



EE210A: Adaptation and Learning

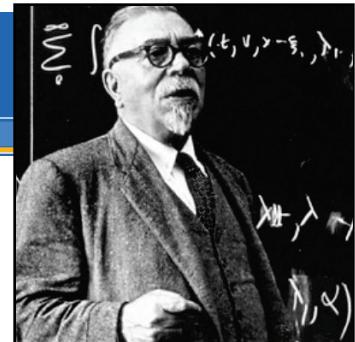
Professor Ali H. Sayed



LECTURE #08

WIENER FILTERING

(Material from handout on Wiener Filtering)



Norbert Wiener
(1894-1964)

Z-Spectra

In preparation for the development of the Wiener filter theory, we consider two zero-mean jointly wide-sense stationary random processes $\{x(n), y(n)\}$ and assume that their auto- and cross-correlation sequences $\{r_x(\ell), r_y(\ell), r_{xy}(\ell)\}$ are exponentially bounded so that the corresponding z -spectra $\{S_x(z), S_y(z), S_{xy}(z)\}$ have well-defined ROCs that include the unit circle

$$r_x(\ell) \triangleq \mathbb{E} x(n)x^*(n-\ell), \quad S_x(z) = \sum_{\ell=-\infty}^{\infty} r_x(\ell)z^{-\ell} \quad (35.1a)$$

$$r_y(\ell) \triangleq \mathbb{E} y(n)y^*(n-\ell), \quad S_y(z) = \sum_{\ell=-\infty}^{\infty} r_y(\ell)z^{-\ell} \quad (35.1b)$$

$$r_{xy}(\ell) \triangleq \mathbb{E} x(n)y^*(n-\ell), \quad S_{xy}(z) = \sum_{\ell=-\infty}^{\infty} r_{xy}(\ell)z^{-\ell} \quad (35.1c)$$

Regular Spectra

To avoid degenerate situations, we shall assume that $S_y(z)$ has no zeros on the unit circle so that

$$S_y(e^{j\omega}) > 0, \quad \omega \in [0, 2\pi] \quad (35.1d)$$

Moreover, to facilitate the calculations of the solution, we shall further assume that the spectra $\{S_x(z), S_y(z), S_{xy}(z)\}$ are rational functions in the arguments $\{z, z^{-1}\}$.

WIENER SMOOTHING

We consider first the smoothing problem, whose solution turns out to be rather trivial. In this formulation, for every time instant n , we desire to determine the linear least-mean-squares estimator (l.l.m.s.e.) of $\mathbf{x}(n)$ using all observations, i.e., using the entire set $\{\mathbf{y}(m), -\infty < m < \infty\}$. We denote the estimator (or smoother) by $\hat{\mathbf{x}}_s(n)$, with a subscript s , and express it in the form of an infinite sum

$$\hat{\mathbf{x}}_s(n) \triangleq \sum_{m=-\infty}^{\infty} w(n, m) \mathbf{y}(m) \quad (35.2)$$

for some combination coefficients $\{w(n, m)\}$ that we need to determine. Observe that we are assuming, for now, that these coefficients depend on n ; the analysis, however, will show that the joint stationarity of the random processes $\{\mathbf{x}(n), \mathbf{y}(n)\}$ will result in coefficients $\{w(n, m)\}$ that are only dependent on the difference $n - m$.

FILTER DESIGN

According to the orthogonality condition, the estimation error must be orthogonal to the observations that are used to generate the error and, hence, it must hold that:

$$\mathbf{x}(n) - \hat{\mathbf{x}}_s(n) \perp \mathbf{y}(\ell), \quad \text{for each } \ell \quad (35.3a)$$

or, equivalently,

$$\left(\mathbf{x}(n) - \sum_{m=-\infty}^{\infty} w(n, m) \mathbf{y}(m) \right) \perp \mathbf{y}(\ell), \quad \text{for each } \ell \quad (35.3b)$$

Computing the correlation between the estimation error and the observations yields the equations

$$r_{xy}(n - \ell) = \sum_{m=-\infty}^{\infty} w(n, m) r_y(m - \ell), \quad \text{for each } \ell \quad (35.4)$$

FILTER DESIGN

variables:

$$n' = n - \ell \quad \text{and} \quad m' = m - \ell \quad (35.5)$$

Then relation (35.4) becomes

$$r_{xy}(n') = \sum_{m'=-\infty}^{\infty} w(n' + \ell, m' + \ell) r_y(m'), \quad \text{for all } -\infty < \ell < \infty \quad (35.6)$$

Observe that the term on the left-hand side of the above equality does not depend on ℓ .

This fact implies that the coefficients $w(n' + \ell, m' + \ell)$ should not depend on ℓ , i.e.,

$$w(n', m') = w(n' + \ell, m' + \ell), \quad \text{for all } \ell \quad (35.7)$$

FILTER DESIGN

This result in turn means that the apparently two-dimensional sequence, $w(n', m')$, should in fact be one-dimensional and dependent on the difference $n' - m'$ only. To see this, pick any time indices (n'', m'') whose difference is the same as that between (n', m') , i.e.,

$$n'' - m'' = n' - m' \quad (35.8)$$

This means that if we write $n'' = n' + \ell_o$, for some integer ℓ_o , then it also holds that $m'' = m' + \ell_o$ for the same ℓ_o . We then conclude from (35.7) that

$$w(n'', m'') = w(n', m') \quad (35.9)$$

so that we can replace the notation $w(n', m')$ by $w(n' - m')$ and rewrite (35.6) in the form

$$r_{xy}(n') = \sum_{m'=-\infty}^{\infty} w(n' - m') r_y(m'), \quad \text{for all } -\infty < n' < \infty \quad (35.10)$$

