EE210A: Adaptation and Learning

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LECTURE #16

UNITARY TRANSFORMATIONS

Sections: 33.4, 34.1-34.2, 36.1-36.3

BASIS ROTATION

Lemma 33.1 (Basis rotation) Given two $n \times M$ $(n \leq M)$ matrices A and B. Then $AA^* = BB^*$ if, and only if, there exists an $M \times M$ unitary matrix Θ such that $A = B\Theta$.

Proof: One direction is obvious. If $A = B\Theta$, for some unitary matrix Θ, then

$$AA^* = (B\Theta)(B\Theta)^* = B(\Theta\Theta^*)B^* = BB^*$$

One proof for the converse implication follows by using the singular value decompositions of A and B (cf. App. B.6) — see Prob. VIII.2 for another proof:

$$A = U_A \begin{bmatrix} \Sigma_A & 0 \end{bmatrix} V_A^*, \quad B = U_B \begin{bmatrix} \Sigma_B & 0 \end{bmatrix} V_B{}^*$$

where U_A and U_B are $n \times n$ unitary matrices, V_A and V_B are $M \times M$ unitary matrices, and Σ_A and Σ_B are $n \times n$ diagonal matrices with nonnegative entries. The squares of the diagonal entries of Σ_A (Σ_B) are the eigenvalues of AA^* (BB^*). Moreover, U_A (U_B) are constructed from an orthonormal basis for the right eigenvectors of AA^* (BB^*). Hence, it follows from the identity $AA^* = BB^*$ that $\Sigma_A = \Sigma_B$ and $U_A = U_B$. Let $\Theta = V_B V_A^*$. Then $\Theta\Theta^* = \mathrm{I}$ and $B\Theta = A$.

◇

33.4 MOTIVATION FOR ARRAY METHODS

Before plunging into the derivation of RLS array methods, we shall study a simple example in some detail in order to highlight the main ideas underlying the mechanics of array methods. Thus consider two scalars $\{a, b\}$ and assume that we wish to evaluate the positive scalar c that satisfies

$$|c|^2 \;=\; |a|^2 \;+\; |b|^2 \tag{33.10}$$

The first method that comes to mind is to evaluate the squares $|a|^2$ and $|b|^2$, add them, and then compute the square-root of the sum to find c.

Preservation of Norms

A less obvious way for determining c, albeit one that will be more useful for our purposes (especially when we deal with matrix quantities $\{A, B, C\}$ as opposed to scalars $\{a, b, c\}$), is to use an array method. It can be motivated as follows. Observe that the right-hand side of (33.10) is the sum of two squares and it can be expressed as an inner product:

$$|a|^2 + |b|^2 = \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} a^* \\ b^* \end{bmatrix}$$

Similarly, the left-hand side of (33.10) can be expressed as an inner product:

$$|c|^2 = \begin{bmatrix} c & 0 \end{bmatrix} \begin{bmatrix} c^* \\ 0 \end{bmatrix}$$

In this way, relation (33.10) in effect amounts to an equality of the form

$$\begin{bmatrix} c & 0 \end{bmatrix} \begin{bmatrix} c^* \\ 0 \end{bmatrix} = \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} a^* \\ b^* \end{bmatrix} \qquad (33.11)$$

which has the same form as $AA^* = BB^*$ with the identifications

$$A \longleftarrow \begin{bmatrix} c & 0 \end{bmatrix}, \qquad B \longleftarrow \begin{bmatrix} a & b \end{bmatrix} \tag{33.12}$$

Therefore, using (33.9), we conclude that there should exist a 2×2 unitary matrix Θ that maps $[a \;\; b]$ to $[c \;\; 0]$,

$$\underbrace{\begin{bmatrix} a & b \end{bmatrix}}_{B} \Theta = \underbrace{\begin{bmatrix} c & 0 \end{bmatrix}}_{A} \tag{33.13}$$

If we can find Θ, then applying it to the pre-array $[a \;\; b]$ would result in the desired value for c.

Now recall that the proof of Lemma 33.1 provides an expression for the unitary matrix Θ that transforms B to A. However, that expression is in terms of the right singular vectors of $\{A, B\}$ and, therefore, it requires that we know beforehand both A and B. Clearly, this construction is not helpful in situations like (33.13) where A is not known. For this reason, the conclusion (33.9) is useful only in that it guarantees the *existence* of a Θ that performs the required transformation (33.13).

In order to find Θ we would argue differently as follows. Choose *any* unitary Θ that transforms the pre-array, $[a \;\; b]$, to the generic form

$$\left[\begin{array}{cc} a & b \end{array}\right]\Theta = \left[\begin{array}{cc} \times & 0 \end{array}\right] \tag{33.14}$$

That is, choose any unitary Θ that annihilates the second entry of $[a \;\; b]$ and let $\times$ denote the resulting leading entry of the post-array. We explain in Lemma 34.2, and in the remark following it, how such a Θ can be found for any $[a \;\; b]$, e.g., a Givens rotation could be used, which in this case would be given by

$$\Theta = \begin{cases} \dfrac{e^{-j\phi_a}}{\sqrt{1+|\rho|^2}}\left[\begin{array}{cc} 1 & -\rho \\ \rho^* & 1 \end{array}\right] & \text{if } a \neq 0 \;\; \text{and where } \rho = b/a \\ e^{-j\phi_b}\left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right] & \text{if } a = 0 \end{cases}$$

where $\{\phi_a, \phi_b\}$ are the phases of (the possibly complex numbers) $\{a, b\}$.

If we apply this choice of Θ to the pre-array $[a \quad b]$ in (33.14), then it is easy to see by direct calculation that a post-array of the form $[\times \quad 0]$ will result. Specifically, as explained in the remark following Lemma 34.2, we get

$$\left[\begin{array}{cc} \times & 0 \end{array}\right] = \left[\begin{array}{cc} \sqrt{|a|^2+|b|^2} & 0 \end{array}\right]$$

which readily identifies $\times$ as the desired c.

Alternatively, the value of $\times$ could have been identified by using the property (33.7) and without assuming any explicit knowledge of Θ. Indeed, (33.7) states that unitary transformations preserve Euclidean norms, so that the norm of the pre-array $[a \quad b]$ must coincide with the norm of the post-array $[\times \quad 0]$. Therefore, by "squaring" both sides of (33.14), namely, by writing

$$\left[\begin{array}{cc} a & b \end{array}\right] \underbrace{\Theta\Theta^*}_{\mathrm{I}} \left[\begin{array}{c} a^* \\ b^* \end{array}\right] = \left[\begin{array}{cc} \times & 0 \end{array}\right] \left[\begin{array}{c} \times^* \\ 0 \end{array}\right]$$

we get $|a|^2+|b|^2 = |\times|^2$, so that $\times = c$.

Preservation of Inner Products

In order to further appreciate the convenience of the array formulation, assume now that in addition to the scalars $\{a, b, c\}$ satisfying (33.10), we are also given scalars $\{d, e\}$ and that we want to evaluate the scalar f that satisfies

$$fc^* = da^* + eb^* \tag{33.15}$$

Of course, given the $\{a, b\}$ we could first determine c as explained above, and then evaluate f by dividing the right-hand side of (33.15) by c^* (or c since c is real in this example). Alternatively, we could evaluate f in array form, just like we did for c, as follows.

