cdot \\ & & \ddots & \\ & & & S_0^2 \end{bmatrix} , \quad \mathcal{S}_1 = \begin{bmatrix} S_0^1 & S_1^1 & \ldots & S_N^1 \\ & S_0^1 & & \cdot \\ & & \ddots & \\ & & & S_0^1 \end{bmatrix} .$$

Setting $\mathcal{K} = -\gamma\mathcal{S}_2$, it follows that $\mathcal{K}$ is upper triangular and $\mathcal{L} - \mathcal{H}\mathcal{K} = \gamma\mathcal{S}_1$. Moreover, $\mathcal{T}_{\mathcal{K}}(u, v)$ will evaluate to

$$\mathcal{T}_{\mathcal{K}}(u,v) = \gamma \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} \mathcal{S}_1 \\ \mathcal{S}_2 \end{bmatrix} .$$

Now since $\mathcal{S}$ is a strict contraction it follows that $\|\mathcal{T}_{\mathcal{K}}\|_{2,ind} < \gamma$. In summary, we established that $R > 0$ implies the existence of a causal estimator $\mathcal{K}$ that satisfies (1.11).

To prove the converse, assume a solution $\mathcal{K}$ to Prob. 1.1 exists and define $\mathcal{S}_2 = -\gamma^{-1}\mathcal{K}$ and $\mathcal{S}_1 = \gamma^{-1}[\mathcal{L} - \mathcal{H}\mathcal{K}]$. Then $\begin{bmatrix} \mathcal{S}_1 \\ \mathcal{S}_2 \end{bmatrix}$ is a strict contraction that satisfies (1.17), from which we conclude that $\mathcal{V} = \mathcal{U}\mathcal{S}$ and, therefore, that $R = \mathcal{U}\mathcal{U}^* - \mathcal{V}\mathcal{V}^* > 0$.

$\diamondsuit$

The argument in the above proof is interesting in several respects. First, it shows that if the original systems $\mathcal{H}$ and $\mathcal{L}$ are time-invariant (and hence have a Toeplitz structure), then we can find a solution $\mathcal{K}$ that also has a Toeplitz structure (and is therefore time-invariant).

Secondly, the theorem provides a solvability condition in terms of the positivity of a structured matrix R. We can employ the generalized Schur algorithm for this purpose: simply apply the algorithm to the matrices $\{Z, G, J\}$ in (1.12); if the Cholesky factorization of R can be completed, then R is positive-definite and a solution to the $\mathcal{H}_\infty$ problem exists.

Even more importantly, the proof suggests that all solutions $\mathcal{K}$ can be constructed by performing an inverse scattering experiment on the resulting feedback cascade. More specifically, let $\Theta(z)$ be the feedforward cascade that we obtain from applying the generalized Schur algorithm to the data $\{Z, G, J\}$ in (1.12). Let also $\Sigma(z)$ denote the corresponding scattering cascade. Then all solutions $\mathcal{K}$ can be obtained by terminating the scattering cascade by any strictly contractive causal load $\mathcal{Q}$ and by multiplying the scattering function at the two lower left-most terminals by $-\gamma^{-1}$. The situation is depicted schematically in Fig. 1.6. [The construction of the cascade $\Sigma(z)$ can be shown to involve some degrees of freedom. One possibility, for example, is to construct it so as to satisfy a certain maximum-entropy property — see [7].]

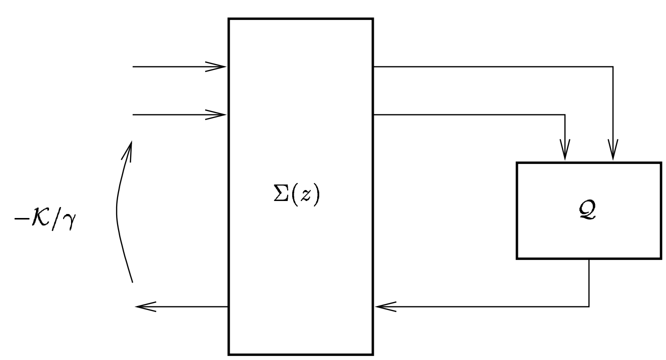

Figure 1.6 Parameterization of $\mathcal{H}_\infty$ solutions.

We therefore see that the solution to the $\mathcal{H}_\infty$ problem in the displacement structure framework is obtained by working with $J-$lossless and lossless transfer matrices $\Theta(z)$ and $\Sigma(z)$, respectively. This construction is related to the so-called dilation techniques in the literature (e.g., [3, 8]), and also to entropy-based designs (e.g., [9]). It also appears to be related to the techniques for $\mathcal{H}_\infty$ problems studied in [10]. These connections will be investigated elsewhere.