FILTER DESIGN

We therefore arrive at a convolution expression indicating that the sequence $\{r_{xy}(\cdot)\}$ is the output of an LTI system with impulse response sequence $\{w(\cdot)\}$ and input sequence $\{r_y(\cdot)\}$. Computing the z -transforms of both sides of the above equality we get

$$S_{xy}(z) = W_s(z)S_y(z) \quad (35.11)$$

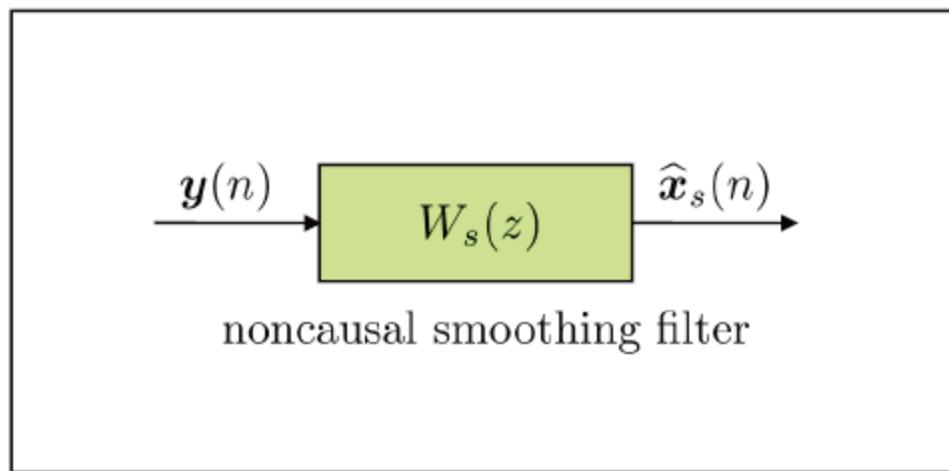
where $W_s(z)$ denotes the transfer function of the smoothing filter whose impulse response sequence is $w(\cdot)$. It follows that the desired filter is given by:

$$W_s(z) = \frac{S_{xy}(z)}{S_y(z)} \quad (\text{Wiener smoothing filter}) \quad (35.12)$$

and its frequency response is

$$W_s(e^{j\omega}) = \frac{S_{xy}(e^{j\omega})}{S_y(e^{j\omega})} \quad (\text{Wiener smoothing filter}) \quad (35.13)$$

SMOOTHING FILTER



$$W_s(z) = \frac{S_{xy}(z)}{S_y(z)}$$

(Wiener smoothing filter) (35.12)

MEAN-SQUARE-ERROR

Mean-Square-Error Performance

$$\tilde{x}_s(n) \triangleq x(n) - \hat{x}_s(n) \quad (35.14)$$

Then:

$$\begin{aligned} \mathbb{E} \tilde{x}_s(n) \tilde{x}_s^*(n-\ell) &= \mathbb{E} (x(n) - \hat{x}_s(n)) \cdot (x(n-\ell) - \hat{x}_s(n-\ell))^* \\ &= \mathbb{E} x(n) x^*(n-\ell) + \mathbb{E} \hat{x}_s(n) \hat{x}_s^*(n-\ell) - \\ &\quad \mathbb{E} x(n) \hat{x}_s^*(n-\ell) - \mathbb{E} \hat{x}_s(n) x^*(n-\ell) \end{aligned} \quad (35.15)$$

We evaluate the last two terms from the following result

$$\begin{aligned} \mathbb{E} x(n) \hat{x}_s^*(n-\ell) &= \mathbb{E} [\tilde{x}_s(n) + \hat{x}_s(n)] \cdot \hat{x}_s^*(n-\ell) \\ &= \mathbb{E} \hat{x}_s(n) \hat{x}_s^*(n-\ell) \\ &\triangleq r_{\hat{x}}(\ell) \end{aligned} \quad (35.16)$$

where in the second step we used the orthogonality condition $\tilde{x}_s(n) \perp \hat{x}_s(n-\ell)$,

MEAN-SQUARE-ERROR

Therefore,

$$r_{\tilde{x}}(\ell) = r_x(\ell) - r_{\hat{x}}(\ell) \quad (35.17)$$

and, consequently, in terms of the corresponding z -spectra:

$$\begin{aligned} S_{\tilde{x}}(z) &= S_x(z) - S_{\hat{x}}(z) \\ &= S_x(z) - W_s(z)S_y(z) \left[W\left(\frac{1}{z^*}\right) \right]^* \end{aligned} \quad (35.19)$$

