DIFFUSION STOCHASTIC LEARNING OVER MULTI-TEAM NETWORK GAMES

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ABSTRACT

This work addresses stochastic multi-network competing problems. Unlike prior studies that focus exclusively on the two-team (or two-network) case, we consider the more general setting with $T \geq 2$ teams. In the multi-team setting, each team consists of a set of partially connected cooperative agents, and there exists at least one link between any two teams to enable information exchange. We propose a new algorithm based on a diffusion learning strategy, which subsumes the existing two-team competing algorithm as a special case. We show that the state vectors generated by the learning algorithm asymptotically converge to a neighborhood of the Nash equilibrium in the sense of first-, second-, and fourth-order error moments. The theoretical claims are supported by simulations on a quadratic game.

***Index Terms*—** Multi-network game, diffusion learning, stability analysis, Nash equilibrium

1. INTRODUCTION

Adaptive networks using diffusion learning have been successesfully applied to a variety of problems involving distributed minimization [1] or minimax formulations [2]. In the latter context, diffusion learning has been studied for the two-team competing network setting, where agents cooperate within their teams but compete against those in the opposing team [3], [4], [5]. However, learning under the more general case of multiple competing teams with $T \geq 2$ remains unaddressed. Competition problems of this type arise in many real-world applications [6], [7], [8], [9], [10], such as Cournot games in economics with multiple firms [6]. The multi-team network can be modeled using two hierarchical graphs: within each team, agents form an intra-team communication graph, while each team is represented as a node in a larger inter-team graph. Each agent communicates only with its immediate neighbors, which allows diffusion-based strategies to be applied in the design of the competing algorithm. The detailed problem formulation under this graph structure is formally introduced in Sec. 2.1.

Similar problems of multi-network games have been studied in [10], [11], [12], [13], [14], [15], [16], where each agent holds their estimates on both the decision vector and gradient of each team. However, these works assume that exact information about the risk value is available. We instead address the general stochastic scenario where the underlying distribution of the data is inaccessible and learning needs to be based on streaming data.

Inspired by the stochastic two-team competing work [4, 5], we propose a new diffusion learning algorithm for solving the more general multiple-network competing case. The novelty of this work comes from both the algorithmic design and the more challenging theoretical analysis, which can be summarized as follows: (1) We propose a new algorithm, called **m**ultiple-network **a**dapt-**t**hen-**c**ombine and **i**nfer-**t**hen-**c**ombine (m-ATC-ITC), for addressing multi-team network games; (2) We establish the long-term dynamics error recursion of m-ATC-ITC in Eqn. (20), and this result provides insights for the theoretical analyses of competing algorithms; and (3) we establish the first-, second-, and fourth-order stability of m-ATC-ITC based on the long-term dynamics under some reasonable assumptions; the analysis is highly nontrivial and details are omitted due to space limitations.

2. NETWORK MODEL AND ALGORITHMS

This section presents the formulation of multi-network competing problems and introduces the m-ATC-ITC algorithm.

2.1. Game Formulation

Let us consider a network consisting of K agents. We use the index set $\mathcal{N} = [K] = \{1, \cdots, K\}$ to denote the *unique* identifier of each agent. The network is partitioned into $T \geq 2$ mutually disjoint teams, indexed by $\mathcal{N}^{(1)}, \cdots, \mathcal{N}^{(T)}$, respectively, where $\mathcal{N}^{(t)}$ is the index set of agents in team t. Each team consists of $K_t := |\mathcal{N}^{(t)}| \geq 1$ agents, and $K = K_1 + \cdots + K_T$. The game over multiple networks is formulated as a problem in which each network optimizes its cost function (e.g., the negative of a payoff function) under the influence of other networks. Formally, each network aims to solve the following problem in cooperation with its own teammates

$$\min_{x^{(t)} \in \mathbb{R}^{M_t}} J^{(t)}(x) = \min_{x^{(t)} \in \mathbb{R}^{M_t}} \sum_{k \in \mathcal{N}^{(t)}} p_k^{(t)} J_k^{(t)}(x), \tag{1a}$$