We start by noting that the right-hand side of (33.15) can be interpreted as the inner product between two vectors:

$$da^* + eb^* = \begin{bmatrix} d & e \end{bmatrix} \begin{bmatrix} a^* \\ b^* \end{bmatrix}$$

ARRAY METHODS

Now we explained above how to find a unitary transformation Θ that takes $[a \;\; b]$ to $[c \;\; 0]$. In addition, we know from our discussions in Sec. 33.3, that any such unitary transformation preserves not only vector norms but also inner products between vectors (recall (33.8)). This second property can be used to our advantage here. Assume we apply Θ to both $[a \;\; b]$ and $[d \;\; e]$. We already know that in the first case we obtain $[c \;\; 0]$ as the post-array, whereas in the second case we would obtain some other post-array that we denote by $[\mathsf{y} \;\; \mathsf{z}]$:

$$\begin{aligned} \begin{bmatrix} a & b \end{bmatrix} \Theta &= \begin{bmatrix} c & 0 \end{bmatrix} \\ \begin{bmatrix} d & e \end{bmatrix} \Theta &= \begin{bmatrix} \mathsf{y} & \mathsf{z} \end{bmatrix} \end{aligned}$$

The preservation of inner products then implies that the inner-product of the pre-array vectors should coincide with the inner-product of the post-array vectors, i.e.,

$$\begin{bmatrix} d & e \end{bmatrix} \begin{bmatrix} a^* \\ b^* \end{bmatrix} = \begin{bmatrix} \mathsf{y} & \mathsf{z} \end{bmatrix} \begin{bmatrix} c^* \\ 0 \end{bmatrix}$$

or, equivalently,

$$da^* + eb^* = \mathsf{y}c^*$$

ARRAY METHODS

or, equivalently,

$$da^* + eb^* = \mathsf{y}c^*$$

Comparing with (33.15), we see that we can immediately identify y as the desired f. As for z, while it is not of immediate interest to us here, its value can be identified by noting that $[d \;\; e]$ and $[\mathsf{y} \;\; \mathsf{z}]$ must have identical Euclidean norms, so that

$$|\mathsf{z}|^2 + |f|^2 = |d|^2 + |e|^2$$

Array Description

In conclusion, the above discussion shows that calculations of the type (33.10) and (33.15), aimed at determining $\{c, f\}$ from knowledge of $\{a, b, d, e\}$, can be accomplished in array form as follows. We form the pre-array

$$\mathcal{A} = \begin{bmatrix} a & b \\ d & e \end{bmatrix} \tag{33.16}$$

and choose a unitary matrix Θ that lower triangularizes $\mathcal{A}$, namely, it reduces $\mathcal{A}$ to the form

$$\mathcal{A}\Theta = \begin{bmatrix} \times & 0 \\ \mathsf{y} & \mathsf{z} \end{bmatrix} \tag{33.17}$$

with a positive $\times$. The determination of Θ is solely dependent on the first row of the pre-array $\mathcal{A}$; the entries of the second row of $\mathcal{A}$ are not used to define Θ.

Then the entries $\{\times, \mathsf{y}\}$ can be identified as the desired $\{c, f\}$; this identification follows from the preservation of norms and inner products by unitary matrices. In particular, the identification of $\times$ as c follows from the fact that the top rows of the pre- and post-arrays $\{\mathcal{A}, \mathcal{A}\Theta\}$ must have the same Euclidean norms, while the identification of y as f follows from the fact that the inner product of the rows in the pre- and post-arrays must coincide.

Another way of carrying out this procedure for identifying the entries of the post-array $\mathcal{A}\Theta$ is as follows. Given the pre-array $\mathcal{A}$ as in (33.16), then the entries of the post-array $\mathcal{A}\Theta$ in (33.17) can be identified by "squaring" both sides of (33.17), i.e., by writing

$$\mathcal{A}\underbrace{\Theta\Theta^*}_{\mathrm{I}}\mathcal{A} = \mathcal{A}\mathcal{A}^* = \begin{bmatrix} \times & 0 \\ \mathsf{y} & \mathsf{z} \end{bmatrix}\begin{bmatrix} \times & 0 \\ \mathsf{y} & \mathsf{z} \end{bmatrix}^*$$

and then by comparing terms on both sides of the resulting equality:

$$\begin{bmatrix} a & b \\ d & e \end{bmatrix}\begin{bmatrix} a & b \\ d & e \end{bmatrix}^* = \begin{bmatrix} \times & 0 \\ \mathsf{y} & \mathsf{z} \end{bmatrix}\begin{bmatrix} \times & 0 \\ \mathsf{y} & \mathsf{z} \end{bmatrix}^* \qquad (33.18)$$

Doing so results in the relations

$$|\times|^2 = |a|^2 + |b|^2 \qquad \text{and} \qquad \mathsf{y}\times^* = da^* + eb^*$$

which identify $\times$ as c and y as f.

VECTOR CASE

The above example, with scalar entries $\{a, b, c, d, e, f\}$, illustrates the main ideas behind the derivation of array algorithms. In the context of adaptive filtering, however, we shall encounter vector analogues of relations (33.10) and (33.15), such as determining a lower triangular matrix C, with positive diagonal entries, satisfying

$$CC^* = AA^* + BB^* \tag{33.19}$$

and determining a matrix F satisfying

$$FC^* = DA^* + EB^* \tag{33.20}$$

where $\{A, B, D, E\}$ are generally matrix or vector quantities. The same arguments that we used above will reveal that $\{C, F\}$ can be determined by means of an array method as follows. We form the pre-array (cf. (33.16)):

$$\mathcal{A} = \begin{bmatrix} A & B \\ D & E \end{bmatrix}$$

VECTOR CASE

and reduce it via a unitary transformation Θ to the lower triangular form (cf. (33.17)):

$$\begin{bmatrix} \mathsf{X} & 0 \\ \mathsf{Y} & \mathsf{Z} \end{bmatrix}$$

where X is lower triangular with positive entries along its diagonal; sometimes Z is a square matrix and Θ is also required to generate it in lower-triangular form along with X:

$$\begin{bmatrix} A & B \\ D & E \end{bmatrix} \Theta = \begin{bmatrix} \mathsf{X} & 0 \\ \mathsf{Y} & \mathsf{Z} \end{bmatrix} \qquad (33.21)$$

The matrix Θ is not 2×2 any longer; but it can be implemented as a sequence of elementary (Givens) rotations or Householder reflections, as explained in App. 34, where we show how to lower triangularize a matrix via a sequence of rotations or reflections.

VECTOR CASE

An explicit expression for Θ in (33.21) is not needed. All we need to do is find the right sequence of rotations that yields the desired triangular post-array. Then, by "squaring" both sides of (33.21) we get

$$\begin{bmatrix} A & B \\ D & E \end{bmatrix} \underbrace{\Theta\Theta^*}_{\text{I}} \begin{bmatrix} A & B \\ D & E \end{bmatrix}^* = \begin{bmatrix} \mathsf{X} & 0 \\ \mathsf{Y} & \mathsf{Z} \end{bmatrix} \begin{bmatrix} \mathsf{X} & 0 \\ \mathsf{Y} & \mathsf{Z} \end{bmatrix}^*$$

so that we must have

$$\mathsf{X}\mathsf{X}^* = AA^* + BB^* \qquad \text{and} \qquad \mathsf{Y}\mathsf{X}^* = DA^* + EB^*$$

In this way, X can be identified as the lower triangular Cholesky factor of the matrix $AA^* + BB^*$, and since Cholesky factors are unique, we conclude that X must coincide with the desired C. From the second equality above we conclude that $\mathsf{Y} = F$, so that the array algorithm (33.21) enables us to determine $\{C, F\}$. We may remark that we are not restricted to array methods with two (block) rows in the pre-array and post-arrays as in (33.21). If additional relations are available that satisfy certain norm and inner-product preservation properties, then these could be incorporated into the array algorithm as well. A demonstration to this effect is the QR algorithm of Sec. 35.2.


James W. Givens
(1910-1993)

34.1 GIVENS ROTATIONS

Givens rotations provide an effective way to annihilate specific entries in a vector and it is enough to explain their operation on 2-dimensional row vectors. We consider the case of real-valued data first.

Real data

Consider a 1×2 real-valued vector $z = \begin{bmatrix} a & b \end{bmatrix}$, and assume that we wish to determine a 2×2 matrix Θ that transforms it to the form:

$$\begin{bmatrix} a & b \end{bmatrix} \Theta = \begin{bmatrix} \alpha & 0 \end{bmatrix} \qquad (34.1)$$

for some real number α to be determined, and where Θ is required to be orthogonal, i.e., it should satisfy

$$\Theta\Theta^{\mathsf{T}} = \Theta^{\mathsf{T}}\Theta = \mathrm{I}$$

We refer to $[a \quad b]$ as the pre-array and to $[\alpha \quad 0]$ as the post-array.