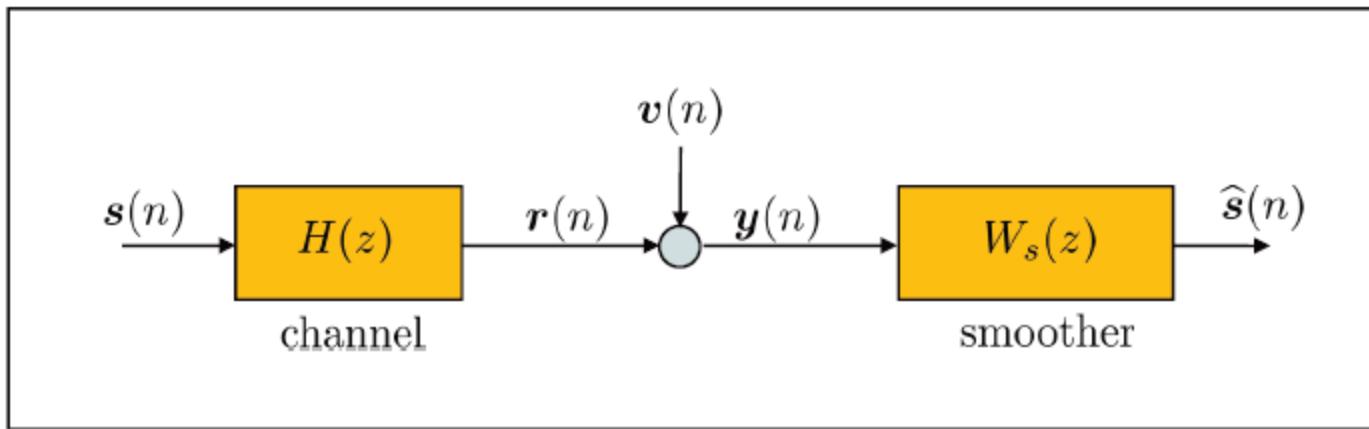
That is,

$$S_{\tilde{x}}(e^{j\omega}) = S_x(e^{j\omega}) - \frac{|S_{xy}(e^{j\omega})|^2}{S_y(e^{j\omega})} \quad (35.20)$$

It follows from (32.185) that the mean-square-error (which is the power or variance of $\tilde{x}_s(n)$) is given by

$$\text{m.m.s.e.} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[S_x(e^{j\omega}) - \frac{|S_{xy}(e^{j\omega})|^2}{S_y(e^{j\omega})} \right] d\omega \quad (35.21)$$

EXAMPLE: EQUALIZATION



The objective is to design a smoother that estimates $s(n)$ by using all measurements $\{y(m), -\infty < m < \infty\}$

$$W_s(z) = \frac{S_{sy}(z)}{S_y(z)} \quad (35.22)$$

Now note that

$$y(n) = r(n) + v(n) \quad (35.23)$$

It then follows that

EXAMPLE: EQUALIZATION

$$\begin{aligned} r_y(\ell) &= \mathbb{E} y(n)y^*(n-\ell) \\ &= \mathbb{E} (r(n) + v(n))(r(n-\ell) + v(n-\ell))^* \\ &= \mathbb{E} r(n)r^*(n-\ell) + \mathbb{E} v(n)v^*(n-\ell) \\ &= r_r(\ell) + r_v(\ell) \end{aligned} \tag{35.24}$$

and, therefore,

$$\begin{aligned} S_y(z) &= S_r(z) + S_v(z) \\ &= H(z) \cdot S_s(z) \cdot \left[H\left(\frac{1}{z^*}\right) \right]^* + \sigma_v^2 \\ &= \sigma_s^2 \cdot H(z) \cdot \left[H\left(\frac{1}{z^*}\right) \right]^* + \sigma_v^2 \end{aligned} \tag{35.25}$$

since

$$S_s(z) = \sigma_s^2 \quad \text{and} \quad S_v(z) = \sigma_v^2 \tag{35.26}$$

EXAMPLE: EQUALIZATION

Moreover, observe that

$$\begin{aligned}\mathbb{E} s(n)y^*(n-\ell) &= \mathbb{E} s(n)r^*(n-\ell) + \mathbb{E} s(n)v^*(n-\ell) \\ &= \mathbb{E} s(n)r^*(n-\ell)\end{aligned}\quad (35.27)$$

so that $r_{sy}(\ell) = r_{sr}(\ell)$ and $S_{sy}(z) = S_{sr}(z)$. Therefore, we need to determine $S_{sr}(z)$, which in view of the result of Example 32.22, is given by

$$S_{sr}(z) = S_s(z) \cdot \left[H\left(\frac{1}{z^*}\right) \right]^* = \sigma_s^2 \cdot \left[H\left(\frac{1}{z^*}\right) \right]^* \quad (35.28)$$

The Wiener smoother is then given by

$$W_s(z) = \frac{\left[H\left(\frac{1}{z^*}\right) \right]^*}{H(z) \cdot \left[H\left(\frac{1}{z^*}\right) \right]^* + \frac{1}{\text{SNR}}} \quad (35.29)$$

where we introduced the signal-to-noise ratio factor, $\text{SNR} = \sigma_s^2/\sigma_v^2$. Observe in particular that as $\text{SNR} \rightarrow \infty$ (e.g., when $\sigma_v^2 \rightarrow 0$), the expression for the equalizer reduces to the expected result

$$W_s(z) = \frac{1}{H(z)}, \quad \text{when } \text{SNR} \rightarrow \infty \quad (35.30)$$

WIENER FILTER

The second problem we consider is a filtering problem. We now desire to determine the linear least-mean-squares estimator (l.l.m.s.e.) of $x(n)$ in a *causal* manner by using only the present and past observations $\{y(m), -\infty < m \leq n\}$. We denote the estimator by $\hat{x}_f(n)$, with a subscript f , and express it in the form of an infinite sum

$$\hat{x}_f(n) \triangleq \sum_{m=-\infty}^n w(n, m) y(m) \quad (35.32)$$

We again appeal to the orthogonality condition (33.89) of linear least-mean-squares estimation, which states that the estimation error must be orthogonal to the observations that are used to generate the error, namely,

$$x(n) - \hat{x}_f(n) \perp y(\ell), \quad \text{for each } -\infty < \ell \leq n \quad (35.33a)$$

or, equivalently,

$$\left(x(n) - \sum_{m=-\infty}^n w(n, m) y(m) \right) \perp y(\ell), \quad \text{for each } -\infty < \ell \leq n \quad (35.33b)$$

FILTER DESIGN

Computing the correlation between the estimation error and the observations yields the equations:

$$r_{xy}(n - \ell) = \sum_{m=-\infty}^n w(n, m) r_y(m - \ell), \quad \text{for each } -\infty < \ell \leq n \quad (35.34)$$