$$\text{where } J_k^{(t)}(x) := J_k^{(t)}(x^{(1)}, \ldots, x^{(T)}) \tag{1b}$$

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for some weighting coefficients $p_k^{(t)} \geq 0$ that sum up to one. Here, $x^{(t)} \in \mathbb{R}^{M_t \times 1}$ is the strategy of team t, and $x = \mathrm{col}\{x^{(t)}\}_{t\in[T]} \in \mathbb{R}^M$ is the collection of strategies of all networks, where $M = M_1 + \cdots + M_T$. More importantly, each agent relies on the local random samples $\{\boldsymbol{\xi}_k\}$ to approximate the true risk $J_k^{(t)}(\cdot) = \mathbb{E}\left[\, Q_k(\cdot; \boldsymbol{\xi}_k)\,\right]$, where we use **boldface** letters to refer to random quantities. The goal of this problem is to find the Nash equilibrium under certain conditions.

Definition 1 (Nash equilibrium)**.** *The decision vector $x^\star = \mathrm{col}\{x^{(t),\star}\}_{t\in[T]} \in \mathbb{R}^M$ is a Nash equilibrium of the network game* (1a) – (1b) *if the following condition holds:*

$$J^{(t)}(x^\star) \leq J^{(t)}\left(x^{(t)}, x^{-(t),\star}\right) \qquad \forall\, x^{(t)} \in \mathbb{R}^{M_t}, \forall\, t \in [T]. \tag{2}$$

The term $x^{-(t),\star}$ denotes $x^\star$ with $x^{(t),\star}$ removed.

2.2. The m-ATC-ITC algorithm

We proceed to introduce the m-ATC-ITC algorithm, which can be seen as the extension of the ATC-ITC algorithm from the two-network case [5] to a more general setup.

At each iteration i, an agent $k \in \mathcal{N}^{(t)}$ utilizes its past strategy $\boldsymbol{x}_{k,i-1}^{(t)}$ as well as its estimate of the strategies of other teams from the previous iteration to update its present strategy. That historical information is essential for an agent to carry out the decision-making process effectively. For simplicity, we denote the estimate of the strategy of team τ made by agent k in team $t \neq \tau$ at the previous iteration by $\boldsymbol{x}_{k,i-1}^{(\tau)}$. We denote the strategy and estimates owned by agent $k \in \mathcal{N}$ at iteration i by $\boldsymbol{x}_{k,i}$. Agent k adjusts its own strategy and infers the other teams' information by performing the following steps:

1. ATC (strategy update): for all $t \in [T]$ and all $k \in \mathcal{N}^{(t)}$,

$$\boldsymbol{x}_{k,i}^{(t)} = \sum_{\ell \in \mathcal{N}^{(t)}} a_{\ell k}^{(t)} \left(\boldsymbol{x}_{\ell,i-1}^{(t)} - \mu^{(t)} \widehat{\nabla_t J}_\ell^{(t)}(\boldsymbol{x}_{\ell,i-1}) \right) \tag{3}$$

   where $\mu^{(t)}$ is a step-size of team t. The gradient ∇_t is applied with respect to the argument $x^{(t)}$. The $\widehat{(\cdot)}$ denotes the stochastic gradient.

2. ITC (estimate update): for all agents $k \in \mathcal{N}^{(t)}$ and $\tau \neq t$

$$\begin{aligned}\boldsymbol{x}_{k,i}^{(\tau)} = &\sum_{\ell \in \mathcal{N}^{(t)}} c_{\ell k}^{(tt)} \left(\boldsymbol{x}_{\ell,i-1}^{(\tau)} - \mu^{(\tau)} \widehat{\boldsymbol{g}}_\ell^{(\tau)}(\boldsymbol{x}_{\ell,i-1}) \right) \\ &+ \sum_{\ell \in \mathcal{N}^{(\tau)}} c_{\ell k}^{(\tau t)} \boldsymbol{x}_{\ell,i}^{(\tau)}.\end{aligned} \tag{4}$$

   The symbol $g_\ell^{(\tau)}$ denotes the local knowledge agent $\ell \in \mathcal{N}^{(t)}$ has about team τ, and we will specify some conditions on it in Assumption 2.