GIVENS ROTATIONS: REAL DATA

Now any orthogonal matrix Θ has the important property that it preserves vector norms. Indeed, it is easy to see from (34.1) that the following equality must hold:

$$\begin{bmatrix} a & b \end{bmatrix} \underbrace{\Theta\Theta^{\mathsf{T}}}_{\mathrm{I}} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \alpha & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ 0 \end{bmatrix}$$

or, equivalently, $a^2 + b^2 = \alpha^2$. In this way, no matter which orthogonal transformation Θ we choose to implement the transformation (34.1), it will always hold that the Euclidean norm of the post-array should coincide with the Euclidean norm of the pre-array. This fact allows us to conclude the value of α, namely,

$$\alpha = \pm\sqrt{a^2 + b^2}$$

even before knowing the expression of any Θ that achieves (34.1). Note that there are two choices for α and which one we pick depends on how Θ is implemented, as explained next.

GIVENS ROTATIONS: REAL DATA

An expression for an orthogonal Θ that achieves the transformation (34.1) is given by

$$\Theta = \frac{1}{\sqrt{1+\rho^2}} \begin{bmatrix} 1 & -\rho \\ \rho & 1 \end{bmatrix} \quad \text{where} \quad \rho = \frac{b}{a}, \quad a \neq 0 \tag{34.2}$$

This choice is known as a Givens or circular rotation. Surely, it can be verified by direct calculation that this Θ is orthogonal and that it leads to

$$\begin{bmatrix} a & b \end{bmatrix} \Theta = \begin{bmatrix} \pm\sqrt{a^2+b^2} & 0 \end{bmatrix}$$

The choice of which sign to pick depends on whether the value of the square root in the expression (34.2) for Θ is chosen to be negative or positive. In general, we choose the positive sign.

GIVENS ROTATIONS: REAL DATA

The reason for the denomination *circular rotation* for Θ can be seen by expressing Θ in the form

$$\Theta = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}, \quad c = \frac{1}{\sqrt{1+\rho^2}}, \quad s = \frac{\rho}{\sqrt{1+\rho^2}}, \quad \rho = \frac{b}{a}$$

in terms of cosine and sine parameters. In this way, we find that the effect of Θ is to rotate any point (x, y) in the two-dimensional Euclidean space along the *circle* of equation

$$x^2 + y^2 = a^2 + b^2$$

When $a = 0$, we simply select Θ to be the permutation matrix

$$\boxed{\Theta = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}} \tag{34.3}$$

We could also choose

$$\Theta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

with $+1$ instead of -1. The effect of this permutation will be the same; we choose (34.3) with a minus sign so that (34.3) can be regarded as the limit of the Givens rotation (34.2) when $\rho \to \infty$. In summary, we have the following result.

Lemma 34.1 (Real Givens rotation) Consider a 1×2 vector $[a \ \ b]$ with real entries. Then choose Θ as in (34.2) to get

$$\begin{bmatrix} a & b \end{bmatrix} \Theta = \pm\sqrt{a^2 + b^2} \begin{bmatrix} 1 & 0 \end{bmatrix}$$

If $a = 0$, then choose Θ as in (34.3) to get $[0 \ \ b]\Theta = [b \ \ 0]$.

GIVENS ROTATIONS: COMPLEX DATA

Complex data

When the entries of $z = \begin{bmatrix} a & b \end{bmatrix}$ are complex-valued, we should seek a unitary (as opposed to orthogonal) matrix Θ that achieves the transformation (34.1), namely, Θ should now satisfy

$$\Theta\Theta^* = \Theta^*\Theta = \mathrm{I}$$

An expression for such a Θ is

$$\boxed{\Theta = \frac{1}{\sqrt{1+|\rho|^2}} \begin{bmatrix} 1 & -\rho \\ \rho^* & 1 \end{bmatrix} \quad \text{where} \quad \rho = \frac{b}{a}, \quad a \neq 0} \tag{34.5}$$

Let $a = |a|e^{j\phi_a}$ denote the polar representation of a. Then it can be verified that we now obtain

$$\begin{bmatrix} a & b \end{bmatrix} \Theta = \begin{bmatrix} \pm e^{j\phi_a}\sqrt{|a|^2 + |b|^2} & 0 \end{bmatrix} \tag{34.6}$$

In other words, the value of α is in general complex and its phase is determined by the phase of a. Again, the choice of the sign depends on whether the value of the square root in (34.5) is chosen to be negative or positive. When $a = 0$, we choose Θ as the permutation matrix (34.3).

GIVENS ROTATIONS: COMPLEX DATA

Lemma 34.2 (Complex Givens rotation) Consider a 1×2 vector $[a \;\; b]$ with possibly complex entries. Then choose Θ as in (34.5) to get

$$\left[\begin{array}{cc} a & b \end{array}\right] \Theta = \pm e^{j\phi_a} \sqrt{|a|^2 + |b|^2} \left[\begin{array}{cc} 1 & 0 \end{array}\right]$$

where ϕ_a denotes the phase of a. If $a = 0$, then choose Θ as in (34.3) to get $[0 \;\; b]\Theta = [b \;\; 0]$.

REAL POST-ARRAY

Remark 34.1 (Real post-array) Usually, it is desirable to obtain a real-valued (and also positive) α even in the complex case (34.6). This property can be enforced by choosing Θ as

$$\Theta = \frac{e^{-j\phi_a}}{\sqrt{1+|\rho|^2}} \begin{bmatrix} 1 & -\rho \\ \rho^* & 1 \end{bmatrix} \quad \text{where} \quad \rho = \frac{b}{a}, \quad a \neq 0$$

To enforce $\alpha > 0$, we simply choose the plus sign in (34.6). If a already happens to be real-valued, then of course $\phi_a = 0$ and the same Θ as before in (34.5) results in a real α. Note further that, in this case, the diagonal entries of Θ will be real.

When $a = 0$, we choose Θ as

$$\Theta = e^{-j\phi_b} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

in terms of the phase of b. This remark applies to all our future discussions for complex-valued data; so it will not be repeated again.

EXAMPLE

Example 34.1 (Using Givens rotations)

Assume we are given a 2×3 pre-array $\mathcal{A}$,

$$\mathcal{A} = \begin{bmatrix} 1 & 0.75 & 0.25 \\ 0.4 & 0.2 & 0.2 \end{bmatrix} \tag{34.7}$$

and that we wish to reduce it to the form

$$\mathcal{A}\Theta = \begin{bmatrix} \times & 0 & 0 \\ \times & \times & 0 \end{bmatrix} \tag{34.8}$$

via a sequence of Givens rotations. This can be obtained, among several possibilities, as follows. We first annihilate the $(1,3)$ entry of $\mathcal{A}$ by pivoting with its $(1,1)$ entry. From the construction (34.2), we know that the orthogonal matrix Θ_1 that achieves this transformation is given by

$$\Theta_1 = \frac{1}{\sqrt{1+\rho_1^2}} \begin{bmatrix} 1 & -\rho_1 \\ \rho_1 & 1 \end{bmatrix} = \begin{bmatrix} 0.9701 & -0.2425 \\ 0.2425 & 0.9701 \end{bmatrix}, \quad \rho_1 = 0.25/1 = 0.25$$

EXAMPLE

Applying Θ_1 to $\mathcal{A}$, and leaving the second column of $\mathcal{A}$ unchanged, leads to

$$\underbrace{\begin{bmatrix} \boxed{1} & 0.75 & \boxed{0.25} \\ 0.4 & 0.2 & 0.2 \end{bmatrix}}_{\mathcal{A}} \begin{bmatrix} 0.9701 & 0 & -0.2425 \\ 0 & 1 & 0 \\ 0.2425 & 0 & 0.9701 \end{bmatrix} = \underbrace{\begin{bmatrix} 1.0307 & 0.7500 & 0 \\ 0.4365 & 0.2000 & 0.0970 \end{bmatrix}}_{\mathcal{A}_1}$$