First, let us introduce the change of variables:

$$n' = n - \ell \quad \text{and} \quad m' = m - \ell \quad (35.35)$$

Then relation (35.34) becomes

$$r_{xy}(n') = \sum_{m'=-\infty}^{n'} w(n' + \ell, m' + \ell) \cdot r_y(m'), \quad \text{for all } n' \geq 0 \quad (35.36)$$

$$\begin{aligned} &= \sum_{m'=-\infty}^{n'} w(n' - m') r_y(m') \\ &= \sum_{m'=0}^{\infty} w(m') r_y(n' - m'), \quad \text{for all } n' \geq 0 \end{aligned} \quad (35.37)$$

WIENER-HOPF EQUATIONS

Dropping the prime notation, we are therefore faced with determining the impulse response sequence $\{w(m), m \geq 0\}$ of a causal filter that solve the equation:

$$r_{xy}(n) = \sum_{m=0}^{\infty} w(m) r_y(n-m), \quad \text{for all } n \geq 0 \quad (35.38)$$

where equality holds only over nonnegative values of n . Relation (35.38) is called the Wiener-Hopf equation. The technique for solving it is called the Wiener-Hopf technique. Introduce the difference sequence

$$q(n) \triangleq r_{xy}(n) - \sum_{m=0}^{\infty} w(m) \cdot r_y(n-m), \quad \text{for all } n \quad (35.39)$$

Then, we know from (35.38) that $q(n)$ is a strictly anti-causal sequence since

$$q(n) = 0, \quad \text{for } n \geq 0 \quad (35.40)$$

Therefore, the z -transform of $q(n)$ is the transform of a strictly anti-causal sequence (which implies that its ROC will be the inside of a circular region).

WIENER-HOPF EQUATIONS

Let $W_f(z)$ denote the transfer function of the causal Wiener filter whose impulse response sequence is $\{w_m, m \geq 0\}$. Evaluating the z -transform of both sides of (35.39) we get

$$Q(z) = S_{xy}(z) - W_f(z)S_y(z) \quad (35.41)$$

This relation does not allow us to determine $W_f(z)$ because we do not know $Q(z)$ — we only know that the samples of $q(n)$ are zero over $n \geq 0$ but do not know the samples of $q(n)$ over $n < 0$.

To arrive at the expression for the Wiener filter $W_f(z)$, we first introduce the spectral factorization of $S_y(z)$ (recall (32.226)):

$$S_y(z) = \gamma \cdot L(z) \left[L\left(\frac{1}{z^*}\right) \right]^* \quad (35.42)$$

in terms of the canonical factor $L(z)$ and the scalar $\gamma > 0$. Recall that $L(z)$ is a stable minimum-phase filter (its poles and zeros are inside the unit circle) and satisfies the normalization $L(\infty) = 1$. Substituting into (35.41) gives

WIENER-HOPF EQUATIONS

$$\frac{Q(z)}{\left[L\left(\frac{1}{z^*}\right)\right]^*} = \frac{S_{xy}(z)}{\left[L\left(\frac{1}{z^*}\right)\right]^*} - \gamma \cdot W_f(z)L(z) \quad (35.43)$$

Now recall from (32.228) that $L(z)$ is causal and causally invertible (i.e., $L(z)$ and its inverse are the z -transforms of causal sequences). Likewise, recall from (32.229) that $[L(1/z^*)]^*$ and its inverse are the z -transforms of anti-causal sequences. It follows that

$$\frac{Q(z)}{\left[L\left(\frac{1}{z^*}\right)\right]^*} = \underbrace{Q(z)}_{\substack{\text{strictly} \\ \text{anticausal} \\ \text{inverse}}} \cdot \underbrace{\frac{1}{\left[L\left(\frac{1}{z^*}\right)\right]^*}}_{\substack{\text{anticausal} \\ \text{inverse}}} \quad (35.44)$$

is the transfer function of a strictly anti-causal sequence; this is because the convolution of a strictly anti-causal sequence with an anti-causal sequence is a strictly anti-causal sequence. On the other hand, the product $W_f(z)L(z)$ is the transform of a causal sequence. We can therefore write

WIENER-HOPF EQUATIONS

$$\underbrace{\frac{Q(z)}{\left[L\left(\frac{1}{z^*}\right)\right]^*}}_{\begin{array}{l} \text{strictly} \\ \text{anticausal} \\ \text{inverse} \end{array}} = \frac{S_{xy}(z)}{\left[L\left(\frac{1}{z^*}\right)\right]^*} - \gamma \cdot \underbrace{W_f(z)L(z)}_{\begin{array}{l} \text{causal} \\ \text{inverse} \end{array}} \quad (35.45)$$

where the first term on the right-hand side is the z -transform of a sequence that includes both causal and anti-causal parts; it is a sequence of mixed type. Therefore, for the equality (35.45) to hold, the Wiener filter $W_f(z)$ must be such that $W_f(z)L(z)$ should match the causal part of the mixed function $S_{xy}(z)/L^*(1/z^*)$. We therefore need to explain how to determine the causal part of a z -transform.

CAUSAL OPERATOR

given a sequence $r(n)$, we denote its causal part by $r^+(n)$. This operation retains the samples of $r(n)$ over $n \geq 0$ and sets all other samples to zero over negative time:

$$r^+(n) = \begin{cases} r(n), & n \geq 0 \\ 0, & \text{otherwise} \end{cases} \quad (35.46)$$

In other words,

$$r^+(n) = r(n)u(n) \quad (35.47)$$

in terms of the step sequence, $u(n)$. For example, consider the sequence $e^{-|n|}$. Then

$$\left\{e^{-|n|}\right\}^+ = e^{-n}u(n) \quad (35.48)$$

CAUSAL OPERATOR

We usually apply the causal operation $\{\cdot\}^+$ directly to a z -transform rather than to the sequence itself. Thus, note that the z -transforms of the sequences $r(n)$ and $r^+(n)$ are given respectively by

$$R(z) = \sum_{n=-\infty}^{\infty} r(n)z^{-n} \quad (35.49a)$$

$$R^+(z) = \sum_{n=0}^{\infty} r(n)z^{-n} \quad (35.49b)$$

where the lower limit of the summation for $R^+(z)$ starts at $n = 0$. The transform $R^+(z)$ is simply the unilateral z -transform of the sequence $r(n)$.