1.4 THE INFINITE-HORIZON $\mathcal{H}_\infty$ PROBLEM

We now show how to handle the infinite-horizon case and thereby obtain a steady-state result. Thus assume $N \to \infty$, in which case the matrices $\{\mathcal{H}, \mathcal{L}, \mathcal{K}, \mathcal{T}_\mathcal{K}\}$ are replaced by semi-infinite operators. For example, $\mathcal{H}$ becomes

$$\mathcal{H} = \begin{bmatrix} \boxed{H_0} & H_1 & H_2 & H_3 & \cdots \\ & H_0 & H_1 & H_2 & \cdots \\ & & H_0 & H_1 & \\ & & & \ddots & \ddots \\ & & & & \ddots \\ & & & & \end{bmatrix}$$

For well-posedness we further assume that $\mathcal{H}$ and $\mathcal{L}$ are bounded operators.

Likewise, the matrices $\{Z, G\}$ also become semi-infinite, with

$$U \triangleq \begin{bmatrix} \gamma I_p & -\gamma H_0 \\ 0 & -\gamma H_1 \\ 0 & -\gamma H_2 \\ \vdots & \vdots \end{bmatrix}, \qquad V \triangleq \begin{bmatrix} L_0 \\ L_1 \\ L_2 \\ \vdots \end{bmatrix},$$

as well as the solution R of the displacement equation (1.12). The following statement provides an explicit necessary and sufficient condition for the existence of an $\mathcal{H}_\infty$ solution $\mathcal{K}$ in terms of the positivity of the matrix R (which is uniquely and fully determined by the given data $\{\mathcal{H}, \mathcal{L}\}$).

Theorem 1.2 (Infinite-horizon solution) *A causal solution $\mathcal{K}$ that achieves $\|\mathcal{T}_\mathcal{K}\|_{2,ind} < \gamma$ exists if, and only if, the semi-infinite matrix R that solves (1.12) is positive-definite. The proof below also provides a constructive procedure for finding one such $\mathcal{K}$.*

Proof: We again introduce the (now semi-infinite) wave operators,

$$\mathcal{U} = [\ \cdots\ Z^2U \quad ZU \quad U\ ]\ , \quad \mathcal{V} = [\ \cdots\ Z^2V \quad ZV \quad V\ ]\ .$$

These operators are well defined (i.e., bounded) in view of the assumed boundedness of $\mathcal{H}$ and $\mathcal{L}$. In [11] we showed that the solution R of (1.12) in the semi-infinite case is still unique and given by $R = \mathcal{U}\mathcal{U}^* - \mathcal{V}\mathcal{V}^*$. Now assume R is positive-definite so that $\mathcal{U}\mathcal{U}^* > \mathcal{V}\mathcal{V}^*$. We also showed in [11] that this condition is again equivalent to the existence of a strictly contractive upper-triangular *Toeplitz* operator $\mathcal{S}$ such that $\mathcal{V} = \mathcal{U}\mathcal{S}$. The construction of $\mathcal{K}$ can now be obtained as in the proof of Thm. 1.1. The proof of the converse statement is also similar to that of Thm. 1.1.

$\diamondsuit$

1.5 THE TIME-VARIANT $\mathcal{H}_\infty$ PROBLEM

The above discussion can be further extended to time-variant models $\{\mathcal{H}, \mathcal{L}\}$ by using the concept of time-variant displacement structure introduced in [11, 13]. We demonstrate this fact here briefly.

In the time-variant case, the matrices (operators) $\{\mathcal{H}, \mathcal{L}\}$ will not be Toeplitz anymore. For example, in the finite-horizon case, $\mathcal{H}$ will be of the form

$$\begin{bmatrix} \boxed{H_{00}} & H_{01} & H_{02} & H_{03} & \cdots & H_{0N} \\ & H_{11} & H_{12} & H_{13} & \cdots & H_{1,N-1} \\ & & H_{22} & H_{23} & & \vdots \\ & & & \ddots & \ddots & \vdots \\ & & & & \ddots & H_{N-1,N} \\ & & & & & H_{N,N} \end{bmatrix},$$

with entries $\{H_{ij}\}$. Likewise for $\mathcal{L}$. To solve Prob. 1.1 in this case, we now need to define the following time-variant quantities (of varying dimensions) for $0 \leq t \leq N$:

$$U(t) \triangleq \begin{bmatrix} \gamma I_p & -\gamma H_{tt} \\ 0 & -\gamma H_{t-1,t} \\ 0 & -\gamma H_{t-2,t} \\ \vdots & \vdots \\ 0 & -\gamma H_{0t} \end{bmatrix}, \qquad V(t) \triangleq \begin{bmatrix} L_{tt} \\ L_{t-1,t} \\ L_{t-2,t} \\ \vdots \\ L_{0t} \end{bmatrix},$$

with

$$G(t) \triangleq [\, U(t) \quad V(t) \,], \quad J = \begin{bmatrix} I_p & & \\ & I_q & \\ & & -I_q \end{bmatrix}. \tag{1.18}$$

Define also

$$F(0) = 0_{p\times p}, \quad F(t) \triangleq \begin{bmatrix} 0 \\ I_{pt} \end{bmatrix}, \ t > 0 .$$