We now annihilate the $(1,2)$ entry of the post-array $\mathcal{A}_1$ by pivoting with its $(1,1)$ entry. For this purpose, we choose the orthogonal matrix as

$$\Theta_2 = \frac{1}{\sqrt{1+\rho_2^2}} \begin{bmatrix} 1 & -\rho_2 \\ \rho_2 & 1 \end{bmatrix} = \begin{bmatrix} 0.8086 & -0.5884 \\ 0.5884 & 0.8086 \end{bmatrix}, \quad \rho_2 = \frac{0.7500}{1.0307}$$

Applying Θ_2 to $\mathcal{A}_1$, and leaving the third column of $\mathcal{A}_1$ unchanged, leads to

$$\underbrace{\begin{bmatrix} \boxed{1.0307} & \boxed{0.7500} & 0 \\ 0.4365 & 0.2000 & 0.0970 \end{bmatrix}}_{\mathcal{A}_1} \begin{bmatrix} 0.8086 & -0.5884 & 0 \\ 0.5884 & 0.8086 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1.2748 & 0 & 0 \\ 0.4707 & -0.0951 & 0.0970 \end{bmatrix}}_{\mathcal{A}_2}$$

EXAMPLE

We finally annihilate the $(2,3)$ entry of $\mathcal{A}_2$ by pivoting with its $(2,2)$ entry. One way to achieve this transformation is to use the orthogonal matrix

$$\Theta_3 \;=\; \frac{1}{\sqrt{1+\rho_3^2}}\begin{bmatrix} 1 & -\rho_3 \\ \rho_3 & 1 \end{bmatrix} = \begin{bmatrix} 0.7001 & 0.7140 \\ -0.7140 & 0.7001 \end{bmatrix}, \quad \rho_3 = \frac{0.0970}{-0.0951}$$

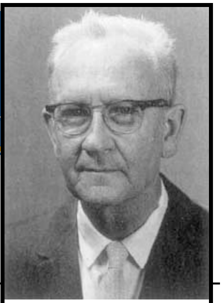
and to apply it to $\mathcal{A}_2$, without modifying its first column, thus leading to

$$\underbrace{\begin{bmatrix} 1.2748 & 0 & 0 \\ 0.4707 & \boxed{-0.0951} & \boxed{0.0970} \end{bmatrix}}_{\mathcal{A}_2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.7001 & 0.7140 \\ 0 & -0.7140 & 0.7001 \end{bmatrix} = \underbrace{\begin{bmatrix} 1.2748 & 0 & 0 \\ 0.4707 & -0.1359 & 0 \end{bmatrix}}_{\mathcal{A}_3}$$

Clearly, the negative entry -0.1359 could have been replaced by a positive entry 0.1359 had we employed $-\Theta_3$ instead of Θ_3.

A more direct way to achieve the last step is to avoid forming Θ_3 altogether and to simply replace the vector $[-0.0951 \quad 0.0970]$ by $[\alpha \quad 0]$, where α is the norm of the vector, i.e., by $[0.1359 \quad 0]$. In this way, the resulting post-array becomes

$$\begin{bmatrix} 1.2748 & 0 & 0 \\ 0.4707 & 0.1359 & 0 \end{bmatrix}$$


Alston Householder (1904-1993)

34.2 HOUSEHOLDER TRANSFORMATIONS

In contrast to Givens rotations, Householder transformations can be used to annihilate multiple entries in a row vector at once. We describe them below for both cases of real and complex data.

Real data

Let $e_0 = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}$ denote the leading row basis vector in $n-$dimensional Euclidean space, and consider a $1 \times n$ real-valued vector z with entries $\{z(i), i = 0, 1, \ldots, n-1\}$. Assume that we wish to transform z to the form

$$\begin{bmatrix} z(0) & z(1) & \ldots & z(n-1) \end{bmatrix} \Theta = \alpha \, e_0 \tag{34.9}$$

for some real scalar α to be determined, and where the transformation Θ is required to be both orthogonal and involutary. That is, Θ should satisfy $\Theta\Theta^{\mathsf{T}} = \mathrm{I}$ and $\Theta^2 = \mathrm{I}$ (or, equivalently, $\Theta = \Theta^{\mathsf{T}}$ and $\Theta^2 = \mathrm{I}$). We remark in passing that matrices Q that satisfy $Q^2 = \mathrm{I}$ are called involutary matrices.

HOUSEHOLDER TRANSFORMATIONS

Of course, the scalar α cannot be arbitrary and its value can be determined even before determining the expression for a matrix Θ that achieves (34.9). Indeed, note from (34.9) and from the orthogonality of Θ that

$$z\Theta\Theta^{\mathsf{T}}z^{\mathsf{T}} = \|z\|^2 = \alpha^2$$

so that we must have $\alpha = \pm\|z\|$. Both values of α are possible (since if Θ achieves $z\Theta = \|z\|e_0$, then $-\Theta$ is orthogonal and achieves $z\Theta = -\|z\|e_0$). One way to achieve the transformation (34.9) is to employ a Householder reflection. We motivate it by means of a geometric argument.

Thus refer to Fig. 34.3, which shows the vector z and its destination αe_0. Since $\alpha = \|z\|$, the triangle with sides z and αe_0 is isosceles and we denote its base by $g = z - \alpha e_0$. If we drop a perpendicular from the origin of z to g, it will divide g into two equal parts, with the upper part being the projection of z onto g and is equal to $zg^{\mathsf{T}}\|g\|^{-2}g$.

HOUSEHOLDER TRANSFORMATIONS

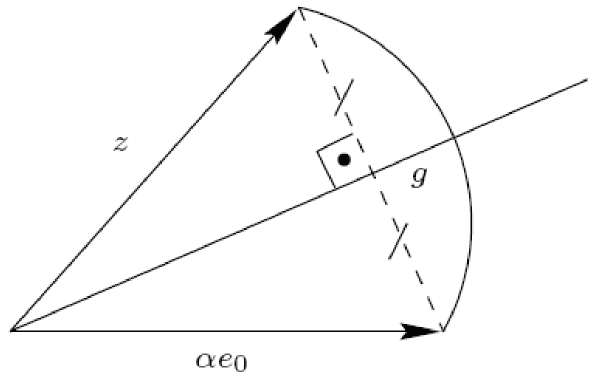


FIGURE 34.3 The vector z is aligned with e_0 by reflecting it across the line that bisects the angle between the sides z and αe_0. This construction provides a geometric interpretation for the Householder transformation.

This means that $g = 2zg^{\mathsf{T}}\|g\|^{-2}g$ and, consequently,

$$\alpha e_0 = z - 2zg^{\mathsf{T}}(gg^{\mathsf{T}})^{-1}g = z\underbrace{\left[\mathrm{I} - 2\frac{g^{\mathsf{T}}g}{gg^{\mathsf{T}}}\right]}_{\Theta} \tag{34.10}$$

where

$$\boxed{\Theta \stackrel{\Delta}{=} \mathrm{I} - 2\frac{g^{\mathsf{T}}g}{gg^{\mathsf{T}}}} \tag{34.11}$$

HOUSEHOLDER TRANSFORMATIONS

We thus have a matrix Θ that maps z to αe_0. It is straightforward to verify that this Θ is orthogonal and involutary, as desired. The matrix Θ so defined is called an (elementary) Householder transformation or reflection; it is a reflection since its effect is to reflect z across the line that bisects the angle between the sides z and αe_0 — see Fig. 34.3. Moreover, for the Householder matrix Θ in (34.11), we see that it is a rank-one modification of the identity matrix. Therefore, it has $(n-1)$ eigenvalues at 1 and a single eigenvalue at -1. Therefore, $\det\Theta = -1$, which again confirms that it is a reflector. In summary, we established the following result.