CAUSAL OPERATOR

Constants. The causal part of a constant transfer function is the constant transfer function again, i.e., for any constant c :

$$\{c\}^+ = c \quad (35.50)$$

This is because the inverse transform is $c\delta(n)$, which is causal.

First-order model. Consider a first-order auto-regressive transfer function of the form:

$$R(z) = \frac{1}{z + a} \quad (35.51)$$

whose ROC includes the unit circle. Two cases are possible depending on the value of a , which determines the location of the pole of $R(z)$ relative to the unit circle. If $|a| < 1$, then the ROC must be $|z| > |a|$ and, therefore, the corresponding sequence $r(n)$ must be causal. It follows that

$$\left\{ \frac{1}{z + a} \right\}^+ = \frac{1}{z + a}, \quad \text{when } |a| < 1 \quad (35.52)$$

CAUSAL OPERATOR

On the other hand, when $|a| > 1$, then the ROC must be $|z| < |a|$. In this case, the sequence $r(n)$ must be anti-causal; its sample at $n = 0$ can be found by evaluating $R(z)$ at $z = 0$,

$$r(0) = R(z)|_{z=0} = \frac{1}{a} \quad (35.53)$$

Therefore,

$$\left\{ \frac{1}{z+a} \right\}^+ = \frac{1}{a}, \quad \text{when } |a| > 1 \quad (35.54)$$

CAUSAL OPERATOR

Higher-order models. Consider a p -th order auto-regressive transfer function of the form:

$$R(z) = \frac{1}{(z + a)^p} \quad (35.55)$$

whose ROC includes the unit circle. Again, two cases are possible depending on the value of a . The same arguments from the previous case lead to the following conclusion:

$$\left\{ \frac{1}{(z + a)^p} \right\}^+ = \begin{cases} \frac{1}{(z + a)^p}, & \text{when } |a| < 1 \\ 1/a^p, & \text{when } |a| > 1 \end{cases} \quad (35.56)$$

EXAMPLE

Let us evaluate the causal part of the transfer function

$$R(z) = \frac{5/2}{z^2 + \frac{7}{2}z + \frac{3}{2}}, \quad \frac{1}{2} < |z| < 3 \quad (35.57)$$

Using partial fractions we have

$$R(z) = \frac{1}{z + \frac{1}{2}} - \frac{1}{z + 3} \quad (35.58)$$

Therefore,

$$\begin{aligned} \{R(z)\}^+ &= \left\{ \frac{1}{z + \frac{1}{2}} \right\}^+ - \left\{ \frac{1}{z + 3} \right\}^+ \\ &= \frac{1}{z + \frac{1}{2}} - \frac{1}{3} \\ &= \frac{5 - 2z}{3 + 6z} \end{aligned} \quad (35.59)$$



SOLVING WIENER-HOPF EQUATIONS

$$\frac{Q(z)}{\left[L\left(\frac{1}{z^*}\right)\right]^*} = \frac{S_{xy}(z)}{\left[L\left(\frac{1}{z^*}\right)\right]^*} - \gamma \cdot W_f(z)L(z) \quad (35.43)$$

Returning to (35.43) and applying the causal operator $\{\cdot\}^+$ to both sides of the equation we must have

$$\begin{aligned} \left\{ \frac{Q(z)}{\left[L\left(\frac{1}{z^*}\right)\right]^*} \right\}^+ &= \left\{ \frac{S_{xy}(z)}{\left[L\left(\frac{1}{z^*}\right)\right]^*} \right\}^+ - \gamma \cdot \{W_f(z)L(z)\}^+ \\ &= \left\{ \frac{S_{xy}(z)}{\left[L\left(\frac{1}{z^*}\right)\right]^*} \right\}^+ - \gamma W_f(z)L(z) \end{aligned} \quad (35.60)$$

SOLVING WIENER-HOPF EQUATIONS

But we already know that the term on the left-hand side has a strictly anticausal inverse sequence and, therefore, its causal part is zero:

$$\left\{ \frac{Q(z)}{\left[L\left(\frac{1}{z^*} \right) \right]^*} \right\}^+ = 0 \quad (35.61)$$

We therefore arrive at the following expression for the desired Wiener filter:

$$W_f(z) = \frac{1}{\gamma L(z)} \cdot \left\{ \frac{S_{xy}(z)}{\left[L\left(\frac{1}{z^*} \right) \right]^*} \right\}^+ \quad (35.62)$$

SOLVING WIENER-HOPF EQUATIONS

$$W_f(z) = \frac{1}{\gamma L(z)} \cdot \left\{ \frac{S_{xy}(z)}{\left[L\left(\frac{1}{z^*}\right) \right]^*} \right\}^+ \quad (35.62)$$

Now recall that, by assumption, the ROC of $S_{xy}(z)$ includes the unit circle. Moreover, the ROC of the factor $1/L^*(1/z^*)$ includes the unit circle. Consequently, the ROC of the ratio $S_{xy}(z)/L^*(1/z^*)$ includes the unit circle. It follows that the ROC of its causal part also includes the unit circle:

$$\text{ROC of } \left\{ \frac{S_{xy}(z)}{\left[L\left(\frac{1}{z^*}\right) \right]^*} \right\}^+ \text{ includes the unit circle} \quad (35.63)$$

But since the ROC of $1/L(z)$ includes the unit circle as well, we conclude that the ROC of $W_f(z)$ includes the unit circle. As a result, we find that the resulting Wiener filter $W_f(z)$ is a BIBO stable (and causal) filter.