The combination coefficients $a_{\ell k}^{(t)}$, $c_{\ell k}^{(tt)}$, and $c_{\ell k}^{(\tau t)}$ are non-negative and form the matrices defined below.

Definition 2 (Combination matrices)**.** *For all $t, \tau \in [T]$: define $A^{(t)}$ as having its (ℓk) entry being $a_{\ell k}^{(t)}$, and define $C^{(\tau t)}$ as having its (ℓk) entry being $c_{\ell k}^{(\tau t)}$.*

Assumption 1 (Network structure)**.** *The combination matrices $A^{(t)}$ and $C^{(\tau t)}$ satisfy the following assumptions:*

1. *For all $t \in [T]$, the matrix $A^{(t)}$ is left-stochastic and primitive.*
2. *For all $t \in [T]$, the matrix $C^{(tt)}$ is irreducible.*
3. *For all $t, \tau \in [T]$ and $t \neq \tau$, the matrix $[C^{(tt)}; C^{(\tau t)}] \in \mathbb{R}^{(K_t + K_\tau)\times K_t}$ is left stochastic.*
4. *For all $t, \tau \in [T]$, the matrix $C^{(\tau t)}$ is non-zero.*

The above assumption is standard in the literature of competing networks as they define a graph structure such that the information of any team can flow across the whole network.

Assumption 2 (Local knowledge for m-ATC-ITC)**.** *For agent $k \in \mathcal{N}^{(t)}$ and $\tau \in [T]$,*

$$g_k^{(\tau)}(x) := \nabla_\tau J_{\ell_k}^{(\tau)}(x) \tag{5}$$

for some correspondence $\ell_k \in \mathcal{N}^{(\tau)}$. As an abuse of notation, for the case where $\tau = t$, it is required that $\ell_k = k$.

The above assumption is also made in [5]. Such an assumption is practical, for instance, where firms share similar loss functions in a Cournot game. This information can serve as local knowledge that helps agents track opponents' strategies and respond effectively.

3. THEORETICAL ANALYSIS

In this section, we establish the convergence guarantee for m-ATC-ITC. Specifically, we show that the deviation between the state vectors of all networks and the Nash equilibrium is mean-stable in the first-, second-, and fourth-orders.

3.1. Assumptions of the game

Some assumptions are required to establish the stability of the m-ATC-ITC algorithm over multi-team network games.

Assumption 3 (Step-sizes)**.** *The step-sizes $\mu^{(t)} > 0$ are fixed over iterations. Their ratios satisfy $\frac{\min_t \mu^{(t)}}{\max_t \mu^{(t)}} \geq \varepsilon_{\mathrm{m}}$ for some $\varepsilon_{\mathrm{m}} > 0$. We denote $\mu_{\max} := \max_t \mu^{(t)}$.*

Assumption 4 (Loss functions)**.** *For all $t, \tau \in [T]$ and $k \in \mathcal{N}^{(t)}$,* *1. The local risk functions $J_k^{(t)}$ are twice differentiable with respect to all arguments.*

2. Smoothness: there exist $\delta_k > 0$ such that for all $x_1, x_2 \in \mathbb{R}^{M\times 1}$,

$$\left\|\nabla_t J_k^{(t)}(x_1) - \nabla_t J_k^{(t)}(x_2)\right\| \le \delta_k \|x_1 - x_2\|. \quad (6)$$

3. Hessian matrices are locally Lipschitz continuous around the Nash equilibrium:

$$\left\|\nabla^2_{\tau t} J_k^{(t)}(x^\star + \Delta x) - \nabla^2_{\tau t} J_k^{(t)}(x^\star)\right\| \le \kappa \|\Delta x\| \quad (7)$$

for some $\kappa > 0$ and any small $\|\Delta x\| \le \varepsilon_{\mathrm{H}}$.