Lemma 34.3 (Real Householder reflection) Consider an $n-$dimensional vector z with real-valued entries. Choose $g = z \pm \|z\| e_0$ and

$$\Theta = \mathrm{I}_n - 2\,\frac{g^{\mathsf{T}} g}{g g^{\mathsf{T}}}$$

to get $z\Theta = \mp\|z\|e_0$. Here, $e_0 = [1\ 0\ 0 \ldots\ 0]$ is the first basis vector.

Usually, the sign in the expression for g is chosen to be the same as the sign of the leading entry of z in order to avoid a vector g with small Euclidean norm.

Complex data

More generally, consider a $1 \times n$ vector z with possibly complex entries, and assume that we wish to determine a transformation Θ that transforms it to the form (34.9) with a scalar α that is possibly complex-valued, and where Θ is required to be a Hermitian unitary matrix, i.e., it should satisfy $\Theta = \Theta^*$ and $\Theta^2 = \text{I}$.

Again, the scalar α cannot be arbitrary and its value can be determined even before determining the expression for a matrix Θ that achieves (34.9). Indeed, note from (34.9), and from the unitarity of Θ, that $|\alpha|^2 = \|z\|^2$ so that $|\alpha| = \|z\|$. Moreover, from the equality $z\Theta z^* = \alpha z_0^*$, and from the fact that $z\Theta z^*$ is real (since Θ is Hermitian), we conclude that αz_0^* must be real. If we introduce the polar representation of the first entry of z, namely, $z(0) = |a|\ e^{j\phi_a}$, then it follows that α is given by

$$\alpha = \pm\|z\|e^{j\phi_a}$$

The prior geometric construction of Θ can be repeated in the complex case and it leads to the following conclusion.

Lemma 34.4 (Complex Householder reflection) Consider a $1 \times n$ vector z with possibly complex-valued entries. Choose $g = z \pm \|z\| e^{j\phi_a} e_0$ and

$$\Theta = \mathrm{I}_n - 2\,\frac{g^* g}{g g^*}$$

to get $z\Theta = \mp \|z\| e^{j\phi_a} e_0$. Here,$e_0 = [1\ 0\ 0 \ldots\ 0]$ and ϕ_a is the phase of the leading entry of z.

Proof: Besides the geometric argument, we can establish the result of the lemma algebraically as follows. We express g as $g = z + \alpha e_0$, where α is a scalar that satisfies $|\alpha|^2 = \|z\|^2$ and αa^* is real. Then direct calculation shows that

$$\begin{aligned} \|g\|^2 &= 2\|z\|^2 + 2\alpha a^* \\ z g^* g &= z\|z\|^2 + \alpha\|z\|^2 e_0 + \alpha^* a z + \alpha(\alpha a^*) e_0 \end{aligned}$$

and, hence,

$$z\Theta = z - \frac{2 z g^* g}{\|g\|^2} = -\alpha e_0$$

EXAMPLE

Example 34.2 (Using Householder transformations)

We re-consider the pre-array of numbers $\mathcal{A}$ in (34.7) and now show how to transform it to the form (34.8) by means of Householder transformations.

Let x_1 denote the top row of $\mathcal{A}$, i.e., $x_1 = \begin{bmatrix} 1 & 0.75 & 0.25 \end{bmatrix}$. Our first step is to annihilate the last two entries of x_1. From expression (34.11), we know that this transformation can be achieved by using the following 3×3 Householder transformation:

$$\Theta_1 = \mathrm{I}_3 - 2\,\frac{g_1^{\mathsf{T}} g_1}{g_1 g_1^{\mathsf{T}}}$$

where $g_1 = x_1 \pm \|x_1\| \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$. We initially choose the sign in the expression for g_1 to be the same as the sign of the leading entry of x_1, which is positive, so that

$$g_1 = \begin{bmatrix} 1 & 0.75 & 0.25 \end{bmatrix} + \begin{bmatrix} 1.2748 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2.2748 & 0.1500 & 1.0000 \end{bmatrix}$$

EXAMPLE

Applying Θ_1 to the two rows of $\mathcal{A}$ gives

$$x_1\Theta_1 = x_1 - 2\frac{x_1 g_1^{\mathsf{T}}}{g_1 g_1^{\mathsf{T}}} g_1 \;=\; \begin{bmatrix} -1.2748 & 0 & 0 \end{bmatrix}$$

and

$$x_2\Theta_1 = x_2 - 2\frac{x_2 g_1^{\mathsf{T}}}{g_1 g_1^{\mathsf{T}}} g_1 \;=\; \begin{bmatrix} -0.4707 & -0.0871 & 0.1043 \end{bmatrix}$$

In other words,

$$\underbrace{\begin{bmatrix} 1 & 0.75 & 0.25 \\ 0.4 & 0.2 & 0.2 \end{bmatrix}}_{\mathcal{A}} \Theta_1 = \underbrace{\begin{bmatrix} -1.2748 & 0 & 0 \\ -0.4707 & -0.0871 & 0.1043 \end{bmatrix}}_{\mathcal{A}_1}$$

EXAMPLE

Of course, had we chosen the sign in the expression for g_1 to be the negative sign, the signs of all entries in the above post-array would have been switched, say

$$\underbrace{\begin{bmatrix} 1 & 0.75 & 0.25 \\ 0.4 & 0.2 & 0.2 \end{bmatrix}}_{\mathcal{A}} \Theta_1 = \underbrace{\begin{bmatrix} 1.2748 & 0 & 0 \\ 0.4707 & 0.0871 & -0.1043 \end{bmatrix}}_{\mathcal{A}_1}$$

In order to annihilate the $(2,3)$ entry of $\mathcal{A}_1$ we can replace the vector $[0.0871 \quad 0.1043]$ by one of the form $[\alpha \quad 0]$, where α is the norm of the vector, i.e., by $[0.1359 \quad 0]$. In this way, it is seen that the resulting post-array in (34.8) can be taken as

$$\begin{bmatrix} 1.2748 & 0 & 0 \\ 0.4707 & 0.1359 & 0 \end{bmatrix}$$

HYPERBOLIC ROTATIONS: MOTIVATION

It is sometimes necessary, especially when deriving fast least-squares algorithms (as we shall discuss in the next chapter), to employ $J-$unitary (also called hyperbolic) transformations, as opposed to unitary transformations, in order to annihilate certain entries in a pre-array of numbers. A $J-$unitary transformation Θ is one that satisfies

$$\boxed{\Theta J \Theta^* = \Theta^* J \Theta = J}$$

for some signature matrix J, i.e., a diagonal matrix with ± 1 entries. The special case $J = I$ corresponds to unitary transformations and was studied in Chapter 34. In this chapter, we extend the results to the $J-$unitary case, starting with Givens rotations and followed by Householder transformations.

36.1 HYPERBOLIC GIVENS ROTATIONS

Real Data

Thus consider a 1×2 real-valued vector $z = \begin{bmatrix} a & b \end{bmatrix}$, and assume that we wish to determine a 2×2 matrix Θ that transforms it to the form:

$$\begin{bmatrix} a & b \end{bmatrix} \Theta = \begin{bmatrix} \alpha & 0 \end{bmatrix} \tag{36.1}$$

for some nonzero real number α to be determined, and where Θ is required to be hyperbolic, i.e., it should satisfy

$$\Theta J \Theta^{\mathsf{T}} = \Theta^{\mathsf{T}} J \Theta = J \qquad \text{where} \qquad J = \text{diag}\{1, -1\}$$

Unfortunately, and in contrast to the case of orthogonal Givens transformations in Sec. 34.1, the transformation (36.1) is not always possible. To see this, note from (36.1) that

$$\begin{bmatrix} a & b \end{bmatrix} \underbrace{\Theta J \Theta^{\mathsf{T}}}_{J} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \alpha & 0 \end{bmatrix} J \begin{bmatrix} \alpha \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \alpha \\ 0 \end{bmatrix}$$

i.e., $a^2 - b^2 = \alpha^2$.