MEAN-SQUARE-ERROR

We can evaluate the z -spectrum (and the power spectral density) of the resulting estimation error process defined by

$$\tilde{x}_f(n) = x(n) - \hat{x}_f(n) \quad (35.64)$$

as follows:

$$\begin{aligned} S_{\tilde{x}}(z) &= S_x(z) - S_{\hat{x}}(z) \\ &= S_x(z) - W_f(z)S_y(z) \left[W_f \left(\frac{1}{z^*} \right) \right]^* \end{aligned} \quad (35.65)$$

so that the error power spectral density is

$$\begin{aligned} S_{\tilde{x}}(e^{j\omega}) &= S_x(e^{j\omega}) - S_{\hat{x}}(e^{j\omega}) \\ &= S_x(e^{j\omega}) - W_f(e^{j\omega})S_y(e^{j\omega})W_f^*(e^{j\omega}) \\ &= S_x(e^{j\omega}) - \left| \frac{1}{\gamma \cdot L(e^{j\omega})} \left\{ \frac{S_{xy}(e^{j\omega})}{L^*(e^{j\omega})} \right\}^+ \right|^2 \cdot \gamma \cdot |L(e^{j\omega})|^2 \end{aligned} \quad (35.66)$$

MEAN-SQUARE-ERROR

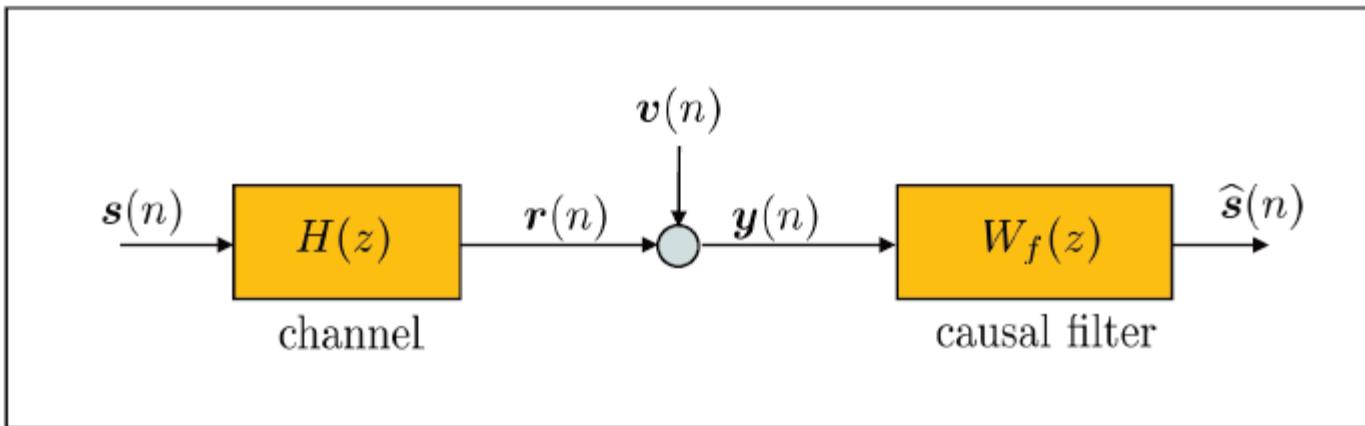
That is,

$$S_{\tilde{x}}(e^{j\omega}) = S_x(e^{j\omega}) - \frac{1}{\gamma} \cdot \left| \left\{ \frac{S_{xy}(e^{j\omega})}{L^*(e^{j\omega})} \right\}^+ \right|^2 \quad (35.67)$$

It follows from (32.185) that the mean-square-error (which is the power or variance of $\tilde{x}_f(n)$) is:

$$\text{m.m.s.e.} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[S_x(e^{j\omega}) - \frac{1}{\gamma} \cdot \left| \left\{ \frac{S_{xy}(e^{j\omega})}{L^*(e^{j\omega})} \right\}^+ \right|^2 \right] d\omega \quad (35.68)$$

EXAMPLE: EQUALIZATION



we are interested in estimating $s(n)$ in a causal manner
from present and past observations $\{y(m), -\infty < m \leq n\}$

EXAMPLE: EQUALIZATION

The Wiener filter is given by

$$W_f(z) = \frac{1}{\gamma \cdot L(z)} \left\{ \frac{S_{sy}(z)}{\left[L\left(\frac{1}{z^*}\right) \right]^*} \right\}^+ \quad (35.69)$$

in terms of the canonical spectral factor of $S_y(z)$, namely,

$$S_y(z) = \sigma_s^2 \cdot H(z) \cdot \left[H\left(\frac{1}{z^*}\right) \right]^* + \sigma_v^2 \triangleq \gamma \cdot L(z) \cdot \left[L\left(\frac{1}{z^*}\right) \right]^* \quad (35.70)$$

Moreover,

$$S_{sy}(z) = S_{sr}(z) = \sigma_s^2 \cdot \left[H\left(\frac{1}{z^*}\right) \right]^* \quad (35.71)$$

Therefore, the Wiener filter is given by

$$W_f(z) = \frac{1}{\gamma \cdot L(z)} \left\{ \frac{\sigma_s^2 \cdot \left[H\left(\frac{1}{z^*}\right) \right]^*}{\left[L\left(\frac{1}{z^*}\right) \right]^*} \right\}^+ \quad (35.72)$$

EXAMPLE: EQUALIZATION

Assume the channel is FIR and given by $H(z) = 1 + az^{-1}$ for some real scalar a . Let us examine some special cases:

No noise. Assume initially that there is no noise and set $\sigma_v^2 = 0$. Then

$$\begin{aligned} S_y(z) &= \sigma_s^2 \cdot H(z) \cdot \left[H\left(\frac{1}{z^*}\right) \right]^* \\ &= \sigma_s^2 \cdot (1 + az^{-1}) \cdot (1 + az) \end{aligned} \tag{35.73}$$

Assume first that $|a| < 1$. We can rewrite $S_y(z)$ as

$$S_y(z) = \sigma_s^2 \cdot (z + a) \cdot (z^{-1} + a) \tag{35.74}$$

so that

$$L(z) = z^{-1}(z + a) = 1 + az^{-1}, \quad \gamma = \sigma_s^2 \tag{35.75}$$

EXAMPLE: EQUALIZATION

Note that $L(\infty) = 1$ and the zero and pole of $L(z)$ lie inside the unit circle. Substituting into the expression for $W_f(z)$ we get

$$W_f(z) = \frac{1}{\sigma_s^2 \cdot (1 + az^{-1})} \left\{ \frac{\sigma_s^2 \cdot (1 + az)}{1 + az} \right\}^+ \quad (35.76)$$

That is,

$$W_f(z) = \frac{1}{1 + az^{-1}}, \quad \text{when } |a| < 1 \text{ and } \sigma_v^2 = 0 \quad (35.77)$$

Thus, in this case, the Wiener filter is simply the inverse of the channel.

EXAMPLE: EQUALIZATION

Assume next that that $|a| > 1$. Then,

$$\begin{aligned} S_y(z) &= \sigma_s^2 \cdot (1 + az^{-1}) \cdot (1 + az) \\ &= \sigma_s^2 \cdot a^2 \cdot \left(z + \frac{1}{a} \right) \cdot \left(z^{-1} + \frac{1}{a} \right) \end{aligned} \quad (35.78)$$

so that now

$$L(z) = z^{-1} \left(z + \frac{1}{a} \right) = 1 + \frac{1}{a} z^{-1}, \quad \gamma = \sigma_s^2 \cdot a^2 \quad (35.79)$$

EXAMPLE: EQUALIZATION

Substituting into the expression for $W_f(z)$ we now get

$$\begin{aligned} W_f(z) &= \frac{1}{\sigma_s^2 \cdot a^2 \cdot (1 + \frac{1}{a}z^{-1})} \left\{ \frac{\sigma_s^2 \cdot (1 + az)}{1 + \frac{1}{a}z} \right\}^+ \\ &= \frac{1}{a^2 \cdot (1 + \frac{1}{a}z^{-1})} \left[\left\{ \frac{1}{1 + \frac{1}{a}z} \right\}^+ + a \cdot \left\{ \frac{z}{1 + \frac{1}{a}z} \right\}^+ \right] \\ &= \frac{1}{a^2 \cdot (1 + \frac{1}{a}z^{-1})} \left[a \cdot \left\{ \frac{1}{z+a} \right\}^+ + a^2 \cdot \left\{ \frac{z}{z+a} \right\}^+ \right] \\ &= \frac{1}{a^2 \cdot (1 + \frac{1}{a}z^{-1})} \left[a \cdot \left\{ \frac{1}{z+a} \right\}^+ + a^2 \cdot \left\{ 1 - \frac{a}{z+a} \right\}^+ \right] \\ &= \frac{1}{a^2 \cdot (1 + \frac{1}{a}z^{-1})} \left[a \cdot \frac{1}{a} + a^2 \cdot 1 - a^2 \cdot a \cdot \frac{1}{a} \right] \\ &= \frac{1}{a^2} \cdot \frac{1}{1 + \frac{1}{a}z^{-1}} \end{aligned} \tag{35.80}$$

EXAMPLE: EQUALIZATION

That is,

$$W_f(z) = \frac{1}{a} \cdot \frac{1}{z^{-1} + a}, \quad \text{when } |a| > 1 \text{ and } \sigma_v^2 = 0 \quad (35.81)$$

In this case, the Wiener filter is not the inverse of the channel.

Large SNR. Let us consider the case in which the SNR is large, where $\text{SNR} = \sigma_s^2 / \sigma_v^2$. If we examine the original expression for $S_y(z)$ we see that we can write

$$\begin{aligned} S_y(z) &= \sigma_s^2 \cdot H(z) \cdot \left[H\left(\frac{1}{z^*}\right) \right]^* + \sigma_v^2 \\ &= \sigma_s^2 \cdot \left(H(z) \cdot \left[H\left(\frac{1}{z^*}\right) \right]^* + \frac{1}{\text{SNR}} \right) \end{aligned} \quad (35.82)$$

Therefore, as $\text{SNR} \rightarrow \infty$, the expression for $S_y(z)$ reduces to the form we studied in the earlier case of zero noise, namely,

$$S_y(z) \rightarrow \sigma_s^2 \cdot H(z) \cdot \left[H\left(\frac{1}{z^*}\right) \right]^* \quad (35.83)$$

EXAMPLE: ONE-STEP PREDICTION

Consider a zero-mean wide-sense stationary process $\{y(n)\}$ with z -spectrum $S_y(z)$. We wish to determine the Wiener filter that predicts $y(n+1)$ from present and past observations $\{y(m), -\infty < m \leq n\}$. In this problem, the variable that we want to estimate is $x(n) = y(n + 1)$. Therefore, according to the Wiener filter expression (35.62), the desired filter is given by

$$W_f(z) = \frac{1}{\gamma \cdot L(z)} \left\{ \frac{S_{xy}(z)}{\left[L\left(\frac{1}{z^*}\right) \right]^*} \right\}^+ \quad (35.84)$$