That is, for $t > 0$, each $F(t)$ is a rectangular matrix of dimensions $p(t+1) \times pt$, e.g.,

$$F(0) = 0\,, \quad F(1) = \begin{bmatrix} 0 \\ I_p \end{bmatrix}, \quad F(2) = \begin{bmatrix} 0 & 0 \\ I_p & 0 \\ 0 & I_p \end{bmatrix}, \ \ldots$$

Introduce further the time-variant displacement equation

$$R(t) - F(t)R(t-1)F^*(t) = G(t)JG^*(t)\,. \tag{1.19}$$

[Since $F(0) = 0$, the value of $R(-1)$ plays no role. Note further that the dimensions of $R(t)$ also change with t: $R(0)$ is $p \times p$, $R(1)$ is $2p \times 2p$, etc.]

Theorem 1.3 (Solvability condition) *A causal (upper-triangular) estimator $\mathcal{K}$ that solves Prob. 1.1 exists if, and only if, $R(t) > 0$ for all $0 \leq t \leq N$. The proof below also provides a constructive procedure for finding one such $\mathcal{K}$.*

Proof: Introduce again the wave operators

$$\begin{aligned}\mathcal{U}(t) &= [\ \ldots\ F(t)F(t-1)U(t-2)\ \ F(t)U(t-1)\ \ U(t)\ ]\ ,\\ \mathcal{V}(t) &= [\ \ldots\ F(t)F(t-1)V(t-2)\ \ F(t)V(t-1)\ \ V(t)\ ]\ .\end{aligned}$$

Then the solution $R(t)$ of (1.19) is unique and given by

$$R(t) = \mathcal{U}(t)\mathcal{U}^*(t) - \mathcal{V}(t)\mathcal{V}^*(t)\ .$$

Now assume $R(t)$ in (1.19) is positive-definite for $0 \leq t \leq N$. From Thm. 3.1 of [11], we can again conclude that there exists a a strictly contractive upper triangular matrix $\mathcal{S}$ such that $\mathcal{V}(t) = \mathcal{U}(t)\mathcal{S}$. The argument now proceeds as in the proof of Thm. 1.1.

◊

In the infinite-horizon case, when $N \to \infty$, we similarly have the following steady-state result. Assume the operators $\{\mathcal{H}, \mathcal{L}\}$ are bounded and define

$$U(t) \triangleq \begin{bmatrix} \gamma I_p & -\gamma H_{tt} \\ 0 & -\gamma H_{t-1,t} \\ 0 & -\gamma H_{t-2,t} \\ \vdots & \vdots \end{bmatrix}, \qquad V(t) \triangleq \begin{bmatrix} L_{tt} \\ L_{t-1,t} \\ L_{t-2,t} \\ \vdots \end{bmatrix}.$$

Also for all $t > 0$, let $F(t) = Z$; the infinite shift operator, and $R(-1) = 0$. The matrix $R(t)$ becomes semi-infinite as well. The following statement provides an explicit necessary and sufficient condition for the existence of an $\mathcal{H}_\infty$ solution $\mathcal{K}$ in terms of the uniform positivity of the sequence of matrices $\{R(t)\}$.

Theorem 1.4 (Infinite-horizon solution) *A causal estimator $\mathcal{K}$ exists if, and only if, $R(t)$ is uniformly positive-definite, i,e.,*

$$R(t) > \epsilon I > 0\ , \tag{1.20}$$

for all $t \geq 0$ and for some $\epsilon > 0$.

Proof: Introduce again the (now semi-infinite) wave operators

$$\begin{aligned}\mathcal{U}(t) &= [\ \ldots\ Z^2U(t-2)\ \ ZU(t-1)\ \ U(t)\ ]\ ,\\ \mathcal{V}(t) &= [\ \ldots\ Z^2V(t-2)\ \ ZV(t-1)\ \ V(t)\ ]\ .\end{aligned}$$

In [11, 12] we showed that the condition (1.20) is equivalent to the existence of a strict contraction $\mathcal{S}$ such that $\mathcal{V}(t) = \mathcal{U}(t)\mathcal{S}$. The result now follows as in the proof of the last theorem.

◊

Once we have established that the solvability of the $\mathcal{H}_\infty$ problem is related to the displacement equation (1.19), we can use the time-variant generalized Schur algorithm of [11, 13] to construct the corresponding scattering cascade. We omit the details here.

1.6 CONCLUDING REMARKS

There are several issues that can be pursued. Here we focused on a description of the systems $\{\mathcal{H}, \mathcal{L}\}$ in terms of their impulse responses (Markov parameters). When state-space descriptions are available, these introduce additional structure into the elements of the generator matrices and can lead to further simplifications in the array algorithm. More explicit connections with the dilation technique of [8], the maximum entropy technique of [9], and the $J-$conjugation technique of [10] merit further investigation.

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