REAL DATA

i.e., $a^2 - b^2 = \alpha^2$. Now since $\alpha^2 > 0$, no matter what α is, this means that the transformation (36.1) is only possible if $|a| > |b|$. When this is not the case, i.e., if $|a| < |b|$, then we should seek instead a hyperbolic rotation Θ that transforms z into the alternative form

$$\begin{bmatrix} a & b \end{bmatrix} \Theta = \begin{bmatrix} 0 & \alpha \end{bmatrix} \tag{36.2}$$

In this case, the transformation (36.2) will guarantee $a^2 - b^2 = -\alpha^2$, which is consistent with the fact that $a^2 - b^2 < 0$. So let us examine the cases $|a| > |b|$ and $|a| < |b|$ separately.

$\boxed{|a| > |b|}$ (**first entry of the vector is dominant**)

In this case, an expression for Θ that achieves (36.1) is given by

$$\boxed{\Theta = \frac{1}{\sqrt{1-\rho^2}} \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix} \quad \text{where} \quad \rho = \frac{b}{a}} \tag{36.3}$$

REAL DATA

It is a straightforward exercise to verify that

$$\begin{bmatrix} a & b \end{bmatrix} \Theta = \begin{bmatrix} \pm\sqrt{a^2 - b^2} & 0 \end{bmatrix}$$

where the sign of the resulting α depends on whether the value of the square-root in the expression (36.3) for Θ is chosen to be negative or positive.

The reason for the denomination *hyperbolic rotation* for Θ can be seen by expressing Θ in the form

$$\Theta = \begin{bmatrix} \mathsf{ch} & -\mathsf{sh} \\ -\mathsf{sh} & \mathsf{ch} \end{bmatrix}, \quad \mathsf{ch} = \frac{1}{\sqrt{1-\rho^2}}, \quad \mathsf{sh} = \frac{\rho}{\sqrt{1-\rho^2}}$$

in terms of hyperbolic cosine and sine parameters, ch and sh, respectively. In this way, we find that the effect of Θ is to rotate any point (x, y) in the two-dimensional Euclidean space along the *hyperbola* of equation $x^2 - y^2 = a^2 - b^2$ — see Fig. 36.1.

REAL DATA

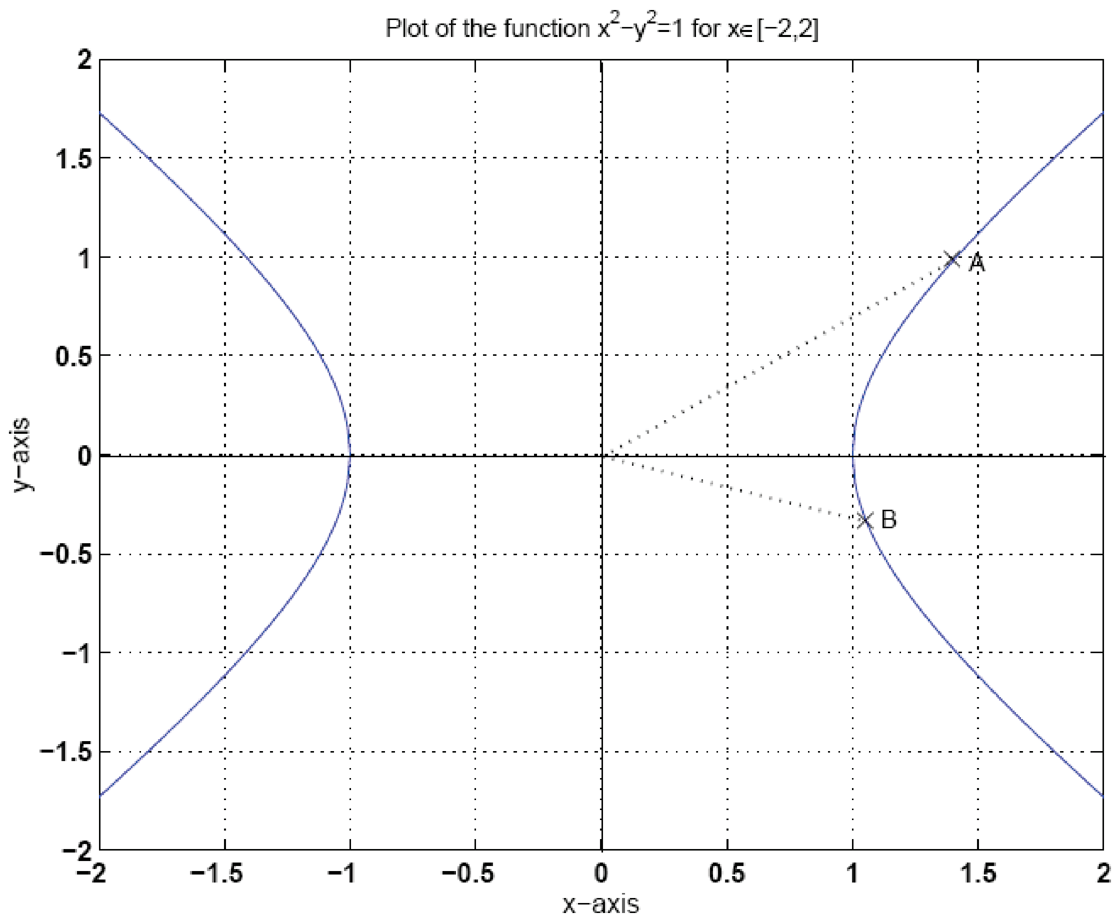


FIGURE 36.1 A hyperbolic rotation in $2-$dimensional Euclidean space moves points along the hyperbola of equation $x^2 - y^2 = a^2 - b^2$. In the figure, $a^2 - b^2 = 1$ and point A is moved into point B.

REAL DATA

$\boxed{|b| > |a|}$ (**second entry of the vector is dominant**)

In this second case, an expression for Θ that achieves (36.2) is given by

$$\boxed{\Theta = \frac{1}{\sqrt{1-\rho^2}} \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix} \quad \text{where} \quad \rho = \frac{a}{b}} \tag{36.4}$$

which now leads to the result

$$\begin{bmatrix} a & b \end{bmatrix} \Theta = \begin{bmatrix} 0 & \pm\sqrt{b^2 - a^2} \end{bmatrix}$$

REAL DATA

Lemma 36.1 (Real hyperbolic Givens) Consider a 1×2 vector $[a \; b]$ with real entries. If $|a| > |b|$, then choose Θ as in (36.3) to get

$$\begin{bmatrix} a & b \end{bmatrix} \Theta = \pm\sqrt{a^2 - b^2} \begin{bmatrix} 1 & 0 \end{bmatrix}$$

If, on the other hand, $|a| < |b|$, then choose Θ as in (36.4) to get

$$\begin{bmatrix} a & b \end{bmatrix} \Theta = \pm\sqrt{b^2 - a^2} \begin{bmatrix} 0 & 1 \end{bmatrix}$$

COMPLEX DATA

More generally, consider a 1×2 vector $z = \begin{bmatrix} a & b \end{bmatrix}$ with possibly complex entries, and assume that we wish to determine an elementary 2×2 matrix Θ that transforms it to either forms (36.1) or (36.2) with a possibly complex-valued α, and where Θ is now required to satisfy

$$\Theta J \Theta^* = \Theta^* J \Theta = J \qquad \text{where} \qquad J = \text{diag}\{1, -1\}$$

We again need to distinguish between two cases.

$|a| > |b|$ (**first entry of the vector is dominant**)

In this case, we choose Θ as

$$\Theta = \frac{1}{\sqrt{1 - |\rho|^2}} \begin{bmatrix} 1 & -\rho \\ -\rho^* & 1 \end{bmatrix} \quad \text{where} \quad \rho = \frac{b}{a} \tag{36.5}$$

This choice achieves the transformation (36.1). Specifically, it gives

$$\begin{bmatrix} a & b \end{bmatrix} \Theta = \begin{bmatrix} \pm e^{j\phi_a} \sqrt{|a|^2 - |b|^2} & 0 \end{bmatrix}$$

where ϕ_a denotes the phase of a.