Now, since $x(n) = y(n + 1)$, we can regard $x(n)$ as the output of a system with transfer function equal to z and input sequence equal to $y(n)$. Therefore,

$$S_{xy}(z) = zS_y(z) \quad (35.85)$$

EXAMPLE: ONE-STEP PREDICTION

Substituting into $W_f(z)$ we get

$$W_f(z) = \frac{1}{\gamma \cdot L(z)} \left\{ \frac{zS_y(z)}{\left[L\left(\frac{1}{z^*}\right) \right]^*} \right\}^+ \quad (35.86)$$

Replacing $S_y(z)$ by its spectral factorization, $S_y(z) = \gamma L(z)L^*(1/z^*)$, we obtain

$$W_f(z) = \frac{1}{\gamma \cdot L(z)} \{ \gamma z L(z) \}^+ \quad (35.87)$$

Now recall that, by definition, the canonical spectral factor $L(z)$ is a causal system satisfying $L(\infty) = 1$. This means that $L(z)$ has the form

$$L(z) = 1 + \ell(1)z^{-1} + \ell(2)z^{-2} + \ell(3)z^{-3} + \dots \quad (35.88)$$

with impulse response sequence $\{\ell(n)\}$. Consequently,

$$\begin{aligned} \{zL(z)\}^+ &= \{z + \ell(1) + \ell(2)z^{-1} + \ell(3)z^{-2} + \dots\}^+ \\ &= \ell(1) + \ell(2)z^{-1} + \ell(3)z^{-2} + \dots \\ &= zL(z) - z \end{aligned} \quad (35.89)$$

EXAMPLE: ONE-STEP PREDICTION

and the expression for $W_f(z)$ simplifies to

$$W_f(z) = \frac{1}{L(z)}(zL(z) - z) \quad (35.90)$$

$$= z \cdot \left(1 - \frac{1}{L(z)}\right) \quad (35.91)$$

This is the filter that predicts $y(n+1)$ from $\{y(m), -\infty < m \leq n\}$. It follows that the filter that predicts $y(n)$ from $\{y(m), -\infty < m \leq n-1\}$ is given by

$$W_p(z) = 1 - \frac{1}{L(z)} \quad (35.92)$$

That is because $\hat{y}(n)$ is delayed by one time instant relative to $\hat{y}(n+1)$. Therefore, the filter that generates $\hat{y}(n)$ should be equal to $W_f(z)$ multiplied by z^{-1} , i.e., $W_p(z) = z^{-1}W_f(z)$.

EXAMPLE: ONE-STEP PREDICTION

For illustration purposes, assume $y(n)$ is a random process with an exponential auto-correlation sequence, as was studied in Example 32.24. Then,

$$S_y(z) = \frac{1 - \rho^2}{(1 - \rho z^{-1})(1 - \rho z)}, \quad |\rho| < |z| < \frac{1}{|\rho|} \quad (35.93)$$

and

$$L(z) = \frac{1}{1 - \rho z^{-1}}, \quad \gamma = 1 - \rho^2 \quad (35.94)$$

It follows that the prediction filter is given by

$$W_p(z) = \rho z^{-1} \quad (35.95)$$

In other words,

$$\hat{y}(n) = \rho \cdot y(n-1) \quad (35.96)$$

EXAMPLE: NOISY DATA

A zero-mean wide-sense stationary process $x(n)$ is observed under additive white noise $v(n)$ with power σ_v^2 :

$$y(n) = x(n) + v(n) \quad (35.102)$$

The noise samples are independent of the process $\{x(n)\}$. We wish to derive a filter that estimates $x(n)$ from present and past measurements $\{y(m), -\infty < m \leq n\}$. The Wiener filter is given by

$$W_f(z) = \frac{1}{\gamma \cdot L(z)} \left\{ \frac{S_{xy}(z)}{\left[L\left(\frac{1}{z^*}\right) \right]^*} \right\}^+ \quad (35.103)$$

in terms of the canonical spectral factor of $S_y(z)$. Now note that

$$S_y(z) = S_x(z) + S_v(z) = S_x(z) + \sigma_v^2 \quad (35.104)$$

and

$$S_{xy}(z) = S_x(z) \quad (35.105)$$

EXAMPLE: NOISY DATA

so that

$$\begin{aligned} W_f(z) &= \left\{ \frac{S_x(z)}{\left[L\left(\frac{1}{z^*}\right) \right]^*} \right\}^+ \cdot \frac{1}{\gamma \cdot L(z)} \\ &= \left\{ \frac{S_y(z) - \sigma_v^2}{\left[L\left(\frac{1}{z^*}\right) \right]^*} \right\}^+ \cdot \frac{1}{\gamma \cdot L(z)} \\ &= \left\{ \frac{\gamma \cdot L(z) \cdot \left[L\left(\frac{1}{z^*}\right) \right]^* - \sigma_v^2}{\left[L\left(\frac{1}{z^*}\right) \right]^*} \right\}^+ \cdot \frac{1}{\gamma \cdot L(z)} \\ &= \left[\gamma \cdot \{L(z)\}^+ - \left\{ \frac{\sigma_v^2}{\left[L\left(\frac{1}{z^*}\right) \right]^*} \right\}^+ \right] \cdot \frac{1}{\gamma \cdot L(z)} \\ &= [\gamma \cdot L(z) - \sigma_v^2] \cdot \frac{1}{\gamma \cdot L(z)} \end{aligned} \tag{35.106}$$

and we conclude that

$$W_f(z) = 1 - \frac{\sigma_v^2}{\gamma} \cdot \frac{1}{L(z)} \tag{35.107}$$

EXAMPLE: NOISY DATA

where we used the fact that $L(z)$ is causal so that

$$\{L(z)\}^+ = L(z) \quad (35.108)$$

and $L^*(1/z^*)$ is anti-causal with a sample equal to 1 at time $n = 0$ since $L^*(0) = 1$ so that

$$\left\{ \frac{1}{[L(\frac{1}{z^*})]^*} \right\}^+ = 1 \quad (35.109)$$