4. There exist $\varepsilon^{(t)} > 0$ such that for all $\{x_k\}_{k\in\mathcal{N}^{(t)}}$,

$$\sum_{k\in\mathcal{N}^{(t)}} p_k^{(t)} \nabla_t^2 J_k^{(t)}(x_k) \succeq \varepsilon^{(t)} I_{M_t}. \quad (8)$$

5. There exist $\delta^{(t\tau)} > 0$ such that

$$\delta^{(t\tau)} := \sup_{\{x_k\}_{k\in\mathcal{N}^{(t)}}} \left\| \sum_{k\in\mathcal{N}^{(t)}} p_k^{(t)} \nabla^2_{\tau t} J_k^{(t)}(x_k) \right\|. \quad (9)$$

6. The constants defined above and the step-sizes satisfy

$$\varepsilon^{(t)} - \sum_{\tau\neq t} \delta^{(t\tau)} > 0 \text{ and} \quad (10)$$

$$\mu^{(t)}\varepsilon^{(t)} - \sum_{t'\neq t} \mu^{(t')}\delta^{(t't)} > 0 \quad (11)$$

for all $t \in [T]$.

These assumptions ensure the existence and uniqueness of *the* Nash equilibrium [17, Thm. 2.3.3] as they imply a strong monotonicity condition on the global gradient

$$F(x) = \mathrm{col}\left\{\nabla_t J^{(t)}(x)\right\}_{t\in[T]}. \quad (12)$$

Let us define the gradient noise of agent $k \in \mathcal{N}^{(t)}$ on strategy x^τ as

$$\boldsymbol{s}_{k,i}^{(\tau)} = \widehat{\nabla_\tau J}_{\ell_k}^{(\tau)}(\boldsymbol{x}_{k,i-1}) - \nabla_\tau J_{\ell_k}^{(\tau)}(\boldsymbol{x}_{k,i-1}). \quad (13)$$

Assumption 5 (Gradient noise process)**.** *Define the filtration*

$$\boldsymbol{\mathcal{F}}_{i-1} := \left\{ \boldsymbol{x}_{k,j} \,\middle|\, k\in\mathcal{N}, j\in[i-1] \right\}. \quad (14)$$

For all $k, \ell \in \mathcal{N}$, for all $t, \tau \in [T]$, and all i, we assume:

$$\mathbb{E}\left[\boldsymbol{s}_{k,i}^{(t)} \,\middle|\, \boldsymbol{\mathcal{F}}_{i-1}\right] = 0, \quad (15)$$

$$\mathbb{E}\left[\boldsymbol{s}_{k,i}^{(t)} \boldsymbol{s}_{\ell,w,i}^{(\tau)\mathsf{T}} \,\middle|\, \boldsymbol{\mathcal{F}}_{i-1}\right] = 0 \quad \forall\, k\neq \ell \text{ or } t\neq\tau, \quad (16)$$

$$\mathbb{E}\left[\left\|\boldsymbol{s}_{k,i}^{(t)}\right\|^4 \,\middle|\, \boldsymbol{\mathcal{F}}_{i-1}\right] \le \left(\overline{\beta}_k^{(t)}\right)^4 \|\boldsymbol{x}_{k,i-1}\|^4 + \left(\overline{\sigma}_k^{(t)}\right)^4 \quad (17)$$

for some $\overline{\beta}_k^{(t)} > 0$ and $\overline{\sigma}_k^{(t)} > 0$.