COMPLEX DATA

$\boxed{|a| < |b|}$ **(second entry of the vector is dominant)**

Now we choose Θ as

$$\boxed{\Theta = \frac{1}{\sqrt{1-|\rho|^2}} \begin{bmatrix} 1 & -\rho \\ -\rho^* & 1 \end{bmatrix} \quad \text{where} \quad \rho = \frac{a^*}{b^*}} \tag{36.6}$$

This choice achieves the transformation (36.2). Specifically, it gives

$$\begin{bmatrix} a & b \end{bmatrix} \Theta = \begin{bmatrix} 0 & \pm e^{j\phi_b}\sqrt{|b|^2 - |a|^2} \end{bmatrix}$$

where ϕ_b denotes the phase of b.

COMPLEX DATA

Lemma 36.2 (Complex hyperbolic Givens) Consider a 1×2 vector $[a\ b]$ with possibly complex entries. If $|a| > |b|$, then choose Θ as in (36.5) to get

$$\begin{bmatrix} a & b \end{bmatrix} \Theta = \pm\, e^{j\phi_a} \sqrt{|a|^2 - |b|^2} \begin{bmatrix} 1 & 0 \end{bmatrix}$$

If, on the other hand, $|a| < |b|$, then choose Θ as in (36.6) to get

$$\begin{bmatrix} a & b \end{bmatrix} \Theta = \pm\, e^{j\phi_b} \sqrt{|b|^2 - |a|^2} \begin{bmatrix} 0 & 1 \end{bmatrix}$$

36.2 HYPERBOLIC HOUSEHOLDER TRANSFORMATIONS

In contrast to hyperbolic Givens rotations, Householder transformations can be used to annihilate multiple entries in a row vector at once. We describe them below for both cases of real and complex data.

Real Data

Let e_0 denote the leading row basis vector in $n-$dimensional Euclidean space, $e_0 = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}$, and consider a $1 \times n$ real-valued vector z with entries $\{z(i), i = 0, 1, \ldots, n-1\}$. Assume that we wish to transform z to the form

$$\begin{bmatrix} z(0) & z(1) & \ldots & z(n-1) \end{bmatrix} \Theta = \alpha \, e_0 \tag{36.7}$$

for some nonzero real scalar α to be determined, and where the transformation Θ is required to be both J-orthogonal and involutary.

REAL DATA

By a $J-$orthogonal transformation we mean one that satisfies $\Theta J \Theta^{\mathsf{T}} = \Theta^{\mathsf{T}} J \Theta = J$ for some given signature matrix J with $\pm$ diagonal entries, usually of the form

$$J = (\mathrm{I}_p \oplus -\mathrm{I}_q), \quad p \geq 1, \quad q \geq 1$$

and by an involutary matrix Θ we mean one that satisfies $\Theta^2 = \mathrm{I}$.

Again, and in contrast to the case of orthogonal Householder transformations in Sec. 34.1 the transformation (36.7) is not always possible. To see this, note from (36.7) that

$$z \underbrace{\Theta J \Theta^{\mathsf{T}}}_{J} z^{\mathsf{T}} \; = \; \alpha e_0 J e_0^{\mathsf{T}} \alpha \; = \; \alpha^2$$

i.e., $zJz^{\mathsf{T}} = \alpha^2$. Now since $\alpha^2 > 0$, no matter what the value of α is, this means that the transformation (36.7) is only possible whenever $zJz^{\mathsf{T}} > 0$. If, this is not the case, i.e., if $zJz^{\mathsf{T}} < 0$, then we should seek instead a $J-$unitary transformation Θ that transforms z to the alternative form

$$\begin{bmatrix} z(0) & z(1) & \ldots & z(n-1) \end{bmatrix} \Theta = \alpha\, e_{n-1} \tag{36.8}$$

REAL DATA

where e_{n-1} is the last basis vector, $e_{n-1} = \begin{bmatrix} 0 & \ldots & 0 & 1 \end{bmatrix}$. In this case, the transformation (36.8) will guarantee $zJz^{\mathsf{T}} = -\alpha^2$, which is consistent with the fact that $zJz^{\mathsf{T}} < 0$. So let us examine the cases $zJz^{\mathsf{T}} > 0$ and $zJz^{\mathsf{T}} < 0$ separately.

$\boxed{zJz^{\mathsf{T}} > 0}$ (**positive value**)

To determine an expression for Θ that meets the requirement (36.7), we can follow the same geometric argument that we used in the orthogonal case in Chapter 34, except that we replace gg^{T} by gJg^{T} and the inner product zg^{T} by zJg^{T}. Therefore, the first step is to choose $\alpha = \pm\sqrt{zJz^{\mathsf{T}}}$ and then to write

$$\alpha e_0 = z - 2zJg^{\mathsf{T}}(gJg^{\mathsf{T}})^{-1}g = z \underbrace{\left[\mathrm{I} - 2J\frac{g^{\mathsf{T}}g}{gJg^{\mathsf{T}}} \right]}_{\triangleq\, \Theta} \tag{36.9}$$

The indicated matrix Θ is called a hyperbolic Householder transformation; it is both J-unitary and involutary.

REAL DATA

$\boxed{zJz^{\mathsf{T}} < 0}$ (**negative value**)

In this case, we can also follow the same geometric argument to determine an expression for Θ to achieve (36.8). The first steps are now to choose $\alpha = \pm\sqrt{-zJz^{\mathsf{T}}}$ and $g = z - \alpha e_{n-1}$, and then to write

$$\alpha e_{n-1} = z - 2zJg^{\mathsf{T}}(gJg^{\mathsf{T}})^{-1}g = z \underbrace{\left[\mathrm{I} - 2J\frac{g^{\mathsf{T}}g}{gJg^{\mathsf{T}}} \right]}_{\triangleq\,\Theta} \tag{36.10}$$

The indicated matrix Θ is also a hyperbolic Householder transformation; it is both J-unitary and involutary.

REAL DATA

Lemma 36.3 (Real hyperbolic Householder) Consider an $n-$dimensional vector z with real-valued entries. If $zJz^{\mathsf{T}} > 0$, then choose $g = z \pm \sqrt{zJz^{\mathsf{T}}}e_0$ and Θ as in (36.9) to get

$$z\Theta = \mp\sqrt{zJz^{\mathsf{T}}}e_0$$

If, on the other hand, $zJz^{\mathsf{T}} < 0$, then choose $g = z \pm \sqrt{-zJz^{\mathsf{T}}}e_{n-1}$ and Θ as in (36.10) to get

$$z\Theta = \mp\sqrt{-zJz^{\mathsf{T}}}\; e_{n-1}$$

Complex Data

More generally, consider a $1 \times n$ vector z with possibly complex entries, and assume that we wish to determine a transformation Θ that transforms it to either forms (36.7) or (36.8) with a possibly complex-valued α, and where Θ is now required to satisfy

$$\Theta J \Theta^* = \Theta^* J \Theta = J \quad \text{and} \quad \Theta^2 = \mathrm{I}$$

for some given signature matrix J and with the same involutary condition. We again need to distinguish between two cases.