3.2. Main results

To analyze the evolution of state vectors across agents in the competing process, we concatenate all state vectors into

$$\boldsymbol{X}_i := \mathrm{col}\left\{\mathrm{col}\left\{\boldsymbol{x}_{k,i}^{(\tau)}\right\}_{k\in\mathcal{N}}\right\}_{\tau\in[T]}. \quad (18)$$

The goal is to show that the above vector converges to the Nash equilibrium, namely,

$$X^\star := \mathrm{col}\left\{\mathbf{1}_K \otimes x^{(\tau),\star}\right\}_{\tau\in[T]}, \quad (19)$$

where $\mathbf{1}_K \in \mathbb{R}^{K\times 1}$ is the vector of ones.

Lemma 1 (Error state recursion)**.** *Define the error state vector as $\widetilde{\boldsymbol{X}}_i := X^\star - \boldsymbol{X}_{i-1}$. Running the m-ATC-ITC algorithm, it follows that the following error recursion holds:*

$$\widetilde{\boldsymbol{X}}_i = \mathcal{P}\left(I_{KM} - \mathcal{M}\boldsymbol{\mathcal{H}}_{i-1}\right)\widetilde{\boldsymbol{X}}_{i-1} + \mathcal{P}\mathcal{M}G^\star + \mathcal{P}\mathcal{M}\boldsymbol{S}_i, \quad (20)$$

where $I_{KM} \in \mathbb{R}^{KM\times KM}$ is the identity matrix. The terms in Eqn. (20) are defined as follows: let $0_{m\times n}$ denote the zero matrix of size $m\times n$, for $\tau, \tau', t, t' \in [T]$, then

$$[\mathcal{P}]_{\tau\tau',tt'} := \begin{cases} A^{(\tau)\mathsf{T}} \otimes I_{M_\tau}, & \tau=\tau'=t=t'; \\ C^{(tt)\mathsf{T}} \otimes I_{M_\tau}, & \tau=\tau'\neq t=t'; \\ (A^{(\tau)}C^{(\tau t)})^{\mathsf{T}} \otimes I_{M_\tau}, & \tau=\tau'=t'\neq t; \\ 0_{K_tM_\tau\times K_{t'}M_{\tau'}}, & \textit{otherwise}, \end{cases} \quad (21a)$$

$$\mathcal{M} := \mathrm{blkdiag}\left\{\mu^{(\tau)} I_{KM_\tau}\right\}_{\tau\in[T]}, \quad (21b)$$

$$[\boldsymbol{\mathcal{H}}_{i-1}]_{\tau\tau',tt'} := \begin{cases} \boldsymbol{\mathcal{H}}^{(t)}_{\tau\tau',i-1}, & t=t'; \\ 0_{K_tM_\tau\times K_{t'}M_{\tau'}}, & \textit{otherwise}, \end{cases} \quad (21c)$$

$$[G^\star]_{\tau,t} := \mathrm{col}\left\{g_k^{(\tau)}(x^\star)\right\}_{k\in\mathcal{N}^{(t)}}, \quad (21d)$$

$$[\boldsymbol{S}_i]_{\tau,t} := \mathrm{col}\left\{s_{k,i}^{(\tau)}\right\}_{k\in\mathcal{N}^{(t)}}, \quad (21e)$$

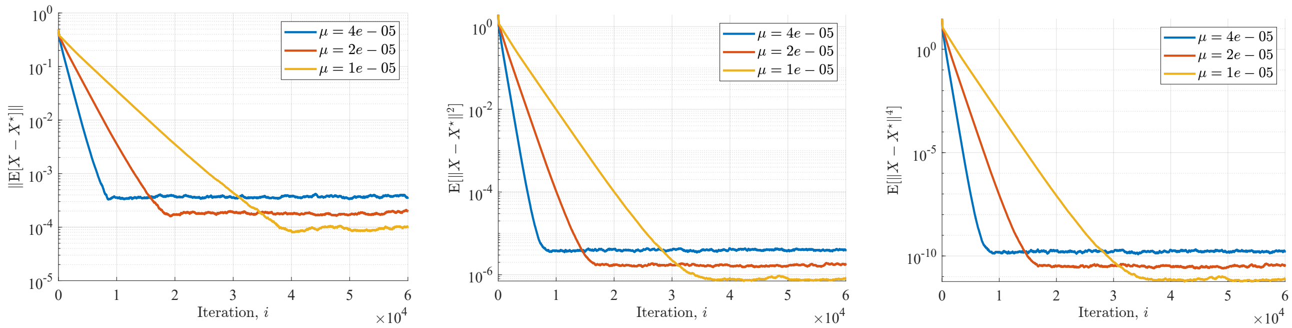
where these terms are associated with the combination matrices, the step-sizes, the second-order derivatives, the local gradient at optimality, and the stochastic gradient noises, respectively. Let $\widetilde{\boldsymbol{x}}_{k,i-1} := x^\star - \boldsymbol{x}_{k,i-1}$, the matrices $\boldsymbol{\mathcal{H}}^{(t)}_{\tau\tau',i-1}$ are defined as

$$\boldsymbol{\mathcal{H}}^{(t)}_{\tau\tau',i-1} := \mathrm{blkdiag}\{\boldsymbol{H}_{k,\tau\tau',i-1}\}_{k\in\mathcal{N}^{(t)}}, \quad (22a)$$

$$\boldsymbol{H}_{k,\tau\tau',i-1} = \int_0^1 \nabla^2_{\tau'\tau} J_{\ell_k}^{(\tau)}\left(x^\star - \eta\widetilde{\boldsymbol{x}}_{k,i-1}\right) \mathrm{d}\eta. \quad (22b)$$

Proof. Proof omitted due to page limitations. □

The matrices defined above are block-partitioned: within each block $(\tau\tau')$, a smaller partitioning by (tt') is made. The



(a) First moment stability. The limit superior is numerically proportional to $\mu^{0.98}$.

(b) Second moment stability. The limit superior is numerically proportional to $\mu^{1.18}$.

(c) Fourth moment stability. The limit superior is numerically proportional to $\mu^{2.28}$.

Fig. 1: Performance of m-ATC-ITC in quadratic game averaged over 200 samples. The supremum of the last one tenth iterations from each curve are used to plot a regression curve against the step-sizes, obtaining the exponent to μ.

following is an example of $T = 2$ with the index $(\tau\tau', tt')$ of each block marked:

$$\left[\begin{array}{cc|cc} 11,11 & 11,12 & 12,11 & 12,12 \\ 11,21 & 11,22 & 12,21 & 12,22 \\ \hline 21,11 & 21,12 & 22,11 & 22,12 \\ 21,21 & 21,22 & 22,21 & 22,22 \end{array}\right].$$

Our stability analysis relies on analyzing recursion (20). Using Assumptions 1 – 5, we establish the following results.

Theorem 1 (Stability of error moment)**.** *The error state is stable in the sense of:*

$$\limsup_{i\to\infty} \mathbb{E}\left[\left\|\widetilde{\boldsymbol{X}}_i\right\|^2\right] = O(\mu_{\max}), \tag{23}$$

$$\limsup_{i\to\infty} \mathbb{E}\left[\left\|\widetilde{\boldsymbol{X}}_i\right\|^4\right] = O(\mu_{\max}^2), \tag{24}$$

$$\limsup_{i\to\infty} \left\|\mathbb{E}\left[\widetilde{\boldsymbol{X}}_i\right]\right\| = O(\mu_{\max}) \tag{25}$$

for step-sizes small enough.

Proof. The main idea of the proof is given as follows: First, one transforms Eqn. (20) to the eigen-basis of $\mathcal{P}$. The recursion can then be bounded by using Jensen's inequality and bounding the spectral norm of $\mathcal{P}(I_{KM} - \mathcal{M}\boldsymbol{\mathcal{H}}_{i-1})$ [18]. □

The proof of the theorem is similar to the stability analysis of the diffusion algorithm for multi-agent networks [1].