$\boxed{zJz^* > 0}$ (**positive value**)

Introduce the polar representation of the first entry of z, namely, let $z(0) = |a|e^{j\phi_a}$. Then choose $g = z \pm \sqrt{zJz^*}\; e^{j\phi_a} e_0$ and Θ as

$$\boxed{\Theta \stackrel{\Delta}{=} \mathrm{I} - 2\frac{Jg^*g}{gJg^*}} \tag{36.11}$$

This choice gives

$$z\Theta = \mp\sqrt{zJz^*}\; e^{j\phi_a} e_0$$

COMPLEX DATA

$\boxed{zJz^* < 0}$ (**negative value**)

Introduce the polar representation of the last entry of z, namely, let $z(n-1) = |b|e^{j\phi_b}$. Then choose $g = z \pm \sqrt{-zJz^*}\, e^{j\phi_b} e_{n-1}$ and Θ as in (36.11). This leads to

$$z\Theta = \mp\sqrt{-zJz^*}\, e^{j\phi_b} e_{n-1}$$

COMPLEX DATA

Lemma 36.4 (Complex hyperbolic Householder) Consider a $1 \times n$ vector z with possibly complex-valued entries. If $zJz^* > 0$, then choose $g = z \pm \sqrt{zJz^*}e^{j\phi_a}\, e_0$ and Θ as in (36.11) to get

$$z\Theta = \mp e^{j\phi_a}\sqrt{zJz^*}e_0$$

If, on the other hand, $zJz^* < 0$, then choose $g = z \pm \sqrt{-zJz^*}e^{j\phi_b}e_{n-1}$ and Θ as in (36.11) to get

$$z\Theta = \mp e^{j\phi_b}\sqrt{-zJz^*}\; e_{n-1}$$

Here $\{ae^{j\phi_a}, be^{j\phi_b}\}$ denote the polar representations of the leading and trailing entries of z, respectively.

36.3 HYPERBOLIC BASIS ROTATIONS

We now extend the result of Lemma 33.1 by replacing the equality $AA^* = BB^*$ by $AJA^* = BJB^*$, for some signature matrix J. We start with the following statement.

Lemma 36.5 (More columns than rows with a full-rank requirement) Consider two $n \times m$ matrices A and B with $n \leq m$ (i.e., the matrices have more columns than rows). Let $J = (\mathrm{I}_p \oplus -\mathrm{I}_q)$ be a signature matrix and assume that AJA^* has full rank. If $AJA^* = BJB^*$, then there should exist a J-unitary matrix Θ that maps B to A, i.e., $A = B\Theta$.

HYPERBOLIC BASIS ROTATION

Proof: Let $S^{-1} = AJA^*$. Then S^{-1} is $n \times n$ Hermitian. Moreover, $S^{-1} = BJB^*$. Let $\text{In}(S^{-1}) = \{\alpha, \beta\}$ denote the inertia of S^{-1}, with $\alpha + \beta = n$, and introduce the two block triangular factorizations:[20]

$$\begin{bmatrix} S^{-1} & A \\ A^* & J \end{bmatrix} = \begin{bmatrix} \text{I} & \\ A^*S & \text{I} \end{bmatrix} \begin{bmatrix} S^{-1} & \\ & J - A^*SA \end{bmatrix} \begin{bmatrix} \text{I} & \\ A^*S & \text{I} \end{bmatrix}^*$$

$$= \begin{bmatrix} \text{I} & AJ \\ & \text{I} \end{bmatrix} \begin{bmatrix} \underbrace{S^{-1} - AJA^*}_{=0} & \\ & J \end{bmatrix} \begin{bmatrix} \text{I} & AJ \\ & \text{I} \end{bmatrix}^*$$

Using Sylvester's law of inertia (cf. Lemma B.5), we conclude that the center matrices in the above factorizations must have the same inertia. In other words, it must hold that

$$\text{In}\{J - A^*SA\} = \text{In}\{J\} - \text{In}\{S^{-1}\} = \{p - \alpha, q - \beta, n\}$$

where, from the definition of J, $p + q = m$. Similarly, $\text{In}\{J - B^*SB\} = \{p - \alpha, q - \beta, n\}$. We therefore find that $J - A^*SA$ and $J - B^*SB$ are $m \times m$ matrices with n zero eigenvalues and $m - n$ nonzero eigenvalues. These matrices can then be factored as $J - A^*SA = XJ_1X^*$ and $J - B^*SB = YJ_1Y^*$, where $J_1 = (\text{I}_{p-\alpha} \oplus -\text{I}_{q-\beta})$ and $\{X, Y\}$ are $m \times (m - n)$. Now define the square matrices

$$\Sigma_1 = \begin{bmatrix} A \\ X^* \end{bmatrix}, \qquad \Sigma_2 = \begin{bmatrix} B \\ Y^* \end{bmatrix}$$

HYPERBOLIC BASIS ROTATION

Then it follows that

$$\Sigma_1^*(S \oplus J_1)\Sigma_1 = J \qquad \text{and} \qquad \Sigma_2^*(S \oplus J_1)\Sigma_2 = J \tag{36.12}$$

so that Σ_1 and Σ_2 are invertible. Multiplying the first equality by $J\Sigma_1^*$ from the right we get $\Sigma_1^*(S \oplus J_1)\Sigma_1(J\Sigma_1^*) = J(J\Sigma_1^*) = \Sigma_1^*$, so that $\Sigma_1 J\Sigma_1^* = (S^{-1} \oplus J_1)$. Likewise, $\Sigma_2 J\Sigma_2^* = (S^{-1} \oplus J_1)$ and, consequently, $\Sigma_1 J\Sigma_1^* = \Sigma_2 J\Sigma_2^*$. From (36.12) we have that $J\Sigma_1^* = \Sigma_1^{-1}(S^{-1} \oplus J)$ so that $\Sigma_1 = \Sigma_2[J\Sigma_2^*(S \oplus J_1)\Sigma_1]$. If we set $\Theta = [J\Sigma_2^*(S \oplus J_1)\Sigma_1]$, then Θ is J-unitary and, from the equality of the first block row of $\Sigma_1 = \Sigma_2\Theta$, we get $A = B\Theta$.

$\diamond$

In the above statement, the matrices A and B were either square or had more columns than rows (since $n \leq m$). We can establish a similar result when $n \geq m$ and A and B have full ranks. For this purpose, we first note that if A is $n \times m$ and has full rank, with $n \geq m$, then its SVD has the form

$$A = U \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} V^*$$

where Σ is $n \times n$ and invertible. The pseudo inverse of A is $A^\dagger \stackrel{\Delta}{=} V \begin{bmatrix} \Sigma^{-1} & 0 \end{bmatrix} U^*$ and it satisfies $A^\dagger A = \mathrm{I}_m$. In other words, the matrix A admits a right inverse. A similar conclusion holds for B.

Lemma 36.6 (More rows than columns) Consider two $n \times m$ matrices A and B with $n \geq m$ (i.e., the matrices have more rows than columns). Let $J = (\mathrm{I}_p \oplus -\mathrm{I}_q)$ be a signature matrix and assume that A and B have full rank. The equality $AJA^* = BJB^*$ holds if, and only if, there exists a $m \times m$ $J-$unitary matrix Θ such that $A = B\Theta$.

HYPERBOLIC BASIS ROTATION

Proof: The "if" statement is immediate. If $A = B\Theta$, for some $J-$unitary Θ, then clearly $AJA^* = BJB^*$. For the converse statement, assume $AJA^* = BJB^*$ and define $\Theta = B^\dagger A$. Then this choice of Θ is $J-$unitary and it maps B to A. Indeed, note that

$$\underbrace{B^\dagger A}_{\Theta} J \underbrace{A^* B^{\dagger *}}_{\Theta^*} = \underbrace{B^\dagger B}_{\mathrm{I}_m} J \underbrace{B^\dagger B}_{\mathrm{I}_m} \underbrace{B^\dagger B}_{\mathrm{I}_m} = J$$

Moreover, from the relations $AJA^* = BJB^*$, $B^\dagger B = \mathrm{I}_m$, and $A^\dagger A = \mathrm{I}_m$ we obtain the following equalities:

$$AJA^* \;=\; B\underbrace{(B^\dagger B)}_{\mathrm{I}_m} JB^* \;=\; BB^\dagger(BJB^*) \;=\; BB^\dagger AJA^*$$

which, upon further multiplication from the right by $A^{\dagger *}$, give

$$AJ\underbrace{A^* A^{\dagger *}}_{\mathrm{I}_m} \;=\; BB^\dagger AJ\underbrace{A^* A^{\dagger *}}_{\mathrm{I}_m} = B\underbrace{B^\dagger A}_{\Theta} J$$

That is, $AJ = B\Theta J$. But since J is invertible we arrive at $A = B\Theta$, as desired. $\diamond$