4. SIMULATIONS

We consider a simple quadratic game where the local loss function of each agent $k \in \mathcal{N}^{(t)}$ ($t \in [T]$) is

$$J_k^{(t)}(x) = \mathbb{E}\left[(x^{(t)})^\mathsf{T}\boldsymbol{Q}_k x + \boldsymbol{L}_k^\mathsf{T} x^{(t)}\right]. \tag{26}$$

Consider $T = 3$, $[K_1, K_2, K_3] = [3, 2, 3]$, $[M_1, M_2, M_3] = [1, 1, 2]$ with $\mathcal{N}^{(1)} = \{1, 2, 3\}$, $\mathcal{N}^{(2)} = \{4, 5\}$, and $\mathcal{N}^{(3)} = \{6, 7, 8\}$. The combination matrices and parameters are

$$A^{(1)} = \begin{bmatrix} 1/3 & 1/2 & 1/2 \\ 1/3 & 1/2 & 0 \\ 1/3 & 0 & 1/2 \end{bmatrix}, A^{(2)} = \begin{bmatrix} 1/2 & 3/4 \\ 1/2 & 1/4 \end{bmatrix}, A^{(3)} = \begin{bmatrix} 1/4 & 1/2 & 1/3 \\ 1/4 & 0 & 1/3 \\ 1/2 & 1/2 & 1/3 \end{bmatrix},$$

$$C^{(11)} = \begin{bmatrix} 3/10 & 1/2 & 1/2 \\ 3/10 & 1/2 & 0 \\ 3/10 & 0 & 1/2 \end{bmatrix}, C^{(21)} = \begin{bmatrix} 1/10 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, C^{(31)} = \begin{bmatrix} 0 & 0 & 0 \\ 1/10 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$C^{(12)} = \begin{bmatrix} 0 & 1/3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, C^{(22)} = \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & 1/3 \end{bmatrix}, \begin{matrix} C^{(32)} = C^{(12)}, C^{(13)} = C^{(31)}, \\ C^{(23)} = C^{(21)}, C^{(33)} = C^{(11)}, \end{matrix}$$

$$Q_1 = \begin{bmatrix} 13 & 2 & 3 & 3 \end{bmatrix}, Q_2 = \begin{bmatrix} 12 & 1 & -1 & -1 \end{bmatrix}, Q_3 = \begin{bmatrix} 12 & 0 & 1 & 1 \end{bmatrix},$$

$$Q_4 = \begin{bmatrix} 3 & 16 & 3 & 1 \end{bmatrix}, Q_5 = \begin{bmatrix} 3 & 15 & 3 & 2 \end{bmatrix},$$

$$Q_6 = \begin{bmatrix} 0 & 1 & 11 & 0 \\ -1 & 1 & 0 & 18 \end{bmatrix}, Q_7 = \begin{bmatrix} -1 & 2 & 10 & 0 \\ -1 & 2 & 0 & 20 \end{bmatrix}, Q_8 = \begin{bmatrix} -2 & 3 & 10 & 0 \\ -1 & 2 & 0 & 15 \end{bmatrix},$$

$$L_1 = L_2 = 5, L_3 = 4, L_4 = -3, L_5 = -4, L_6 = \begin{bmatrix} -4 & 1 \end{bmatrix}, L_7 = L_8 = \begin{bmatrix} -5 & -2 \end{bmatrix}.$$

The noise added onto each element of Q_k and L_k is uniformly distributed on $[-0.5, 0.5]$. The step-sizes are chosen to be the same for all teams and are denoted by μ. The simulation results are shown in Fig. 1.

5. CONCLUSIONS

In this work, we proposed the m-ATC-ITC algorithm, a diffusion learning strategy over multi-network competing problems with gradient noise. We established the asymptotic convergence of the algorithm to a neighborhood around the Nash equilibrium. We have also shown that the long-term dynamics of the algorithm is stable in the first-, second-, and fourth-order error moments.

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